[C]

(1) $f(x) = x\left(3^{\frac{1}{x}} - 2^{\frac{1}{x}}\right) = \underbrace{2^{\frac{1}{x}}}_{P_1} \cdot \underbrace{x\left(\left(\frac{3}{2}\right)^{\frac{1}{x}} - 1\right)}_{P_2} \to \ln\frac{3}{2}$ ✓

$$P_1 = e^{\frac{1}{x}\ln 2} \to e^0 = 1 \quad (\text{spojitost } e^y)$$

$$P_2 = \frac{e^{\frac{1}{x}\ln 3/2} - 1}{\frac{1}{x}\cdot \ln 3/2} \cdot \ln 3/2 \to \ln 3/2\,;$$

neboť $\frac{1}{x}\ln 3/2 \to 0$; $x \to +\infty$

$\neq 0$ na $P(+\infty)$.

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(2) $$f(x) = \frac{x^x - a^a}{x-a} = \frac{x^x - a^x}{x-a} + \frac{a^x - a^a}{x-a} = P_1 + P_2 \underset{x\to a}{\to} a^a(\ln a + 1)$$

$P_2 \to a^a \ln a$; viz (A17).

$$P_1 = a^x \cdot \frac{\left(\frac{x}{a}\right)^x - 1}{x-a} = a^x \cdot \underbrace{\frac{e^{x\ln\frac{x}{a}} - 1}{x\cdot\ln\frac{x}{a}}}_{\to 1} \cdot \underbrace{\frac{\ln\frac{x}{a}}{x-a}}_{\to \frac{1}{a}} \cdot x \to a^a$$

viz (A16)

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(3) $$\frac{\ln(1+\sin^2 x)}{\ln(1+\operatorname{tg}^2 x)} = \frac{\ln(1+\sin^2 x)}{\sin^2 x} \cdot \frac{\operatorname{tg}^2 x}{\ln(1+\operatorname{tg}^2 x)} \cdot \cos^2 x \to 1$$

neboť: $\cos y$ spojitá v $y = 0$

$\frac{\ln(1+y)}{y} \to 1$; $y \to 0$

$\sin^2 x \to 0$; $x \to 0$

$\neq 0$ na $P(0)$; podobně $\operatorname{tg} x$

(4) $a, b \neq 0$: $\dfrac{\ln \cos ax}{\ln \cos bx} = \underbrace{\dfrac{\ln \cos ax}{\cos ax - 1}}_{P_1} \cdot \underbrace{\dfrac{\cos bx - 1}{\ln \cos bx}}_{P_2} \cdot \underbrace{\dfrac{\cos ax - 1}{\cos bx - 1}}_{P_3}$

$P_2 : P_1$: $\left.\begin{array}{l} \dfrac{\ln R}{R-1} \to 1;\ R \to 1 \\ \cos ax \to 1;\ x \to 0 \\ \quad \cos \neq 1 \text{ na } P(0, \frac{\pi}{2}) \end{array}\right\} \underset{V.2.6\,(b)}{\Rightarrow}$ $\quad P_1 = 1$

$\dfrac{1}{P_2} \to 1 \Rightarrow P_2 \to 1.$

P_3 : víme: $\dfrac{\cos R - 1}{R^2} \to \dfrac{1}{2}$; $R \to 0$; proto píšeme

$$P_3 = \frac{\cos ax - 1}{(ax)^2} \cdot \frac{(bx)^2}{\cos bx - 1} \cdot \left(\frac{a}{b}\right)^2 \longrightarrow \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{-1} \frac{a^2}{b^2}.$$

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(5) $f(x) = \exp h(x)$; $h(x) = \dfrac{1}{\sin^2 x} \ln(1+x^2)$

$= \dfrac{\ln(1+x^2)}{x^2} \cdot \left(\dfrac{x}{\sin x}\right)^2 \longrightarrow 1$; neboť $\dfrac{\ln(1+R)}{R} \to 0;\ R \to 0$

$x^2 \to 0;\ x \to 0$

leč $x^2 \neq 0$ na $P(0,1)$.

odtud: $f(x) \to e^1,\ x \to 0$

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(6) $f(x) = \exp h(x)$; $h(x) = \dfrac{1}{1+\sqrt{x}} \cdot \ln\left(\dfrac{1+x}{2+x}\right) \to \dfrac{1}{+\infty} \cdot \ln 1 = 0.$

$f(x) \to 1$

varianta: $f(x) = \left(\dfrac{1+x}{2+x}\right)^{1+\sqrt{x}} \longrightarrow 1$ též.

$$h(x) = \underbrace{\frac{\ln\left(\frac{1+x}{2+x}\right)}{\frac{1+x}{2+x} - 1}}_{\to 1;} \cdot \underbrace{\frac{(-2)(1+\sqrt{x})}{2+x}}_{\to 0}$$

viz B8

(7) $f(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{1+\cos x}}$; $x \to \pi+$ $= \exp h(x)$;

$$h(x) = \frac{1}{1+\cos x} \cdot \ln\left(\frac{x}{\pi}\right)$$; typ „$\frac{0}{0}$".

$$= \frac{\ln \frac{x}{\pi}}{\frac{x}{\pi} - 1} \cdot \frac{(x-\pi)}{\pi(1+\cos x)} = P_1 \cdot \frac{1}{\pi} \cdot P_2 \; ; \quad P_1 \to 1$$

(snadno z V.2.6)

$P_2 = \frac{x-\pi}{1+\cos x}$; $x \to \pi+$: totéž jako $\lim_{y \to 0+}$ | $x = y + \pi$

$$\tilde{P}_2 = \frac{y}{1+\cos(y+\pi)} = \frac{y}{1-\cos y} = \frac{y^2}{1-\cos y} \cdot \frac{1}{y} = R_1 \cdot R_2$$

$R_1 \to 2$ (zákl. lim. pro $\cos x$)

$R_2 \to +\infty$ (Věta 2.8 (a); limita typu $\frac{1}{0+}$).

celkem: $h(x) \to +\infty$; $f(x) = e^{h(x)} \to +\infty$; $x \to \pi+$.

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(8) $f(x) = \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x = \exp h(x)$; $h(x) = x \ln\left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)$

$$\lim_{x \to +\infty} h(x) = \lim_{y \to 0+} h\left(\frac{1}{y}\right); \quad h\left(\frac{1}{y}\right) = \frac{\ln(\sin y + \cos y)}{y}$$

$$h\left(\frac{1}{y}\right) = \underbrace{\frac{\ln(\sin y + \cos y)}{\sin y + \cos y - 1}}_{\to 1} \cdot \underbrace{\frac{\sin y + \cos y - 1}{y}}_{P_2}$$

? ověření $\sin y + \cos y \neq 1$ na $P_+(0, \delta)$

$$P_2 = \frac{\sin y}{y} + \frac{\cos y - 1}{y^2} \cdot y$$

$$\to 1 + \left(-\frac{1}{2}\right) \cdot 0 = 1.$$

celkem: $f(x) = e^1 = e$.

(9) $f(x) = \exp\big(\underbrace{\tfrac{1}{x}\ln(x+e^x)}_{h(x)}\big)$; $x \to 0$.

$$h(x) = \frac{\ln(x+e^x)}{x+e^x-1} \cdot \frac{x+e^x-1}{x} = P_1 \cdot P_2$$

$$\left.\begin{array}{l} \dfrac{\ln r}{r-1} \to 1;\ r \to 1 \\ x+e^x \to 1;\ x \to 0 \\ (??) \neq 1 \text{ na } P(0,\delta) \end{array}\right\} \xRightarrow[\text{V.2.6(b)}]{} \quad P_1 \to 1$$

celkem $f(x) \to e^2$,

$x \to 0$.

$$P_2 = 1 + \frac{e^x-1}{x} \to 2;\ x \to 0 \ :$$

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(10) $f(x) = \exp\left(\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)\right) = \exp h(x)$; $a,b \neq 1$
$a,b > 0$

$h(x) = \frac{1}{x}\ln\frac{a^x+b^x}{2}$; $\quad a^x = e^{x\ln a} \to 1;\ x \to 0$

leč $\neq 1 \ \forall x \in P(0,\delta)$

(neboť e^y je rostoucí v $\mathbb{R}$)

$$= \frac{\ln\left(\frac{a^x+b^x}{2}\right)}{\frac{a^x+b^x}{2}-1} \cdot \frac{\frac{a^x+b^x}{2}-1}{x} = P_1 \cdot P_2;$$

$P_1 \to 1$; neboť $\frac{a^x+b^x}{2} \to 1;\ x \to 0$ $\quad ??\ \frac{a^x+b^x}{2} \neq 1$ na $P(0,\delta)$

(pomocí derivace)

$$P_2 = \frac{1}{2}\left(\frac{a^x-1}{x} + \frac{b^x-1}{x}\right) \to \frac{1}{2}(\ln a + \ln b) = \ln\sqrt{ab}$$

celkem: $f(x) \to \sqrt{ab}$, $x \to 0$.

(11) $f(x) = \exp h(x)$; $h(x) = \frac{1}{x^2} \ln\left(\frac{1+x2^x}{1+x3^x}\right)$;

$$h(x) = \frac{\ln\left(\frac{1+x2^x}{1+x3^x}\right)}{\frac{1+x2^x}{1+x3^x} - 1} \cdot \frac{\frac{1+x2^x}{1+x3^x} - 1}{x^2} = P_1 \cdot P_2 ;$$

$P_1 \to 1$; $x \to 0$; neboť (V.2.6.(b)): $\frac{\ln z}{z-1} \to 1$; $z \to 1$

$$\frac{1+x2^x}{1+x3^x} \to \frac{1+0\cdot 2^0}{1+0\cdot 3^0} = 1; \ x \to 0;$$

$$\frac{1+x2^x}{1+x3^x} \neq 1: \ 1+x2^x \neq 1+x3^x$$

$$2^x \neq 3^x$$

$$\exp(\underbrace{x \ln 2/3}_{\neq 0}) \neq 1 \ ; x \in P(0,\delta)$$

o.k.

$$P_2 = \frac{1}{1+x3^x} \cdot \frac{2^x - 3^x}{x} = \frac{1}{1+x3^x} \cdot \left(\frac{2^x-1}{x} - \frac{3^x-1}{x}\right)$$

$\to 1 \cdot (\ln 2 - \ln 3)$; $x \to 0$; celkem $f(x) \to \frac{2}{3}$.

(12) $f(x) = e^{h(x)}$; $h(x) = \frac{1}{x} \ln \cos\sqrt{x}$; $x \to 0+$.

$$h(x) = \frac{\ln \cos\sqrt{x}}{\cos\sqrt{x} - 1} \cdot \frac{\cos\sqrt{x} - 1}{(\sqrt{x})^2} = P_1 \cdot P_2 .$$

$P_1 \to 1$; neboť $\cos\sqrt{x} \to 1$; $x \to 0+$

pro $x \in P_+(0,1)$: $\sqrt{x} \in (0,1) \Rightarrow \cos\sqrt{x} < 1$

$P_2 \to -\frac{1}{2}$; viz důvod; $\frac{\cos y - 1}{y^2} \to -\frac{1}{2}$; $y \to 0$

celkem: $f(x) \to e^{-1/2}$; $x \to 0+$

(13) $f(x) = \exp h(x)$; $h(x) = x^2 \ln\left(\frac{x+2}{2x+3}\right)$;

$$h(x) = x^2 \cdot \ln\left(\frac{x+2}{2x+3}\right) = P_1 \cdot P_2 \to -\infty, \; x \to +\infty$$

$$P_1 \to (+\infty)(+\infty) = +\infty; \quad x \to +\infty$$

$$P_2 = \ln\left(\frac{1+\frac{2}{x}}{2+\frac{3}{x}}\right) \to \ln\left(\frac{1+0}{2+0}\right) = \ln 1/2 < 0$$

dle VOAL & spojitosti $\ln y$ v $y_0 = \frac{1}{2}$;

celkem $f(x) \to 0$; $x \to +\infty$; neboť $e^y \to 0$, $y \to -\infty$.

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(14) $f(x) = \left(\frac{x^2-1}{x^2+1}\right)^{\frac{1}{x^2}} = \exp h(x)$; $h(x) = \frac{1}{x^2} \ln \frac{x^2-1}{x^2+1}$; $x \to +\infty$

$$\frac{1}{x^2} \to \frac{1}{(+\infty)^2} = 0; \quad x \to +\infty$$

$$\ln\left(\frac{x^2-1}{x^2+1}\right) = \ln\left(\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\right) \to \ln\left(\frac{1-0}{1+0}\right) = 0;$$

díky VOAL & spojitosti $\ln y$ v $y_0 = 1$;

celkem $f(x) \to e^{0 \cdot 0} = 1$.

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(15) $f(x) = \exp h(x)$; $h(x) = \frac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \ln\left(\frac{x+1}{x-1}\right)$; $x \to +\infty$

$$\frac{1-\sqrt{x}}{1+\sqrt{x}} = \frac{\frac{1}{\sqrt{x}} - 1}{\frac{1}{\sqrt{x}} + 1} \to -1; \quad x \to +\infty$$

$$\ln\left(\frac{x+1}{x-1}\right) = \ln\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right) \to 0; \quad x \to +\infty$$ (viz př. 14)

celkem $f(x) \to e^{-1 \cdot 0} = 1$; $x \to +\infty$.