* **Ex 5.1.** Let $S(t)$ be a c_0-semigroup in X. Show that the following are equivalent:

(1) $S(t) = e^{tA}$ for some $A \in \mathcal{L}(X)$

(2) $S(t)$ is uniformly continuous, i.e. $S(t) \to I$ in $\mathcal{L}(X)$ for $t \to 0+$

Ex 5.2. Let $u(t) \in L^2(I; W_0^{1,2}) \cap C(I; L^2)$ be the (unique) weak solution to the heat equation

$$\frac{d}{dt}u - \Delta u = 0, \qquad u(0) = u_0$$

Verify that the solution operators $S(t) : u_0 \mapsto u(t)$ form a c_0-semigroup in L^2.

Ex 5.3. Let $(A, \mathcal{D}(A))$ be an unbounded operator in X, which is closed, and let $\mathcal{D}(A)$ be dense in X.

1. Let $v'(t) = \lim_{h \to 0} \frac{1}{h}(v(t+h) - v(t)$ be the classical derivative in X. Assuming that $u'(t)$ and $(Au)'(t)$ exist, show that $u'(t) \in \mathcal{D}(A)$ and $A(u'(t)) = (Au)'(t)$.
2. Assume that $u(t) : I \to \mathcal{D}(A)$ be Bochner integrable, where $\mathcal{D}(A)$ is equipped with the graph-norm $\|u\|_X + \|Au\|_X$.

   Show that both $u(t) : I \to X$ and $Au(t) : I \to X$ are Bochner integrable, and $A\big(\int_I u(t)\,dt\big) = \int_I Au(t)\,dt$.

Ex 5.4. Let $X = L^2(\mathbb{R})$ and define the "shift" operators $S(t) : X \to X$ by $S(t) : f(x) \mapsto f(x+t)$.

1. Verify that $S(t)$ form a c_0-semigroup
2. Show that $\|S(t) - I\|_{\mathcal{L}(X)} = 2$ for any $t > 0$, hence the semigroup is not uniformly continuous
3. Prove that if $f(x) \in W^{1,2}(\mathbb{R})$, then $\frac{1}{h}(S(h)f(x) - f(x)) \to \frac{d}{dx}f(x)$ in $L^2(\mathbb{R})$, as $h \to 0+$.
4. Prove conversely that if $f(x)$, $g(x) \in L^2(\mathbb{R})$ are such that $\frac{1}{h}(S(h)f(x) - f(x)) \to g(x)$ in $L^2(\mathbb{R})$, as $h \to 0+$, then $f(x) \in W^{1,2}(\mathbb{R})$ and $\frac{d}{dx}f(x) = g(x)$
5. Observe that the above assertions imply that the generator of $S(t)$ is the operator $A : f(x) \mapsto \frac{d}{dx}f(x)$ with the domain of definition $\mathcal{D}(A) = W^{1,2}(\mathbb{R})$.

HINTS.

Ex. 5.3.2. Let $u_n(t)$ be simple functions and $u_n(t) \to u(t)$ in the norm of $\mathcal{D}(A)$ for a.e. $t \in I$, ...

Ex. 5.4.

2. Consider suitable $f(x) \in L^2(\mathbb{R})$ with compact support.

3. Working with AC representative, we have $f(x+h) - f(x) = \int_0^h g(x+s)\, ds$, where $g = \frac{d}{dx} f$. Deduce that $\frac{1}{h}\big(f(x+h) - f(x)\big)$ can be written as convolution of g with suitable kernels, and use Lemma 1.1, part 4.

4. Let $\varphi(x) \in C_c^\infty(\mathbb{R})$ be given test function and $h > 0$ be fixed. Prove that
$$\int_{\mathbb{R}} \frac{f(x+h) - f(x)}{h} \varphi(x)\, dx = \int_{\mathbb{R}} f(x) \frac{\varphi(x-h) - \varphi(x)}{h}\, dx$$
Using the assumptions, show that you can take the limit $h \to 0+$ on both sides, to obtain that $\frac{d}{dx} f(x) = g(x)$ in the sense of weak derivative.