1 $A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$

(i) $a = 0 \Rightarrow A^2 = 0$, hence

$$e^{tA} = I + tA = \begin{pmatrix} 1 & 0 \\ bt & 0 \end{pmatrix}$$

(ii) $a \neq 0$ .... observe $A^2 = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} a^3 & 0 \\ a^2 b & 0 \end{pmatrix}$

in general: $A^k = \begin{pmatrix} a^k & 0 \\ ba^{k-1} & 0 \end{pmatrix} \Rightarrow e^{tA} = \begin{pmatrix} a(t) & 0 \\ b(t) & 0 \end{pmatrix}$

where: $a(t) = 1 + at + a^2 \frac{t^2}{2} + \dots = e^{at}$

$$b(t) = bt + ba\frac{t^2}{2} + ba^2\frac{t^3}{3!} + \dots$$

$$= \frac{b}{a}\Big(\underbrace{at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \dots}_{e^{at}-1}\Big)$$

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<u>Another method</u>: clearly $\sigma(A) = \{a, 0\}$

$\Rightarrow \exists$ regular matrix $C$ s.t.

$A = CDC^{-1}$, where $D = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

hence $\Rightarrow$ $e^{tA} = Ce^{tD}C^{-1}$, and $e^{tD} = \begin{pmatrix} e^{at} & 0 \\ 0 & 1 \end{pmatrix}$

we know from LA that $C = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix}$

eigenvectors corresponding to $\lambda = a, 0$.

$$\Rightarrow \quad C^{-1} = \begin{pmatrix} 1 & 0 \\ -b/a & 1 \end{pmatrix}$$

hence finally: $$e^{tA} = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix} \begin{pmatrix} e^{ta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b/a & 1 \end{pmatrix}$$

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(2) $$A = \begin{pmatrix} 0 & 1 & 10 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$ ... observe that $A^3 = 0$

$$\Rightarrow \quad e^{tA} = I + t^2 A + \frac{t^2}{2} A^2 = \begin{pmatrix} 1, & t, & 10t - \frac{t^2}{2} \\ 0, & 1, & -t \\ 0, & 0, & 1 \end{pmatrix}$$

(3) $A = \alpha I + B$, where $B = \begin{pmatrix} 0, & -\beta \\ \beta, & 0 \end{pmatrix}$

now $I$ and $B$ commute (i.e. $IB = BI$)

$$\Rightarrow \quad e^{tA} = e^{\alpha t I} e^{tB} = e^{\alpha t} e^{tB}$$

remains to be done

... observe that $B^2 = -\beta^2 I$, $B^3 = -\beta^2 B$, $B^4 = \beta^4 I$

in general: $B^{2\ell} = (-1)^\ell \beta^{2\ell} I$, $B^{2\ell+1} = (-1)^\ell \beta^{2\ell+1} J$,

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\Rightarrow e^{tB} = \sum_{r=0}^{\infty} \frac{t^r}{r!} B^r = \underbrace{\sum_{\ell=0}^{\infty} \frac{t^{2\ell}}{(2\ell)!} B^{2\ell}}_{(1)} + \underbrace{\sum_{\ell=0}^{\infty} \frac{t^{2\ell+1}}{(2\ell+1)!} B^{2\ell+1}}_{(2)}$$

Separate even and odd powers

$$(1) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\beta t)^{2\ell}}{(2\ell)!} I = \cos(\beta t) I$$

$$(2) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(\beta t)^{2\ell+1}}{(2\ell+1)!} J = \sin(\beta t) J$$

$$\begin{pmatrix} \cos\beta t, & -\sin\beta t \\ \sin\beta t, & \cos\beta t \end{pmatrix}$$

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4

$$p(\lambda) = \lambda^4 - 16 = (\lambda^2 - 4)(\lambda^2 + 4)$$

$$\Rightarrow \lambda = \pm 2, \pm 2i \quad \ldots \quad e^{2t}, e^{-2t}, \underbrace{e^{2it}, e^{-2it}}$$

can be replaced by real/imaginary parts: $\boxed{\cos 2t, \sin 2t}$

general solution: $x(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos 2t + c_4 \sin 2t$

5. $p(\lambda) = \lambda^4$ ... $\lambda = 0$ (multiple root!)

$e^{0t} = 1$, but also: $t, t^2, t^3$

$\Rightarrow$ general solution $x(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3$

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6. $p(\lambda) = \lambda^2 + 3\lambda - 40 = (\lambda+8)(\lambda-5)$

$\Rightarrow$ general solution: $x(t) = c_1 e^{-8t} + c_2 e^{5t}$

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7. $p(\lambda) = (\lambda-1)^3$ $\Rightarrow$ $\lambda = 1$ is triple root

$\{e^t, te^t, t^2 e^t\}$