8. Aproximace funkcí polynomy

Definice. Nechť $f(x)$, $g(x)$ jsou definovány na $P(x_0)$.

Řekneme, že $f(x)$ je „malé o" od $g(x)$ pro $x \to x_0$, jestliže

$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$. Značíme $f(x) = o(g(x))$, $x \to x_0$.

Řekneme, že $f(x)$ je „velké O" od $g(x)$ pro $x \to x_0$, jestliže

$\exists \delta > 0$ tak, že $\frac{f(x)}{g(x)}$ je omezená na $P(x_0)$. Značíme $f(x) = O(g(x))$, $x \to x_0$.

Řekneme, že $f(x)$ je řádově rovno $g(x)$ pro $x \to x_0$, jestliže

$\lim_{x \to x_0} \frac{f(x)}{g(x)}$ existuje a je konečná, nenulová. Značíme $f(x) \sim g(x)$, $x \to x_0$.

Analogicky pro $x \to x_0+$, $x \to x_0-$.

Poznámky. $f(x) = o(g(x))$ … „$f(x)$ mnohem menší než $g(x)$" (pro $x \to x_0$)

$f(x) = O(g(x))$ … $|f(x)| \leq C|g(x)|$ … $f(x)$ srovnatelné s $g(x)$

$f(x) \sim g(x)$ … f, g se chovají v nějakém okolí stejně …

Příklady: ① $\ln x = o(\sqrt{x})$ pro $x \to \infty$;

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \underset{\text{l'Hosp.}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0.$$

② $\frac{\sin x}{x^2+1} = O\left(\frac{1}{x^2}\right)$ pro $x \to \infty$

$$\left| \frac{\frac{\sin x}{x^2+1}}{\frac{1}{x^2}} \right| = \left| \frac{x^2}{x^2+1} \cdot \sin x \right| \leq 1 \text{ na } P(\infty).$$

③ $\sin x \sim x$ pro $x \to 0$: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$1 - \cos x \sim x^2$ pro $x \to 0$:

$\ln(1+x) \sim x$ pro $x \to 0$:

Poznámky • $f(x) = o(1)$ pro $x \to x_0$ $\iff$ $\lim_{x\to x_0} f(x) = 0$.

• $f(x) = O(1)$ pro $x \to x_0$ $\iff$ $f(x)$ omezená na nějakém $P(x_0,\delta)$.

• $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ existuje, konečná $\Rightarrow$ $f(x) = O(g(x))$ pro $x \to x_0$

důsledky: $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$; $x \to x_0$

$f(x) \sim g(x) \Rightarrow f(x) = O(g(x))$

$g(x) = O(g(x))$ pro $x \to x_0$.

Definice: [Derivace vyšších řádů.] $f^{(k)}(x)$; $k \geq 0$ celé:

$f'(x) = f^{(1)}(x)$

$f''(x) = \{f'(x)\}' = f^{(2)}(x)$

indukcí: $f^{(n+1)}(x) = [f^{(n)}(x)]'$; $\frac{d^n}{dx^n} f(x)$;

$f(x) = f^{(0)}(x)$ .. ztotožním s 0-derivací $f^{(0)}(x) = f(x)$.

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idea aproximace funkce polynomem:

$f(x)$ definované na $U(x_0,\delta)$:

sestrojme polynom $p(x)$ tak, že

$$\boxed{\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ p''(x_0) &= f''(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}}$$

= $p(x)$ velmi dobře aproximuje $f(x)$ pro x blízké x_0

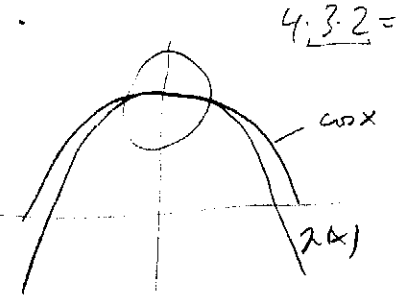
tím lépe, čím n je větší.

$4 \cdot 3 \cdot 2 = 24$

$f(x) = \cos x$ v okolí $x_0 = 0$;

$f(0) = 1$ $\quad f''(0) = -1$. $\quad p(x) = 1 - \frac{x^2}{2} + \frac{x^4}{\underset{24}{4!}}$

$f'(0) = 0$;



idea: $f(x)$ daná   sestrojíme polynom $p(x)$ tak, že

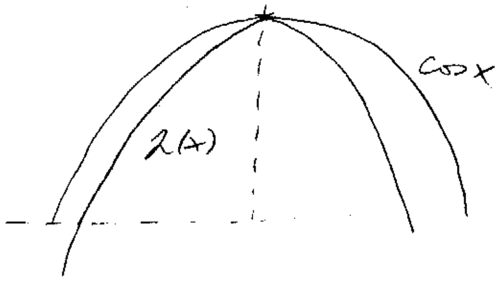
$$\left.\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}\right\} \quad (1)$$

pak $p(x)$ „dobře aproximuje" $f(x)$ pro x blízko x_0.

Příklad: $f(x) = \cos x$. $f(0) = 1$;

?

$$f'(0) = -\sin 0 = 0$$
$$f''(0) = -\cos 0 = 1$$
$$p(x) = 1 - \frac{x^2}{2}$$



Lemma 8.1. Pro $k \geq 0$ celé definujme $Q_{k,x_0}(x) = \frac{1}{k!}(x-x_0)^k$.

Platí: (i) $Q_{k,x_0}(x)$ je polynom stupně k

(ii) $Q'_{k,x_0}(x) = Q_{k-1,x_0}(x)$; $\forall k \in \mathbb{N}$.

(iii)
$$Q^{(l)}_{k,x_0}(x_0) = \begin{cases} 1 & k = l \\ 0 & k \neq l; \end{cases}$$

Dk: (i) zřejmé. .. $Q_{x_0,k} = 1, (x-x_0); \frac{(x-x_0)^2}{2!}, \frac{(x-x_0)^3}{3!}, \dots$

(ii)
$$\left[Q_{k,x_0}(x)\right]' = \underbrace{\frac{k}{k!}}_{=\frac{1}{(k-1)!}}(x-x_0)^{k-1} = Q_{k-1,x_0}(x).$$

(iii)
$$\left[Q_{k,x_0}(x)\right]^{(l)} = \begin{cases} 0 \; ; \; l > k & \text{(polynom stupně } k \text{ derivujeme } k+1\text{ krát)} \\ Q_{k-l;x_0}(x) = \frac{1}{(k-l)!}(x-x_0)^{k-l} \; ; & l \leq k. \end{cases}$$

$$\text{pro } x = x_0: \begin{cases} 1 & k = l: \text{ (definice: } 0^0 = 1) \\ 0 & l < k: \frac{1}{(k-l)!} 0^{k-l>0} = 0. \end{cases}$$

Definice. Nechť I je otevřený interval. Řekneme, že $f(x)$ je třídy C^m na I, jestliže funkce $f(x), f'(x), f''(x) \dots f^{(m)}(x)$ jsou spojité na I.
Značíme $f(x) \in C^m(I)$. Řekneme, že $f(x)$ je třídy C^∞ na I, pokud
($f^{(k)}(x)$ je spojitá v I pro $\forall k \geq 0$ celé.)
Speciálně: $C^0(I) = C(I)$ je množina všech funkcí, které jsou spojité v I.

Definice Nechť $f(x)$ je třídy C^m na nějakém $U(x_0)$. Potom

$$p(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

se nazývá m-tý Taylorův polynom funkce $f(x)$ o středu x_0.
Značí se $T^f_{x_0,m}(x)$.

Věta 8.1. Nechť $f(x)$ je třídy C^m na nějakém $U(x_0)$. Potom

$$f(x) - T^f_{x_0,m}(x) = o\left((x-x_0)^m\right) \text{ pro } x \to x_0. \qquad (*)$$

Navíc $T^f_{x_0,m}(x)$ je jediný polynom stupně $\leq m$, který má vlastnost $(*)$.

Dk.

$$p(x) = T^f_{x_0,m}(x) = \sum_{k=0}^{m} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = \sum_{k=0}^{m} f^{(k)}(x_0) \cdot Q_{k,x_0}(x).$$

krok (1)

$$p^{(l)}(x_0) = \sum_{k=0}^{m} f^{(k)}(x_0) \cdot \underbrace{Q^{(l)}_{k,x_0}(x_0)}_{\text{Lemma 8.1.}\ \begin{cases} 0 & l \neq k \\ 1 & l = k \end{cases}} = \begin{cases} 0 & l > m \\ f^{(l)}(x_0) \ ; & 0 \leq l \leq m. \end{cases}$$

$0 \quad l > m : (\text{st } p \leq m)$

$$p^{(k)}(x_0) = f^{(k)}(x_0), \ \forall k = 0, 1, \dots m.$$

2. krok: Platí (?): $\lim\limits_{x\to x_0} \dfrac{f(x)-p(x)}{(x-x_0)^n} = 0.$

$$\left.\begin{array}{l} f(x)-p(x) \to f(x_0)-p(x_0)=0 \\ (x-x_0)^n \to 0 \end{array}\right| \text{ l'Hosp. typ } \text{„}\tfrac{0}{0}\text{"}.$$

$$\frac{f'(x)-p'(x)}{n(x-x_0)^{n-1}} \quad \dots$$

$$f'(x)-p'(x) \to f'(x_0)-p'(x_0)$$

$$n\cdot(x-x_0)^{n-1}\to 0 \;\dots$$

$f^{(k)}(x)\to f^{(k)}(x_0)$; $k=0,1,\dots n$

neboť $f^{(k)}(x)$ spoj. v x_0 (předpoklad)

obecný krok: $$\frac{f^{(k)}(x)-p^{(k)}(x)}{n\cdot(n-1)\cdots(n-k+1)\,(x-x_0)^{n-k}}$$

$\vdots$

l'Hospital n-krát: $$\frac{f^{(n)}(x)-p^{(n)}(x)}{n!} \to \frac{f^{(n)}(x_0)-p^{(n)}(x_0)}{n!} = 0$$

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3. krok: nechť $\exists\, q(x)$ pol. st. $\le n$;
$$f(x)-q(x) = o\big((x-x_0)^n\big), \quad x\to x_0$$
$$q(x) \neq p(x) = T^{f}_{x_0,n}(x). \quad \dots \text{ odvodíme spor.}$$

Ukážeme, že $\lim\limits_{x\to x_0} \dfrac{p(x)-q(x)}{(x-x_0)^n} = 0.$

$$\frac{p(x)-q(x)}{(x-x_0)^n} = \underbrace{\frac{f(x)-q(x)}{(x-x_0)^n}}_{\to 0} - \underbrace{\frac{f(x)-p(x)}{(x-x_0)^n}}_{\to 0} \to 0 \text{ pro } x\to x_0$$

zde: $p(x) = \sum\limits_{k=0}^{n} a_k (x-x_0)^k$;

$$q(x) = \sum_{j=0}^{n} c_k x^k = \sum_{j=0}^{n} c_k\left[(x-x_0)+x_0\right]^k = \sum_{j=0}^{n} c_k \sum_{l=0}^{k} \binom{k}{l}\cdot \boxed{(x-x_0)^l}\, x_0^{k-l}$$

$l \in \{0,1,\dots n\}$.

$$= \sum_{k=0}^{n} b_k (x-x_0)^k \qquad \left[(x-x_0)+x_0\right]^k$$

$f \neq g \quad \exists\, s \in$ [illegible]

nejmenší takové, že $a_s \neq b_s$.

$$\frac{f(x)-g(x)}{(x-x_0)^s} = \underbrace{\frac{f(x)-g(x)}{(x-x_0)^m}}_{\to 0} \cdot \underbrace{(x-x_0)^{m-s}}_{\to \begin{cases} 0 & s<m \\ 1 & s=m \end{cases}} \to 0 \text{ pro } x \to x_0+$$

$$\frac{f(x)-g(x)}{(x-x_0)^s} = \frac{\sum_{k=0}^{m} (a_k-b_k)\cdot(x-x_0)^k}{(x-x_0)^s} = \sum_{k=s}^{m} (a_k-b_k)\cdot(x-x_0)^{k-s}$$

$x-x_0 > 0 \quad x \in P_+(x_0)$

$$= (a_s-b_s) + \underbrace{\sum_{k=s+1}^{m} (a_k-b_k)\cdot(x-x_0)^{k-s}}_{\to 0} \to a_s - b_s \neq 0$$

pro $x \to x_0$

<u>spor</u>.

Příklady: 1 $f(x) = e^x$; $x_0 = 0$.

$$f^{(k)}(x) = e^x ; \quad f^{(k)}(0) = \cancel{0}\, 1$$

$$T_{0,n}^{e^x}(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k ; \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$x \to 0$.

2 $f(x) = \sin x$; $x_0 = 0$

$$f(0) = \sin 0 = 0$$
$$f'(0) = \cos 0 = 1$$
$$f''(0) = -\sin 0 = 0$$
$$f'''(0) = -\cos 0 = -1$$
$$f^{(4)}(0) = \sin 0 = 0$$

$$f^{(l)}(0) = \begin{cases} 0 & l = 2k \\ (-1)^k & l = 2\cancel{n}k+1 . \end{cases}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)$$

$$T_{0,2n+2}^{\sin x}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o\left(x^{2n+2}\right)$$

3 $f(x) = (1+x)^a$; $a \in \mathbb{R}$; $x_0 = 0$.

$$f'(x) = a\,(1+x)^{a-1}$$
$$f^{(k)}(x) = a\cdot(a-1)\dots(a-k+1)\cdot(1+x)^{a-k}$$
$$f^{(k)}(0) = a\cdot(a-1)\dots(a-k+1)$$

$$T_{0,n}^{(1+x)^a}(x) = \sum_{k=0}^{n} \underbrace{\frac{a\cdot(a-1)\dots(a-k+1)}{k!}}_{=\binom{a}{k}} x^k$$

$$(1+x)^a = 1 + ax + \frac{a\cdot(a-1)}{2} x^2 + \frac{a\cdot(a-1)(a-2)}{6} x^3 + \dots + \binom{a}{n} x^n + o(x^n), \; x \to 0.$$

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!} = \frac{n\cdot(n-1)\dots(n-k+1)}{k!}$$

Věta 8.4: [illegible]

Potom 1. $\left\{ T^F_{x_0, n+1}(x) \right\}' = T^f_{x_0, n}(x)$.

2. Naopak: nechť $\lambda(x) = T^f_{x_0,n}(x)$; a $P(x) = \int \lambda(x)\,dx$. Potom lze zvolit $c \in \mathbb{R}$ tak, že $P(x) + c = T^F_{x_0, n+1}(x)$.

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dk. 1. $$T^F_{x_0, n+1}(x) = \sum_{k=0}^{n+1} F^{(k)}(x_0) \cdot Q_{x_0,k}(x) \quad \Big/ \frac{d}{dx}$$

$$\sum_{k=0}^{n+1} F^{(k)}(x_0)\, Q'_{x_0,k}(x) \ ; \quad Q'_{x_0,k}(x) = \begin{cases} 0 & k = 0 \\ Q_{x_0,k-1} \ ; & k \geq 1 \end{cases}$$

$$= \sum_{k=1}^{n+1} F^{(k)}(x_0) \cdot Q_{x_0,k-1}(x) \ ; \quad \begin{matrix} k = l+1 \\ l = 0, \dots n \end{matrix}$$

$$= \sum_{l=0}^{n} \underbrace{F^{(l+1)}(x_0)}_{f^{(l)}(x_0)}\, Q_{x_0,l}(x) = T^f_{x_0,n}(x).$$

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2. $$P(x) = \int T^f_{x_0,n}(x)\,dx = \sum_{k=0}^{n} f^{(k)}(x_0) \cdot \int Q_{x_0,k}(x)\,dx$$

$$= \sum_{k=0}^{n} f^{(k)}(x_0)\, Q_{x_0,k+1}(x) \ ; \quad \Big| \ \begin{matrix} l = k+1 \\ l = 1 \dots n+1 \end{matrix}$$

$$= \sum_{l=1}^{n+1} \underbrace{f^{(l-1)}(x_0)}_{F^{(l)}(x_0)}\, Q_{x_0,l}(x)$$

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stačí volit $c = F(x_0) = F(x_0) \cdot \underbrace{Q_{x_0,0}(x)}_{\equiv 1}$

a $P(x) + c = T^F_{x_0, n+1}$.

Příklady: ① $\cos x = (\sin x)'$

$$T_{0,2n}^{\cos x}(x) = \left\{T_{0,2n+1}^{\sin x}(x)\right\}' = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} \;;$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n}\right), \; x \to 0$$

② $\frac{1}{1+x} = \left[\ln(1+x)\right]'$;

$$T_{0,n-1}^{(1+x)^{-1}}(x) = \sum_{k=0}^{n-1} \binom{-1}{k} \cdot x^k = \sum_{k=0}^{n-1} (-1)^k x^k$$

$$\binom{-1}{k} = \frac{(-1)\cdot(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k \cdot k!}{k!}$$

$$T_{0,n}^{\ln(1+x)}(x) = C + \sum_{k=0}^{n-1} (-1)^k \frac{x^{k+1}}{k+1} \;; \quad C = \ln(1+0) = 0.$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{x^k}{k}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

$x \to 0$.

1. Nechť $f(x)=o(x^n)$, $g(x)=o(x^m)$ pro $x\to 0$; $m\geq n$.

Potom $f(x)\pm g(x)=o(x^n)$ pro $x\to 0$.

2. Nechť $f(x)=o(x^n)$ a $g(x)=o(x^m)$ pro $x\to 0$,

~~nebo g(x)~~ Potom $f(x)\cdot g(x)=o(x^{n+m})$ pro $x\to 0$.

3. Nechť $f(x)=o(x^n)$ pro $x\to 0$; $g(x)=c\cdot x^m$. ~~fx~~

Potom $f(x)g(x)=o(x^{n+m})$.

4. Nechť $f(x)=o(x^n)$; nechť $g(x)\sim x^m$, $m\geq 1$ pro $x\to 0$.

Potom $f(g(x))=o(x^{m\cdot n})$ pro $x\to 0$.

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dk: 1. $$\frac{f(x)+g(x)}{x^n}=\underbrace{\frac{f(x)}{x^n}}_{\to 0}+\underbrace{\frac{g(x)}{x^m}}_{\to 0}\cdot\underbrace{x^{m-n}}_{\substack{\to 0\ m>n\\ \to 1\ m=n}}\to 0 \text{ pro } x\to 0.$$

2. $$\frac{f(x)\cdot g(x)}{x^{m+n}}=\underbrace{\frac{f(x)}{x^n}}_{\to 0}\cdot\underbrace{\frac{g(x)}{x^m}}_{\to 0}\to 0 \text{ pro } x\to 0.$$

3. $$\frac{f(x)\cdot c\cdot x^m}{x^{m+n}}=\frac{c\cdot f(x)}{x^n}\to 0 \text{ pro } x\to 0.$$

4. $$\frac{f(g(x))}{x^{m\cdot n}}=\frac{f(g(x))}{[g(x)]^n}\cdot\Bigg[\underbrace{\frac{g(x)}{x^m}}_{\to c\in\mathbb{R}\setminus\{0\}}\Bigg]^n\to 0\cdot c^n=0$$

$\frac{f(y)}{y^n}\to 0$ pro $y\to 0$

$g(x)\sim x^m$ pro $x\to 0$: $\frac{g(x)}{x^m}\to c\in\mathbb{R}\setminus\{0\}$

$\Rightarrow g(x)\neq 0$ na $P(x_0,\delta)$

<u>Poznámky</u>. některé operace [illegible]

$$2.\quad o(x^n)\cdot o(x^m) = o(x^{m\cdot m})$$

$$3.\quad x^m o(x^n) = o(x^{m+n})$$

$$o(x^n) - o(x^n) = o(x^n);$$

<u>neboli</u> $f(x) = o(x^n);\ g(x) = o(x^n) \Rightarrow f(x) - g(x) = o(x^n),\ x \to 0.$

<u>Příklady.</u> ① $\displaystyle\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$; $\sin x = x - \dfrac{x^3}{3!} + o(x^3),\quad x \to 0$

$$\cos x = 1 - \frac{x^2}{2!} + o(x^2)$$

$$x\cdot\cos x = x - \frac{x^3}{2} + \underbrace{x\, o(x^2)}_{o(x^3)}$$

V. 8.3; (3).

$$f(x) = \frac{1}{x^3}\left(x - \frac{x^3}{2} + o(x^3) - \left(x - \frac{x^3}{6} + o(x^3)\right)\right)$$

$$= \frac{1}{x^3}\left[\frac{x^3}{3} + o(x^3)\right] = \frac{1}{3} + \frac{o(x^3)}{x^3} \to 0$$

② $$\lim_{x\to\infty} x\cdot\left(\sqrt{1+x^2} - 3\sqrt[3]{1+x^3} + 2\sqrt[4]{1+x^4}\right)$$

$$= \lim_{y\to 0+} \frac{1}{y}\left(\sqrt{1+\frac{1}{y^2}} - 3\sqrt[3]{1+\frac{1}{y^3}} + 2\sqrt[4]{1+\frac{1}{y^4}}\right)$$

$$= \lim_{x\to 0+} \frac{1}{x^2}\left(\sqrt{x^2+1} - 3\sqrt[3]{x^3+1} + 2\sqrt[4]{x^4+1}\right)$$

$$(1+y)^a = 1 + ay + o(y);\quad y \to 0$$

$$\sqrt{(1+x^2)} = 1 + \frac{1}{2}x^2 + o(x^2)$$

$$\sqrt[3]{1+x^3} = 1 + \frac{1}{3}x^3 + o(x^3) = 1 + o(x^2)$$

$$\sqrt[4]{1+x^4} = 1 + \frac{1}{4}x^4 + \dots = 1 + o(x^2)$$

$$f(x) = \frac{1}{x^2}\left\{1 + \frac{1}{2}x^2 + o(x^2) - 3\left[1 + o(x^2)\right] + 2\left[1 + o(x^2)\right]\right\}$$

$$= \frac{1}{x^2}\left\{\frac{1}{2}x^2 + o(x^2)\right\} = \frac{1}{2} + o(1) \to \frac{1}{2},\quad x \to 0+.$$

Definice. [illegible]

$$R_{n+1}(x) = f(x) - T^{f}_{x_0,n}(x)$$

se nazývá Taylorův zbytek funkce po n-tém členu.

Poznámka: známe: $R_{n+1}(x) = o\left((x-x_0)^n\right)$ pro $x \to x_0$, n pevné.

jiná otázka: $R_{n+1}(x) \to 0$ pro $n \to \infty$; x pevné.

neboli: $f(x) = \lim\limits_{n\to\infty} T^{f}_{x_0,n}(x)$

Věta 8.4. [Odhad zbytku.] Nechť $f(x) \in C^{n+1}(\mathcal{U}(x_0))$; nechť $x \in \mathcal{U}(x_0)$, $x \neq x_0$ je pevné. Potom uvnitř intervalu mezi x, x_0 existuje θ tak, že

$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1}.$$

Terminologie: Lagrangeův tvar zbytku.

Dk. pro $t \in [x_0, x]$ položme $\varphi(t) = f(x) - T^{f}_{t,n}(x)$
pevné.

$$= f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k.$$

0^k

$$\varphi(x_0) = f(x) - T^{f}_{x_0,n}(x) = R_{n+1}(x).$$

$$\varphi(x) = f(x) - T^{f}_{x,n}(x) = f(x) - f(x) = 0.$$

derivace dle t !!

$$\varphi'(t) = -\left\{ f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \right\}'$$

$$= -f'(t) - \sum_{k=1}^{n}\left[\frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}\right]$$

$$= -f'(t) + \sum_{k=0}^{n-1} f^{(k+1)}(t)\cdot\frac{(x-t)^k}{k!} - \sum_{k=1}^{n} f^{(k+1)}(t)\cdot\frac{(x-t)^k}{k!}$$

$k = 1, \dots n$ máme $\quad = -\dfrac{f^{(n+1)}(t)}{n!}(x-t)^n$

$\psi(t) = (x-t)^{n+1}$ ... , $(x-t)^{n} \neq 0$ pro $t \neq x_0$

$\psi(x) = 0$

$\psi(x_0) = (x - x_0)^{n+1}$

Cauchyho věta o střední hodnotě: (V.6.8)

$\exists\, \theta$ mezi x, x_0 ... $\quad \dfrac{\varphi(x_0) - \varphi(x)}{\psi(x_0) - \psi(x)} = \dfrac{\varphi'(\theta)}{\psi'(\theta)}$

$$\frac{R_{n+1}(x)}{(x-x_0)^{n+1}} = + \frac{f^{(n+1)}(\theta)}{\underbrace{n!}_{(n+1)!}} \cancel{(x-\theta)^{n}} \cdot \frac{1}{+(n+1)\cancel{(x-\theta)^{n}}}$$

úprava: závěr.

Příklady: (1) $f(x) = e^x$; $\quad T_{0,n}^{e^x}(x) = \sum_{k=0}^{n} \dfrac{x^k}{k!}$;

V.8.4: $x > 0$ pevné. $\exists\, \theta \in (0, x)$ tak, že

$$e^x = T_{0,n}^{e^x}(x) + \frac{e^\theta}{(n+1)!} x^{n+1}$$

$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!} x^{n+1} = \frac{e^\theta}{(n+1)!} \cdot x^{n+1}.$$

$$|R_{n+1}(x)| \leq C \cdot \frac{|x|^{n+1}}{(n+1)!} \underset{n \to \infty}{\overset{?}{\longrightarrow}} 0 \; ; \; x \text{ pevné}.$$

Lemma 8.2. Nechť $M > 0$. Potom $\lim\limits_{n \to \infty} \dfrac{M^n}{n!} = 0$.

Dk.: $\dfrac{M^n}{n!}$; $\quad a_{n+1} = a_n \cdot \dfrac{M}{n+1}$;

fixuji $m \in \mathbb{N}$ tak, že $m > 2M$. ; potom pro $n \geq m$ platí:

$$n! = \underbrace{n \cdot (n-1) \cdots (m+1)}_{\substack{n-m \text{ čísel;} \\ \text{každé} > m > 2M}} \cdot m! > (2M)^{n-m} \cdot m!$$

Když: $n > m$: $\quad a_n = \dfrac{M^n}{n!} < \dfrac{M^n}{(2M)^{n-m} \cdot m!} = \underbrace{\dfrac{(2M)^m}{m!}}_{C \,..\, \text{pevné}} \cdot \dfrac{M^n}{(2M)^n}$

$= C \cdot \dfrac{1}{2^n} \to 0$ po $n \to \infty$.

a tedy také $a_n \to 0$ po $n \to \infty$.

$$|R_{n+1}(x)| = e^{\theta} \cdot \frac{x^{n+1}}{(n+1)!} \to 0 \quad \text{po } n \to \infty; \ (x \text{ pevné}).$$

$$e^x = \underbrace{\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}}_{\text{později značíme } \sum_{k=0}^{\infty} \frac{x^k}{k!}} ;$$

$\sum_{k=0}^{\infty} \frac{x^k}{k!}$ .. Taylorova řada pro e^x

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speciálně: $e = e^1 = \lim\limits_{n\to\infty} \left(\sum\limits_{k=0}^{n} \dfrac{1}{k!} \right)$

$e = \lim\limits_{n\to\infty} a_n; \quad a_n = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dots + \dfrac{1}{n!}$

$\{a_n\}$ rostoucí: $\quad a_n < e$ pro $\forall n \in \mathbb{N}$;

$$e - a_n = \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \right)$$

$$= \frac{1}{n!} \left(\frac{1}{n+1} + \frac{1}{(n+2)(n+1)} + \frac{1}{(n+3)(n+2)(n+1)} + \dots \right)$$

$$< \frac{1}{n!} \left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots \right) \qquad q = \frac{1}{n+1}$$

$$q + q^2 + q^3 + \dots = q(1 + q + q^2 + q^3 \dots) = \frac{q}{1-q}; \ |q| < 1.$$

$$= \frac{1}{n!} \cdot \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n!} \cdot \frac{1}{n}$$

Poznámka. Číslo e je iracionální. [illegible] $n \to \infty$

$$a_n = \sum_{k=0}^{n} \frac{1}{k!} \qquad\qquad e = \exp(1) = \lim_{n\to\infty} a_n;$$

Lemma 8.3 Pro $\forall n \in \mathbb{N}$ platí $\displaystyle\sum_{k=0}^{n} \frac{1}{k!} < e < \sum_{k=0}^{n} \frac{1}{k!} + \frac{1}{n!\,n}$

Lemma 8.4. Pro $\forall q$, $|q|<1$ platí $\displaystyle\sum_{k=0}^{\infty} q^k = 1+q+q^2+\cdots = \frac{1}{1-q}$.

$$(1+q+q^2+\cdots+q^n)\cdot(1-q) = 1-q^{n+1}$$

$$\begin{aligned} &1+\cancel{q}+\cancel{q^2}+\cdots+q^n \\ &-\cancel{q}-\cancel{q^2}-q^3\cdots-q^{n+1} \end{aligned} \qquad \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \to \frac{1}{1-q} \text{ pro } n\to\infty$$

$$|-q^{n+1}| = |q|^{n+1} = \exp\Big(\underbrace{(n+1)\cdot\underbrace{\ln|q|}_{<0}}_{\to -\infty}\Big) \to 0.$$

Dk.: 8.3. $\displaystyle e = \lim_{n\to\infty} a_n; \quad a_n = \sum_{k=0}^{n} \frac{1}{k!}$

$\{a_n\}$ rostoucí: $\Rightarrow \displaystyle\lim_{n\to\infty} a_n = \sup\{a_n;\, n\in\mathbb{N}\} = e$

$$a_n < a_{n+1} \le e \quad \text{(první nerovnost.)}$$

$$e - a_n = \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

$$= \frac{1}{(n+1)!}\left[1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \cdots\right]$$

$$< \frac{1}{(n+1)!}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] = \frac{1}{(n+1)!}\,\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n!\,n}$$

$$\underset{q=\frac{1}{n+1}}{[\ \ ]} = 1+q+q^2+\cdots = \frac{1}{1-\frac{1}{n+1}} \qquad \text{(druhá nerovnost.)}$$

$f(x) = T^f_{x_0,n}(x) + R_{n+1}(x)$; $\quad T^f_{x_0,n}(x) = \sum_{k=0} \ldots \frac{\ldots}{k!} \ldots$

$\exists \theta \in (x_0, x)$ $\qquad R_{n+1}(x) = \dfrac{f^{(n+1)}(\theta)}{(n+1)!}(x-x_0)^{n+1}.$

↑ závisí na $\underline{x_0, x, n}$

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(2) $f(x) = \sin x$; $\quad x_0 = 0$; $x \neq 0$ pevné.

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2n+2}(x);$$

$$R_{2n+2}(x) = \frac{f^{(2n+2)}(\theta)}{(2n+2)!}(x - 0)^{2n+2};$$

$$\sin^{(l)}(x) = \pm \begin{matrix} \sin x \\ \cos x \end{matrix} \Rightarrow |R_{2n+2}(x)| \leq \frac{|f^{(2n+2)}(\theta)|}{|2(n+1)|!}|x - 0|^{2n+2}$$

$$\leq \frac{x^2}{n!}(x^2)^n \to 0 \text{ pro } n \to \infty$$

(lemma)

tj. pro $\forall x \in \mathbb{R}$ pevné je

$$\sin x = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

podobně $\displaystyle \cos x = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$

$i^2 = -1$; $i^3 = -i$

$i^4 = 1$; $i^5 = i$

---

pozn.:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!}$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= \cos x + i \sin x$$

Důsledek: Číslo e je iracionální.

sporem: nechť $e = \frac{p}{q}$; $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

užij L. 8.3

$n = q$

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\,n}$$

$$0 < \frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q!\,q} \qquad \Big/ \cdot q!\,q$$

$$0 < \underbrace{q!\,p - \sum_{k=0}^{q} q\,\frac{q!}{k!}}_{} < 1$$

celé číslo: $k \le q$

tedy $k! \mid q!$

celé číslo,

leží v $(0,1)$

spor.