① $$\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)}\,dx =$$

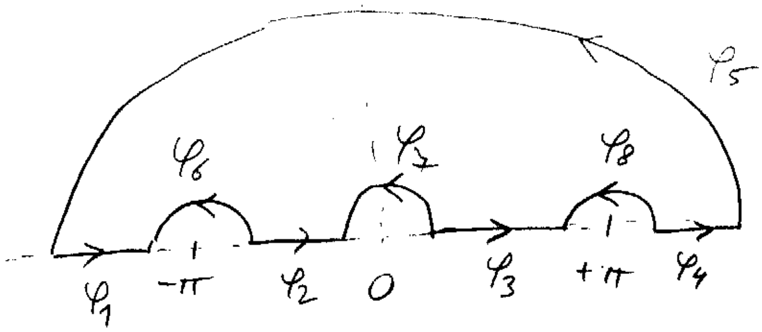
② $$\left[\frac{1}{(x^2 + ix + 2)^2}\right]^{\wedge}(\xi) =$$

③ $$f(z) = \left(z + \exp\left(\frac{1}{z}\right)\right)^2 ;$$

v $z_0 = 0$: 2 členy Laurentova rozvoje
residuum.
typ singularity.

① $g(z) = \dfrac{e^{iz}}{z(\pi^2 - z^2)}$;

singul.: $z = 0, \pm\pi$



$$\varphi := \varphi_1 \dot{+} \varphi_2 \dot{+} \varphi_3 \dot{+} \varphi_4 \dot{+} \varphi_5 \dot{-} \varphi_6 \dot{-} \varphi_7 \dot{-} \varphi_8 .$$

$$g(z) \in \mathcal{H}(\overset{*}{\operatorname{int}} \varphi) \Rightarrow \int_\varphi g = 0 = \sum_{j=1}^{5} \int_{\varphi_j} - \sum_{k=6}^{8} \int_{\varphi_k} .$$

$$\int_{\varphi_5} g \to 0 \ ; \ \text{neboť} \ |g(z)| = \frac{1}{|z||\pi^2 - z^2|} \le \frac{1}{R(R^2-\pi^2)} \sim \frac{1}{R^3}$$

pro $|z| = R$.

$$\sum_{j=1}^{4} \int_{\varphi_j} g = \int_{I_{r,R}} \frac{e^{ix}}{x(\pi^2 - x^2)}\, dx = i \int_{I_{r,R}} \frac{\sin x}{x(\pi^2 - x^2)} \to i\, I ; \quad \begin{matrix} R \to +\infty \\ r \to 0+ \end{matrix} ;$$

$$I_{r,R} = (-R, R) \setminus \{ U(-\pi, r) \cup U(0, r) \cup U(\pi, r) \}.$$

$$\int_{\varphi_7} g(z)\,dz \to i\pi \cdot L_7 , \ \text{kde} \ L_7 = \lim_{z \to 0} z\, g(z)$$

$$= \lim_{z \to 0} \frac{e^{iz}}{\pi^2 - z^2} = \frac{1}{\pi^2}$$

Lemma o <u>malé</u> kružnici.

$$\int_{\varphi_6} g(z)\,dz \to i\pi L_6 ; \ \text{kde} \ L_6 = \lim_{z \to -\pi} (z - (-\pi)) \cdot g(z)$$

$$= \lim_{z \to -\pi} \frac{-e^{iz}}{z(z-\pi)} = \frac{+1}{2\pi^2}$$

podobně: $\displaystyle\int_{\varphi_8} g(z) \to i\pi L_8$; $\quad L_8 = \dfrac{1}{2\pi^2}$.

celkem: $\quad 0 = \sum_{j=1}^{4} \int_{\varphi_j} - \sum_{j=6}^{10} \int_{\varphi_j} \to iI - i\pi\left(\frac{1}{2\pi^2} + \frac{1}{\pi^2} + \frac{1}{2\pi^2}\right)$

$$\Rightarrow \quad I = \frac{2}{\pi}.$$

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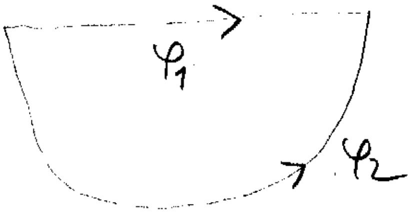
(2) $\quad g(z) = \dfrac{e^{-2\pi i \xi z}}{(z^2 + iz + 2)^2}; \qquad D = -1 - 4\cdot 2 = -9; \quad z_{1,2} = \dfrac{-i \pm 3i}{2} = \begin{cases} -2i \\ i \end{cases}$

(a) $\xi > 0$: $\qquad \varphi := \dot{-}\varphi_1 \dot{+} \varphi_2;$



$$\int_{\varphi} g = \int_{\varphi_2} - \int_{\varphi_1} = 2\pi i \cdot \operatorname{res}_{-2i} g.$$

$z \in \langle \varphi_2 \rangle \Rightarrow \left| e^{-2\pi \xi z} \right| \le 1;$ tedy: $\; |g(z)| \le \dfrac{1}{|z+2i|\cdot|z-i|}$

$\xi > 0$

$$\Rightarrow \int_{\varphi_2} g \to 0; \; R \to +\infty. \qquad \le \frac{1}{(R-2)(R-1)} \sim \frac{1}{R^2}$$

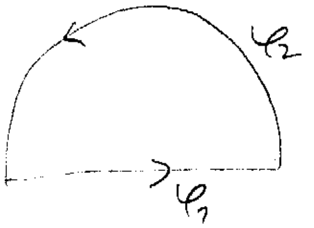
$$\int_{\varphi} g = \int_{-R}^{R} \frac{e^{-2\pi i \xi x}}{(x^2 + ix + 2)^2} \to \hat{f}(\xi); \; R \to +\infty.$$

$$\operatorname{res}_{-2i} = \left(\frac{e^{-2\pi i \xi z}}{(z-i)^2} \right)' \Bigg|_{z=-2i} = \frac{-2}{(z-i)^3} e^{-2\pi i \xi z} \left(i\pi\xi(z-i) + 1 \right) \Bigg|_{z=-2i}$$

$$= \frac{2i}{27} e^{-4\pi\xi} \left(3\pi\xi + 1 \right).$$

celkem: $f(\xi) = \frac{4\pi}{27} e^{-4\pi\xi}(3\pi\xi+1)$; $\xi > 0$.

(b) $\xi < 0$:



podobně: $\int_{\varphi_1} g \to \hat{f}(\xi)$; $\int_{\varphi_2} g \to 0$

$$\Rightarrow \quad \hat{f}(\xi) = 2\pi i \cdot \operatorname{res}_i g .$$

$$\operatorname{res}_i g = \left(\frac{e^{-2\pi i \xi z}}{(z+2i)^2} \right)' \Bigg|_{z=i} = \frac{2i}{27} \cdot e^{2\pi\xi}(3\pi\xi - 1).$$

odtud: $\hat{f}(\xi) = -\frac{4\pi}{27} e^{2\pi\xi}(3\pi\xi - 1)$; $\xi < 0$.

$$\hat{f}(0) = \lim_{\xi \to 0+} \hat{f}(\xi) = \frac{4\pi}{27} \quad (\text{spojitost; } f \in L^1 !!)$$

(3) $z + \exp\left(\frac{1}{z}\right) = z + 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots$

$$f(z) = \left(z + 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots\right) \cdot \left(z + 1 + \frac{1}{z} + \frac{1}{2z^2} + \cdots\right)$$

$$= z^2 + \underbrace{z \cdot 1 + 1 \cdot z}_{2z} + \underbrace{z \cdot \frac{1}{z} + 1 \cdot 1 + \frac{1}{z} \cdot z}_{3} +$$

$$+ \underbrace{z \cdot \frac{1}{2z^2} + 1 \cdot \frac{1}{z} + \frac{1}{z} \cdot 1 + \frac{1}{2z^2} \cdot z}_{= 3 \cdot \frac{1}{z}}$$

reziduum! ; podstatná singularita..