① $F(a) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\operatorname{arctg}\left(\frac{\cos x}{a}\right)}_{f(a,x)} dx \quad ; \quad a \in (0,\infty)$

(a) $f(a,\cdot)$ měřitelné $\Leftarrow$ spojité

$f(\cdot,x)$ spojité [[1]]

majoranta: $|f(a,x)| \le \frac{\pi}{2} =: g(x) \in L^1(-\frac{\pi}{2},\frac{\pi}{2})$

(omezený interval)

$\Rightarrow F(a)$ spojitá v $(0,\infty)$ [[2]]

(b) $f(a,\cdot)$ měřitelné $\Leftarrow$ spojité (1)

$$\frac{\partial f}{\partial a}(a,x) = \frac{1}{1+\frac{\cos^2 x}{a^2}} \cdot \cos x \cdot \left(-\frac{1}{a^2}\right) = \frac{-\cos x}{a^2+\cos^2 x}$$

vlastní derivace $\frac{\partial f}{\partial a}$ $\exists$ pro $\forall x$ jasné. (1)

majoranta: $\eta > 0$ pevné: $\left|\frac{\partial f}{\partial a}\right| \le \frac{1}{a^2} \le \frac{1}{\eta^2} =: g(x) \in L^1(-\frac{\pi}{2},\frac{\pi}{2})$.

$$\Rightarrow F'(a) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-\cos x}{a^2+\cos^2 x} dx \quad \forall a \in (\eta,\infty)$$ (2)

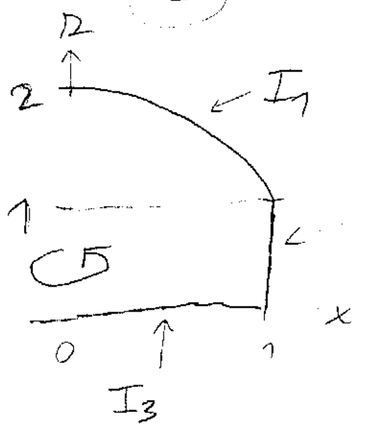
$\eta > 0$ libovolné: platí pro $\forall a > 0$. (1)

dopočet: $F'(a) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x \, dx}{\sin^2 x - (1+a^2)}$ $\left| \begin{array}{l} \sin x = y \\ \cos x \, dx = dy \\ b^2 = 1+a^2 \end{array} \right.$ $= \int_{-1}^{1} \frac{dy}{y^2-b^2}$

$$= \frac{1}{b} \ln\left|\frac{b-1}{b+1}\right| = \frac{1}{\sqrt{1+a^2}} \ln \frac{\sqrt{1+a^2}-1}{\sqrt{1+a^2}+1}.$$ (3)

(2) $\displaystyle\int_{\partial\Omega} \underset{\sim}{F}\cdot d\underset{\sim}{S} = ?$ orientace **dovnitř** $\underset{\sim}{F} = (-x, -y, +z)$

$$\Omega: \begin{cases} x^2+y^2 \le 1 \\ 0 \le z \le 2-(x^2+y^2)^{3/2}. \end{cases}$$



popis řezu: [1]

I_1: $$\varphi: \begin{cases} x = r\cos u \\ y = r \sin u \\ z = 2-r^3 \end{cases} \quad \left.\begin{matrix} u\in(0,2\pi) \\ r\in(0,1) \end{matrix}\right\} \Pi. \qquad \begin{matrix} \partial_r\varphi = (\cos u, \sin u, -3r^2) \\ \partial_u\varphi = (-r\sin u, r\cos u, 0) \end{matrix}$$

$$\partial_r\varphi\times\partial_u\varphi = (3r^3\cos u, 3r^3\sin u, r)$$

tato parametrizace $\boxed{\text{není}}$ ve shodě s orientací.

$$\underset{\sim}{F}\circ\varphi = (-r\cos u, -r\sin u, 2-r^3)$$

$$-I_1 = \int_\Pi \underset{\sim}{F}\circ\varphi \cdot (\partial_r\varphi\times\partial_u\varphi)\, du\, dr = \int_\Pi \left(-3r^4\cos^2 u - 3r^4\sin^2 u + 2r - r^4\right) dr\, du$$

$$= \int_\Pi (2r-4r^4)\, du\, dr = 2\pi\int_0^1 (2r-4r^4)\, dr = 2\pi\left[r^2-\frac{4}{5}r^5\right]_0^1 = \frac{2\pi}{5}.$$

I_2: $$\varphi: \begin{cases} x = \cos u \\ y = \sin u \\ z = v \end{cases} \quad \left.\begin{matrix} u\in(0,2\pi) \\ v\in(0,1) \end{matrix}\right\} \Pi \qquad \begin{matrix} \partial_u\varphi = (-\sin u, \cos u, 0) \\ \partial_v\varphi = (0, 0, 1) \end{matrix}$$

$$\partial_u\varphi\times\partial_v\varphi = (\cos u, \sin u, 0)$$

opět $\boxed{\text{není}}$ ve shodě (ven)

$$\underset{\sim}{F}\circ\varphi = (-\cos u, -\sin u, v)$$

$u=0$: $(1,0,v)$ .. normála $\underline{(1,0,0)}$

$$-I_2 = \int_\Pi \left(-\cos^2 u - \sin^2 u\right) du\, dv = -2\pi$$

$\boxed{I_3}$: (podstava): $\underset{\sim}{n} = (0,0,+1)$ $\Rightarrow$ $I_3 = 0$.

<u>Úvaha:</u> $\underset{\sim}{F} = (-x,-y,0)$

jinak .. $\varphi: \begin{cases} x = u \\ y = v \\ z = 0 \end{cases}$ $(u,v) \in B$ .. jednotkový kruh

$$\partial_u \varphi = (1,0,0)$$
$$\partial_v \varphi = (0,1,0)$$
$$\partial_u \varphi \times \partial_v \varphi = (0,0,1)$$: je ve shodě.
$$\underset{\sim}{F} \circ \varphi = (-u,-v,0)$$

orientace: celkem (2 b)
parametrizace (3 b)
výpočet integrálu (2 b)

$$\bar{I}_3 = \int_B 0 \, du\, dv = 0.$$

Celkem: $\displaystyle\int_{\partial\Omega} \underset{\sim}{F} \cdot d\underset{\sim}{S} = I_1 + I_2 + I_3 = -\frac{2\pi}{5} + 2\pi + 0 = \frac{8\pi}{5}$

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varianta: <u>Gaussova věta</u> : $\displaystyle\int_{\partial\Omega} \underset{\sim}{F} \cdot d\underset{\sim}{S} = -\int_{\Omega} \operatorname{div} \underset{\sim}{F} \, d\lambda_3 = \lambda_3(\Omega)$

neboť $\operatorname{div} \underset{\sim}{F} = -1$. (2)

$$\lambda_3(\Omega) = \int_{\{u^2+v^2<1\}} \left(2 - (x^2+y^2)^{3/2}\right) du\, dv =$$ (2)

polární souřadnice
$x = r\cos u$ $\quad u \in (0,2\pi)$
$y = r \sin u$ $\quad r \in (0,1)$ $\Big\}\ \Pi$
$J = r$
(2)

$$= \int_{\Pi} (2-r^3) r \, du\, dr = 2\pi \cdot \int_0^1 (2r - r^4)\, dr = 2\pi \left[r^2 - \frac{r^5}{5} \right]_0^1 = \frac{8\pi}{5}.$$

(2)

(3) $\Phi(y) = \int_0^1 (y')^2 + y^2 - 2xy \, dx$, $y(0) = 1$

$y(1) = 1 + e$

$f = R^2 + y^2 - 2xy$

$f_R = 2R$ E.L. $(-2y')' + 2y - 2x = 0$

$f_y = 2y - 2x$; $y'' - y = -x$ 2

F.S.: $\{e^{-x}, e^{x}\}$. partikulární řešení: x

obecné řešení: $y = Ae^{x} + Be^{-x} + x$ 1

okr. podm.: $x = 0$: $A + B = 1$

$x = 1$: $Ae + Be^{-1} + 1 = 1 + e$ $\Big) - e$

$Ae + Be^{-1} = e$

$B(e^{-1} - e) = 0$: $B = 0$; $A = 1$.

$\boxed{y_0 = e^{x} + x}$: $\boxed{\text{jediné}}$ extremála 2

↑bod.

podezření: minimum. 1

Jacobiho rce: $P(t) = f_{RR}(t, y_0, y_0') = 2 > 0$

$Q(t) = f_{yy}(t, y_0, y_0') - \left(f_{yR}(t, y_0, y_0')\right)' = 2$

(J): $(-2u')' + u = 0$

$u'' - u = 0$

NE $\exists$ konjugovaný bod $\tilde{x} \in (0, 1]$; tj. $u(0) = u(\tilde{x}) = 0$;

$u \not\equiv 0$.

$u_{obec}(t) = Ae^{x} + Be^{-x}$;

$u(0) = 0$: $u(t) = A(e^{x} - e^{-x})$; $A \neq 0$: $u(t)$ monotónní

$\longrightarrow$ lokální minimum 2

(4) $f(x) = x(\pi - |x|)$; $x \in (-\pi, \pi]$

2π-periodické.

(a) f liché: $\Rightarrow$ $a_k = 0$ $\forall k \geq 0$. [1]

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x(\pi - |x|)\sin kx = \frac{2}{\pi}\underbrace{\int_0^{\pi} x(\pi - x)\sin kx}_{I_k}$$ [1]

$$I_k = \left| \text{per partes} \right| = \left[x(\pi-x)\left(-\frac{\cos kx}{k}\right)\right]_0^{\pi} + \frac{1}{k}\int_0^{\pi}(\pi - 2x)\cos kx\, dx \, ;$$

$$\int_0^{\pi}\cos kx\, dx = 0 \, ; \qquad \Rightarrow \qquad I_k = -\frac{2}{k}\underbrace{\int_0^{\pi} x\cos kx\, dx}_{J_k} \, ,$$

$$J_k = \left| p.p. \right| = \left[x\frac{\sin kx}{k}\right]_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx = -\frac{1}{k}\left[-\frac{\cos kx}{k}\right]_0^{\pi}$$

$$= \frac{1}{k^2}\left((-1)^k - 1\right).$$

celkem: $$b_k = \frac{2}{\pi}\cdot\left(-\frac{2}{k}\right)\cdot\frac{1}{k^2}\left((-1)^k - 1\right)$$

$$= \frac{-4}{\pi k^3}\left((-1)^k - 1\right) = \begin{cases} 0 \, ; & k \text{ sudé} \\ \dfrac{8}{\pi k^3} \, ; & k \text{ liché.} \end{cases}$$ [2]

$$F_f(x) = \frac{8}{\pi}\sum_{l=0}^{\infty}\frac{\sin(2l+1)x}{(2l+1)^3} \, ;$$

Důkaz: $f(x)$ spojité, po částech C^1

$\Rightarrow f(x) = \mathcal{F}_f(x)$ pro $\forall x \in \mathbb{R}$. [1]

$b_k \sim \frac{1}{k^3} \Rightarrow f \in C^1 \setminus C^2$ v $\mathbb{R}$ [1]

Parseval: (obecně): $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ [1]

(zde:) L.S. $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2(\pi - |x|)^2 = \frac{2}{\pi}\int_0^{\pi} x^2(x-\pi)^2$

$= \frac{2}{\pi}\int_0^{\pi} (\pi^2 x^2 - 2\pi x^3 + x^4)\,dx = \frac{2}{\pi}\left[\pi^2 \frac{x^3}{3} - 2\pi \frac{x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^4}{15}$

P.S.: $\frac{64}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^6}$. [1]

vedlejší produkt: $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{\pi^6}{960}$ konec.