NMMA406 – Exercises 4

Ex 4.1. Let X be reflexive, separable, $X \hookrightarrow Z$, and let $u(t) \in L^\infty(I;X) \cap C(I;Z)$. Show that $u(t) \in X$ for *all* $t \in I$ and moreover, $t \mapsto u(t) \in X$ is weakly continuous.

Ex 4.2. Let w_j be the eigenfunctions of $-\Delta u = \lambda u$ with zero Dirichlet b.c. Let P_N be the ON projection (in L^2) on the space $\mathrm{span}\{w_1, \dots, w_N\}$. Clearly P_N is continuous $L^2 \to L^2$ with norm 1.

1. Show that P_N is also continuous $W_0^{1,2} \to W_0^{1,2}$ with norm 1, if $W_0^{1,2}$ is taken as a Hilbert space with scalar product $(\!(u,v)\!) = (\nabla u, \nabla v)$.
2. Show that $\|P_N u\|_{2,2} \le c\|u\|_{2,2}$ for any $u \in W_0^{1,2} \cap W^{2,2}$ (assume $\partial\Omega$ sufficiently regular).

Ex 4.3. Let $\psi(z) : \mathbb{R} \to \mathbb{R}$ be smooth function with a bounded derivative. Show that $u_n \to u$ in $W^{1,2}$ implies $\psi(u_n) \to \psi(u)$ in $W^{1,2}$.

Ex 4.4. [d'Alembert's transform]. Let $u(t) : I \to X$, $g(t) : I \to X$ be integrable functions. Then the following assertions are equivalent:
(i) $\frac{d^2}{dt^2}u(t) = g(t)$ weakly, i.e.

$$\int_I u(t)\varphi''(t)\,dt = \int_I g(t)\varphi(t)\,dt \qquad \forall \varphi(t) \in C_c^\infty(I)$$

(ii) there is $v(t) : I \to X$ integrable such that $\frac{d}{dt}u(t) = v(t)$ and $\frac{d}{dt}v(t) = g(t)$ weakly in I.

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Ex 4.1. $\exists K > 0$, $N \subset I$ s.t. $\lambda(N) = 0$ and $\|u(t)\|_X \leq K$ for all $t \in I \setminus N$. Approximate $t_0 \in N$ with $t_n \to t_0$, $t_n \in I \setminus N$ to show that $\|u(t_0)\|_X \leq K$. Prove continuity by contradiction, using uniqueness of limits in Z.

Ex 4.2. (i) Rewrite P_N as ON (in $W_0^{1,2}$ w.r. to $((\cdot, \cdot))$) projection (ii) Show that $P_N(-\Delta u) = -\Delta P_N u$; use elliptic regularity for the laplacian

Ex 4.3. In view of Lemma 2.4, it is enough to show that $u_n \to u$, $\nabla u_n \to \nabla u$ in L^2 implies $\psi'(u_n)\nabla u_n \to \psi'(u)\nabla u$ in L^2. By taking a subsequence we can in the first step assume $u_n \to u$ a.e. Show further by contradiction (and step one) that convergence takes place even without taking a subsequence.