Theorem 1.9.* Let X be reflexive, separable and $p \in [1, \infty)$. Denote $\mathcal{X} = L^p(I; X)$. Then for any $F \in \mathcal{X}^*$ there is $v(t) \in L^{p'}(I, X^*)$ such that

$$\langle F, u(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \int_I \langle v(t), u(t)\rangle_{X^*,X}\, dt \qquad \forall u(t) \in \mathcal{X}.$$

Moreover, $v(t)$ is uniquely defined, and its norm in $L^{p'}(I; X^*)$ equals to the norm of F in $\mathcal{X}^*$.

Proof. STEP 1. For $\tau \in I$ and $x \in X$ denote $u_{\tau,x}(t) = \chi_{[0,\tau]}(t)x$. Clearly $u_{\tau,x}(t) \in \mathcal{X}$ and $x \mapsto \langle F, u_{\tau,x}(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}}$ is linear continuous. Hence there is $g(\tau) \in X^*$ such that $\langle F, u_{\tau,x}(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \langle g(\tau), x\rangle_{X^*,X}$ for all $x \in X$.
We will show there is $v(t) \in L^{p'}(I; X^*)$ such that

$$g(\tau) = \int_0^\tau v(t)\, dt \tag{1}$$

$$\|v(t)\|_{L^{p'}(I;X^*)} \le \|F\|_{\mathcal{X}^*} \tag{2}$$

Observe that with this we are done: (1) implies that

$$\langle F, u_{\tau,x}(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \left\langle \int_0^\tau v(t)\, dt, x \right\rangle_{X^*,X} = \int_I \langle v(t), u_{\tau,x}(t)\rangle_{X^*,X}\, dt$$

By linearity we have

$$\langle F, u(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \int_I \langle v(t), u(t)\rangle\, dt \tag{3}$$

for any $u(t) = \sum_j \chi_{(\alpha_j,\beta_j]}(t)x_j$. But such functions are dense in $\mathcal{X}$, to which (3) extends, using continuity of F on the left, and Hölder inequality on the right. Furthermore, it now follows from (3) that

$$\|F\|_{\mathcal{X}^*} = \sup_{\|u(t)\|_{\mathcal{X}}=1} \langle F, u(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \sup_{\|u(t)\|_{\mathcal{X}}=1} \int_I \langle v(t), u(t)\rangle_{X^*,X}\, dt \le \|v(t)\|_{L^{p'}(I;X^*)}$$

by Hölder inequality again. Together with (2) we obtain $\|F\|_{\mathcal{X}^*} = \|v(t)\|_{L^{p'}(I;X^*)}$; this also implies that $v(t)$ is uniquely defined.

STEP 2. Towards proving (1), we first show that $g(t) : I \to X^*$ is absolutely continuous. Let $(\alpha_j, \beta_j) \subset I$ be disjoint. It follows from reflexivity of X that there exist $x_j \in X$ with $\|x_j\| = 1$ such that $\|g(\beta_j) - g(\alpha_j)\|_{X^*} = \langle g(\beta_j) - g(\alpha_j), x_j\rangle_{X^*,X}$.
On the other hand

$$\langle g(\beta_j) - g(\alpha_j), x_j\rangle_{X^*,X} = \langle F, u_{\beta_j,x_j}(\cdot) - u_{\alpha_j,x_j}(\cdot)\rangle_{\mathcal{X}^*,\mathcal{X}} = \left\langle F, \chi_{(\alpha_j,\beta_j]}(\cdot)x_j \right\rangle_{\mathcal{X}^*,\mathcal{X}}$$

Hence

$$\begin{aligned}\sum_j \|g(\beta_j) - g(\alpha_j)\|_{X^*} &= \left\langle F, \sum_j \chi_{(\alpha_j,\beta_j]}(\cdot)x_j \right\rangle_{\mathcal{X}^*,\mathcal{X}} \\ &\le \|F\|_{\mathcal{X}^*} \|\sum_j \chi_{(\alpha_j,\beta_j]}(t)x_j\|_{\mathcal{X}} = \|F\|_{\mathcal{X}^*} \left(\sum_j (\beta_j - \alpha_j) \right)^{\frac{1}{p}}\end{aligned} \tag{4}$$

Obviously, this implies $g(t) \in AC(I; X^*)$.

STEP 3. By previous step and the fact that $g(0) = 0$, we see that (1) holds with some $v(t) \in L^1(I; X^*)$. It remains to establish (2).
If $p = 1$, observe that (4) implies $g(t)$ is lipschitz, and in particular $v(t) = g'(t)$ a.e. is essentially bounded by $\|F\|_{\mathcal{X}^*}$. In other words, (2) holds with $p' = \infty$ as required.
If $p \in (1, \infty)$, one can proceed as follows. As in Step 1 we use linearity to deduce (3) for all $u(t) = \sum_j \chi_{(\alpha_j, \beta_j]}(t) x_j$. We now just have $v(t) \in L^1(I; X^*)$, so the density argument only extends to $u(t) \in L^\infty(I; X)$.
We need one more limiting argument: set

$$v_n(t) = \begin{cases} v(t), & \text{if } \|v(t)\|_{X^*} \leq n \\ 0, & \text{otherwise} \end{cases}$$

and $u_n(t) = z(t) \|v_n(t)\|_{X^*}^{p'-1}$, where $z(t) \in X$ are such that $\|z(t)\|_X = 1$ and $\langle v(t), z(t) \rangle_{X^*, X} = \|v(t)\|_{X^*}$. Now $u_n(t)$ are essentially bounded, and $\|u_n(t)\|_X^p = \langle v(t), u_n(t) \rangle_{X^*, X} = \|v_n(t)\|_{X^*}^{p'}$. Plugging $u_n(t)$ into (3) gives, after a simple manipulation, that

$$\left(\int_I \|v_n(t)\|_{X^*}^{p'} \, dt \right)^{\frac{1}{p'}} \leq \|F\|_{\mathcal{X}^*}$$

Since $\|v_n(t)\|_{X^*} \nearrow \|v(t)\|_{X^*}$, estimate (2) follows by Levi theorem.