

$$\textcircled{0} \quad \begin{aligned} x' &= y \\ y' &= -g(y) - \sin x \end{aligned}$$

$$V = \frac{1}{2}y^2 + \underbrace{(1 - \cos x)}_{2\sin^2 \frac{x}{2}}$$

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' = \cancel{\sin x} \cdot y + y(-g(y) - \cancel{\sin x}) \\ &= -yg(y) \leq 0 \end{aligned}$$

$$\textcircled{1} \quad \begin{aligned} x' &= -2y - x^3 \\ y' &= 3x - y^3 \end{aligned}$$

$$\left[V = ax^2 + by^2, \quad a, b > 0 \right]$$

$$\begin{aligned} V' &= \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' = 2ax(-2y - x^3) + 2by(3x - y^3) \\ &= \underbrace{(-4a + 6b)}_{\text{need to cancel out}} xy + \underbrace{(-2ax^4 - 2by^4)}_{\text{always } \leq 0} \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \text{set } a &= 3 \\ b &= 2 \end{aligned}}$$

$$V = 3x^2 + 2y^2$$

$$\dot{V} = -\underbrace{(6x^4 + 4y^4)}_{\text{in fact } < 0 \text{ if } (x,y) \neq (0,0)} \leq 0 \quad \forall x, y$$

$\Rightarrow V$ is strict Lyapunov function

$\Rightarrow (0,0)$ is even asymptotically stable

②
$$\begin{aligned} x' &= -\frac{x}{2} - y^2 \\ y' &= xy - 7x^2y \end{aligned}$$

$$V = ax^2 + by^2$$

... choice of $a=b=1$

$\Rightarrow$ Stability

③
$$\begin{aligned} x' &= -2y^3 \\ y' &= x \end{aligned}$$

$$V = x^2 + y^4$$

$\Rightarrow$ stability

④*
$$\begin{aligned} x' &= -y + 2x^3 \\ y' &= 2x + y^3 \end{aligned}$$

for $V = x^2 + 2y^2$ we have:

$$\dot{V} = 2(x^4 + y^4) > 0 \text{ if } (x,y) \neq (0,0)$$

$\Rightarrow$ $(0,0)$ is not stable