Ex 1.1 Assume that $u(t) \in C(I, X)$ (always $I = [0, T]$) and prove the following propositions:

1. $u(I) \subset X$ is compact, and $u(t) : I \to X$ is uniformly continuous.
2. Prove that $u(t) : I \to X$ is strongly measurable (a) using the Pettis theorem and (b) directly from the definition
3. Prove that $u(t) : I \to X$ is Bochner integrable (a) using the Bochner theorem and (b) directly from the definition
4. Show that $\int_I u(t)\, dt = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} u(jT/n)$

Ex 1.2

1. Let $u(t) : I \to X$ be Bochner integrable, let $F : X \to Y$ be linear, continuous. Then $Fu(t) : I \to Y$ is Bochner integrable, and
$$F\left(\int_I u(t)\, dt\right) = \int_I F\big(u(t)\big)\, dt$$
In particular: $\langle x^*, \int_I u(t)\, dt\rangle = \int_I \langle x^*, u(t)\rangle\, dt$ for any $x^* \in X^*$.
2. Prove the following version of Fatou's lemma: let $u_n(t) : I \to X$ are (strongly) measurable, and $u_n(t)$ converge weakly to $u(t)$ for a.e. $t \in I$. Then $u(t)$ is measurable and $\int_I \|u(t)\|\, dt \le \liminf_{n\to\infty} \int_I \|u_n(t)\|\, dt$. In particular, if $\int_I \|u_n(t)\|\, dt$ are bounded, then $u(t)$ is integrable.

Ex 1.3 Let $\psi_0(t) : \mathbb{R} \to \mathbb{R}$ be convolution kernel, i.e. $\psi_0(t)$ is bounded, zero outside $[-1, 1]$ and $\int_{-1}^{1} \psi_0(t)\, dt = 1$. Let $u(t) : I \to X$ be given, and assume that $u(t)$ is extended by zero outside of I. Let $\psi_n(t) = n\psi_0(nt)$ and finally let $u_n(t) = u * \psi_n(t) = \int_{\mathbb{R}} u(t-s)\psi_n(s)\, ds$.

1. Show that $u_n(t) \in C(I, X)$ and if $\psi_0 \in C^1$, then also $u_n(t) \in C^1(I, X)$ and $u_n'(t) = u * \psi_n'(t)$.
2. Show that norm of $u_n(t)$ is not larger than the norm of $u(t)$ in the spaces $C(I, X)$, $L^p(I, X)$.
3. Show that if $u(t) \in C(J, X)$ for some J strictly larger than I, then $u_n(t) \rightrightarrows u(t)$ in $C(I, X)$, for $n \to \infty$.

Ex 1.4

1. Prove the following Convergence Principle: Let $F_n : X \to X$ be a sequence of linear operators such that the norms $\|F_n\|$ are bounded independently of n. Let there be a dense $S \subset X$ such that $F_n v \to v$ as $n \to \infty$ for any $v \in S$. Then $F_n u \to u$ as $n \to \infty$ for any $u \in X$.
2. Apply the Convergence Principle to prove part 4 of Lemma 1.1.

HINTS.

Ex 1.1

1. Follows from compactness of I just as in the scalar case $X = \mathbb{R}$.
2. (a) compact implies separable, and continuous scalar is measurable; (b) set $u_n(t) = u(jT/n)$ for $t \in [(j-1)T/n, jT/n]$ — these are simple functions and $u_n(t) \rightrightarrows u(t)$ thanks to uniform continuity

   parts 3 and 4 use very similar ideas

Ex 1.2
1. Let $u_n(t)$ be simple functions from the definition of $\int_I u(t)\,dt$. Then $Fu_n(t)$ are simple $\ldots$ In particular: set $Y = \mathbb{R}$.
2. Use the fact that a separable set can be enlarged to a closed and convex (hence weakly closed) set. Use weak lower semicontinuity of the norm and scalar version of Fatou's lemma.
Ex 1.3
1. Rewrite $u_n(t) = \int_{\mathbb{R}} u(s)\psi_n(t-s)\,ds$ and show that usual theorems about dependence of integral on parameter apply. (In fact for ψ_0 smooth enough, the dependence on t is uniform, so the exchange of integral and limit is trivial.)
2. For $p = 1$, this follows by Fubini's theorem.
3. Use uniform continuity of $u(t)$ on the neighborhood of I.
Ex 1.4
1. Fix $u \in X$, and let $\varepsilon > 0$ arbitrary be given. Pick $v \in S$ such that $\|u - v\| < \varepsilon$. Write $F_n u - u = F_n(u - v) + \big(F_n v - v\big) + (v - u)$ and show that each term is estimated by (a multiple of) ε if n is large enough.
2. Set $X = L^p(I; X)$, $F_n u = u * \psi_n$ and $S = C_c(I; X)$. Use the results of Ex 1.3.2 and 1.3.3.