Ex 6.1. Consider space ℓ^2 of sequences $\{u_j\}$ with the norm $\big(\sum_j u_j^2\big)^{1/2}$.
Let λ_j be real numbers. Define operator $A : \mathcal{D}(A) \subset \ell^2 \to \ell^2$ as

$$A : \{u_j\} \mapsto \{\lambda_j u_j\}, \qquad \mathcal{D}(A) = \left\{ \{u_j\} \in \ell^2;\ \sum_j \lambda_j^2 u_j^2 < \infty \right\}$$

1. Observe that $\mathcal{D}(A) = \ell^2$ and $A \in \mathcal{L}(\ell^2) \iff$ the sequence $\{\lambda_j\}$ is bounded.
2. Show that $(A, \mathcal{D}(A))$ is closed, and $\mathcal{D}(A)$ is dense in ℓ^2.
3. Assume that $\lambda_j \le \omega$ for all j. Deduce that any $\lambda > \omega$ belongs to the resolvent set $\rho(A)$. Write an explicit formula for the resolvent $R(\lambda, A)$.
4. By Hille-Yosida theorem, A is a generator of a c_0-semigroup in ℓ^2, which satisfies $\|S(t)\|_{\mathcal{L}(\ell^2)} \le e^{\omega t}$, $t \ge 0$. Compute $S(t)$ explicitely.

Ex 6.2. Let $S(t)$ be a c_0-semigroup in X, satisfying $\|S(t)\|_{\mathcal{L}(X)} \le M e^{\omega t}$, $t \ge 0$.
Let $\tilde{S}(t) = e^{-\omega t} S(t)$. Prove that:

1. $\tilde{S}(t)$ is also a c_0-semigroup, and $\|\tilde{S}(t)\|_{\mathcal{L}(X)} \le M$, $t \ge 0$.
2. Let $(A, \mathcal{D}(A))$, $\tilde{A}, \mathcal{D}(\tilde{A}))$ be the generators of $S(t)$, $\tilde{S}(t)$, respectively. Show that $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$ and $\tilde{A} = A - \omega I$.

*** Ex. 6.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let λ_j, w_j be the eigenvalues and eigenfunctions of the Dirichlet laplacian, with $\|w_j\|_2 = 1$.
By Parseval's theorem, we have $u_0 = \sum_j u_j w_j$, where $u_j = (u, w_j)$ and the sum converges in L^2.

1. Show that $u(t) = \sum_j e^{-\lambda_j t} u_j w_j$ is a weak solution to the heat equation $\frac{d}{dt}u - \Delta u = 0$, $u(0) = u_0$.
2. Show that $\sum_j \lambda_j u_j^2 < \infty$ if and only if $u_0 \in W_0^{1,2}$. Show that $\sum_j \lambda_j^2 u_j^2 < \infty$ if and only if $u_0 \in W_0^{1,2} \cap W^{2,2}$.
3. Observe that the "heat semigroup" (see Exercise 5.2) can thus be identified with a multiplicative semigroup (as in Exercise 6.1). It follows that the generator of heat semigroup is the operator $\Delta = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$, with derivatives understood weakly on the domain of definition $\mathcal{D}(\Delta) = W_0^{1,2} \cap W^{2,2}$.

6.4. Let $S(t)$ be a c_0-semigroup on X.

1. Show that the map $(t,x) \mapsto S(t)x$ is *jointly* continuous $[0,\infty) \times X \to X$.
2. Assuming that $\tilde{S}(t)$ is another c_0-semigroup, show that $y(t) = S(T-t)\tilde{S}(t)x$ is continuous $[0,T] \to X$, where $T > 0$ and $x \in X$ are fixed.
3. Assuming finally that $S(t)$ and $\tilde{S}(t)$ have the same generator, show that $y'(t) = 0$ for any $t \in (0,T)$ if $x \in \mathcal{D}(A)$.
4. Deduce that $S(t) = \tilde{S}(t)$ for all $t \geq 0$, thus establishing Lemma 4.2.

HINTS.

6.1. 2. for density, consider $\mathcal{C} \subset \mathcal{D}(A)$ consisting of $\{u_j\}$ with only finitely many nonzero u_j; 3. $R(\lambda, A) : \{v_j\} \mapsto \{\frac{1}{\lambda-\lambda_j} v_j\}$; 4. Use Yosida approximation. — Alternatively, one can guess that the answer $S(t) : \{u_j\} \mapsto \{e^{\lambda_j t} u_j\}$ and then verify that $S(t)$ is a c_0-semigroup on ℓ^2 and A is its generator. So $S(t)$ must be the sought-for semigroup by Lemma 4.2. For taking limits in ℓ^2, you can use the following version of Lebesgue's theorem (where sum is seen as an integral): if $\sum_j |a_j| < \infty$ and $b_j(t)$ are bounded independently of j and t, and $b_j(t) \to \beta_j$, then $\sum_j b_j(t) a_j \to \sum_j \beta_j a_j$.
6.4. 1. use part 1 of Lemma 4.1; in writing $y(t+h) - y(t)$, add $\pm S(T-(t+h))\tilde{S}(t)x$ and use joint continuity and definition of the generator, and Theorem 4.1; 4. $S(T)x = \tilde{S}(T)x$ if $x \in \mathcal{D}(A)$ and use density