⓪ $x' = y$
$y' = -g(y) - \sin x$

$$V = \frac{1}{2}y^2 + \underbrace{(1-\cos x)}_{2\sin^2\frac{x}{2}}$$

$$\dot{V} = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' = \cancel{\sin x}\cdot y + y\left(-g(y) - \cancel{\sin x}\right)$$
$$= -y\,g(y) \leq 0$$

---

① $x' = -2y - x^3$
$y' = 3x - y^3$

$$V \stackrel{?}{=} ax^2 + by^2, \quad a,b > 0$$

$$V' = \frac{\partial V}{\partial x}\cdot x' + \frac{\partial V}{\partial y}\cdot y' = 2ax(-2y - x^3) + 2by(3x - y^3)$$
$$= \underbrace{(-4a + 6b)}_{\text{need to cancel out}}\underbrace{xy}_{?} + \underbrace{(-2ax^4 - 2by^4)}_{\text{always} \leq 0}$$

$\Rightarrow$ $\boxed{\text{set } a = 3,\ b = 2}$

$$V = 3x^2 + 2y^2$$
$$\dot{V} = \underbrace{-(6x^4 + 4y^4)}_{\text{in fact } < 0 \text{ if } (x,y) \neq (0,0)} \leq 0 \quad \forall x,y$$

$\Rightarrow$ V is strict Lyapunov function $\Rightarrow$ $(0,0)$ is even asymptotically stable

2. $x' = -\frac{x}{2} - y^2$
$y' = xy - 7x^2y$

$V = ax^2 + by^2$

... choice of $a = b = 1$

$\Rightarrow$ stability

---

3. $x' = -2y^3$
$y' = x$

$V = x^2 + y^4$ $\Rightarrow$ stability

4*. $x' = -y + 2x^3$
$y' = 2x + y^3$

for $V = x^2 + 2y^2$ we have:

$\dot{V} = 2(x^4 + y^4) > 0$ if $(x,y) \neq (0,0)$

$\Rightarrow$ $(0,0)$ is not stable