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13. Dynamical systems.

Definition. By *dynamical system* (d.s.) we mean a couple (φ, Ω), where $\varphi = \varphi(t, x) : \mathbb{R} \times \Omega \to \Omega$ is a map, satisfying

(i) $\varphi(0, x) = x$ for $\forall x \in \Omega$

(ii) $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for $\forall s,\, t \in \mathbb{R},\, x \in \Omega$

(iii) $(t, x) \mapsto \varphi(t, x)$ is continuous

While Ω can be any topological space, we will consider mostly open domains in $\mathbb{R}^n$ and smooth $\varphi(t, x)$.

Example. If $\Omega \subset \mathbb{R}^n$ is open and $f = f(x) : \Omega \to \mathbb{R}^n$ of class C^1, then $\varphi(t, x_0) := x(t)$, where $x = x(t)$ is the (unique) maximal solution to

$$x' = f(x), \qquad x(0) = x_0 \tag{1}$$

is a dynamical system with $\varphi \in C^1$. This is a canonical example in the sense that any smooth d.s. arises as a solution operator to the equation (1).

Definition. Let (φ, Ω) be a dynamical system. A set $M \subset \Omega$ is called

- positively invariant, if $\varphi(t, x) \in M$ for $\forall t \geq 0,\, x \in M$
- negatively invariant, if $\varphi(t, x) \in M$ for $\forall t \leq 0,\, x \in M$
- (fully) invariant, if $\varphi(t, x) \in M$ for $\forall t \in \mathbb{R},\, x \in M$

Given a point $x_0 \in M$ we further define

- positive orbit $\gamma^+(x_0) = \{\varphi(t, x_0);\ t \geq 0\}$
- negative orbit $\gamma^-(x_0) = \{\varphi(t, x_0);\ t \leq 0\}$
- (full) orbit $\gamma^-(x_0) = \{\varphi(t, x_0);\ t \in \mathbb{R}\}$

Observe that positive (resp. negative resp. full) orbit is positively (resp. negatively resp. fully) invariant. The set M is positively (resp. negatively resp. fully) invariant, iff for any $x_0 \in M$, the orbit $\gamma^+(x_0)$ (resp. $\gamma^-(x_0)$ resp. $\gamma(x_0)$) is a subset of M.

Definition. Let (φ, Ω) be a dynamical system. We define the *ω-limit set* of a point $x_0 \in \Omega$ as

$$\omega(x_0) = \{y \in \Omega;\ \exists t_n \to +\infty \text{ s.t. } \varphi(t_n, x_0) \to y\}$$

Analogously, we define the *α-limit set* of x_0 as

$$\alpha(x_0) = \{y \in \Omega;\ \exists t_n \to -\infty \text{ t.ž. } \varphi(t_n, x_0) \to y\}$$

Remark. It is easy to see that

$$y \in \omega(x_0) \iff \big(\forall \varepsilon > 0\big)\big(\forall T > 0\big)\big(\exists t \geq T\big)\Big[|y - \varphi(t, x_0)| < \varepsilon\Big],$$

or equivalently

$$y \in \Omega \setminus \omega(x_0) \iff \big(\exists \varepsilon > 0\big)\big(\exists T > 0\big)\big(\forall t \geq T\big)\Big[|y - \varphi(t, x_0)| \geq \varepsilon\Big].$$

Thus $\omega(x_0)$ consists of precisely all the points of Ω, the are relevant for $\varphi(t, x_0)$ as t becomes large.

Lemma 13.1. $\omega(x_0) = \bigcap_{\tau>0} \overline{\gamma^+(\varphi(\tau, x_0))}$

Remark. Recall that the set M is called *connected*, provided there *do not exist* open, disjoint sets $\mathcal{G}$, $\mathcal{H}$ such that $M \subset \mathcal{G} \cup \mathcal{H}$, while $M \cap \mathcal{G} \neq \emptyset$, $M \cap \mathcal{H} \neq \emptyset$.
Furthermore, any interval $I \subset \mathbb{R}$ is connected (in fact a subset of $\mathbb{R}$ is connected iff it is an interval), and a continuous image of a connected set is again connected.

Theorem 13.1. [Properties of $\omega(x_0)$.] Let (φ, Ω) be a dynamical system. Then

1. $\omega(x_0)$ is closed, fully invariant
2. If $\gamma^+(x_0)$ relatively compact in Ω, then $\omega(x_0)$ is non-empty, compact, and connected.

Theorem 13.2.[1] Let (φ, Ω) be a dynamical system, let $K \subset \Omega$ be compact. Then

$$\mathrm{dist}(\varphi(t, x_0), K) \to 0 \qquad \text{for } t \to +\infty\,, \tag{*}$$

if and only if $\emptyset \neq \omega(x_0) \subset K$. In particular, $\omega(x_0) = \{z\}$ iff $\varphi(t, x_0) \to z$ for $t \to +\infty$.

Definition. Dynamical systems (φ, Ω) and (ψ, Θ) are called *topologically conjugate*, if there exists a homeomorphism $h : \Omega \to \Theta$ such that $h(\varphi(t, x)) = \psi(t, h(x))$ for all $t \in \mathbb{R}$, $x \in \Omega$. Equivalently $\varphi(t, \cdot) = h_{-1} \circ \psi(t, \cdot) \circ h$ in Ω, for all t.

Remark. Topological conjugacy preserves the key properties of dynamical systems: stationary points and their stability, periodic orbits, ω-limit sets, ...

Theorem 13.3. [Rectification lemma.] Let $f(x)$ be C^1 in a neighborhood of $x_0 \in \mathbb{R}^n$, let $f(x_0) \neq 0$. Then there exist $\mathcal{V}$ a neighborhood of x_0, $\mathcal{W}$ a neighborhood of $0 \in \mathbb{R}^n$ and a diffeomorphism $g : \mathcal{V} \to \mathcal{W}$ such that $x(t)$ is a solution to (1) in $\mathcal{V}$ iff $y(t) = g(x(t))$ is a solution to

$$y' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2}$$

[1]Proven in exercises.

in $\mathcal{W}$. In terms of the previous definition: d.s. given by (1) and (2) are topologically conjugate (in fact C^1-conjugate) on respective neighborhoods.

Remark. Rectification lemma says that close to non-stationary points there is no interesting dynamics. The following (and considerably more difficult) theorem implies that close to stationary hyperbolic points, there is no nonlinear dynamics.
Recall that a stationary point x_0 to equation (1) is called *hyperbolic*, if $\operatorname{Re}\lambda \neq 0$ for any λ from the spectrum of $A = \nabla f(x_0)$.

Theorem 13.4.[2] [Hartman-Grobman.] Let $f(x)$ be C^1 on some neighborhood of x_0, where x_0 is a hyperbolic stationary point to (1).
Then there exist $\mathcal{V}$ a neighborhood of x_0 and $\mathcal{W}$ a neighborhood of $0 \in \mathbb{R}^n$ such that the d.s. given by (1) and $y' = Ay$ are topologically conjugate on respective neighborhoods.

14. La Salle's invariance principle.

Recall that given a C^1 function $V : \Omega \to \mathbb{R}$ we define the *orbital derivative* – w.r.t. solutions of (1) – as

$$\dot{V}_f(x) = \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x) f_j(x)$$

By chain rule for any $x = x(t)$ a solution of (1) in Ω one has $\frac{\mathrm{d}}{\mathrm{dt}} V(x(t)) = \dot{V}_f(x(t))$.

Example. Consider the mathematical pendulum with friction $x'' + q(x') + \sin x = 0$. Here $x = x(t)$ is the displacement angle, and $q = q(y)$ friction, depending on the velocity $y = x'$. It is natural to assume $q(0) = 0$ and $q(y)y > 0$ for $y \neq 0$. In such a case the equilibrium $(x, y) = (0, 0)$ is stable, using the Lyapunov function $V = y^2/2 + 1 - \cos x$.
But is it even asymptotically stable? If $q'(0) > 0$, this follows by the linearization argument. But the more delicate (in fact, non-hyperbolic) case when $q'(0) = 0$ requires a more subtle argument, which is contained in the following abstract theorem.

Theorem 14.1. [La Salle.] Let (φ, Ω) be the d.s. given by (1). Let there exist $V(x) : \Omega \to \mathbb{R}$ a C^1 function bounded from below, and let there be $\ell \in \mathbb{R}$ such the set $\Omega_\ell = \{x \in \Omega;\ V(x) < \ell\}$ is bounded. Assume finally that $\dot{V}_f(x) \leq 0$ in Ω_ℓ.
Denote

$$R = \{x \in \Omega_\ell;\ \dot{V}_f = 0\}$$
$$M = \{y \in R;\ \gamma(y) \subset R\}$$

Then for any $x_0 \in \Omega_\ell$ one has $\emptyset \neq \omega(x_0) \subset M$.

Remark. M is the largest fully invariant subset of R. In a typical application, M reduces to a single point which (in view of Theorem 13.2) is thus asymptotically stable (in fact it attracts all of Ω_ℓ).

[2] Without proof.

15. Poincaré-Bendixson theory.

The central problem of chapter: (non)existence of periodic solutions in $\mathbb{R}^2$. It is essential that we are in two dimensions only.

Standing assumptions. Throughout this chapter, $\Omega \subset \mathbb{R}^2$ is a domain (i.e. open, connected set), $f(x) : \Omega \to \mathbb{R}^2$ is C^1 and $\varphi = \varphi(t, x)$ is the d.s., given by (1). We also assume that $\varphi(t, x)$ is well-defined for all $t \geq 0$, $x \in \Omega$.

Recall. We say that γ is a *curve*, if $\gamma = \psi([a, b])$, where ψ is injective, continuous. It is a *Jordan curve*, provided that ψ is continuous, injective on $[a, b)$ and $\psi(a) = \psi(b)$. Finally, γ is a *(line) segment*, provided that ψ can be taken affine, i.e. $\psi(t) = at + b$ for some vectors $a \neq 0$ and b.

Note. Orbit (periodic orbit) is a curve (Jordan curve).

Jordan theorem. If $\gamma \subset \mathbb{R}^2$ is a Jordan curve, then $\mathbb{R}^2 \setminus \gamma$ consists precisely of two domains, of which one is bounded and simply connected ("the interior") and the other is unbounded ("the exterior").

Definition. A segment Σ is called transversal, provided that $f(p) \cdot n \neq 0$ for any $p \in \Sigma$, where n is the normal vector.
Geometrically: solutions of (1) traverse Σ with a non-zero speed (and in particular, in the same direction) at all points. Clearly every non-stationary point lies on some transversal.

Lemma 15.1. Let $\Sigma \subset \Omega$ be transversal, $y \in \Sigma$. Then there exist $\mathcal{U} \supset \tilde{\mathcal{U}}$ neighborhoods of u and $\Delta > 0$ such that for any $x_0 \in \tilde{\mathcal{U}}$ and solution $x(t) = \varphi(t, x_0)$ there holds:

(i) $x(t) \in \mathcal{U}$ for all $|t| < \Delta$

(ii) there is a unique $|\tilde{t}| < \Delta/2$ such that $x(\tilde{t}) \in \Sigma \cap \tilde{\mathcal{U}}$

Lemma 15.2. Let $\Sigma \subset \Omega$ be a transversal, let $p \in \Omega$. Then the intersections of $\gamma^+(p)$ with Σ form a monotone sequence. More precisely: if $t_1 < t_2 < t_3$ are such that $\varphi(t_i, p) \in \Sigma$, then either (i) $\varphi(t_1, p) = \varphi(t_2, p) = \varphi(t_3, p)$, or (ii) the point $\varphi(t_2, p)$ lies strictly between $\varphi(t_1, p)$ and $\varphi(t_3, p)$.

Lemma 15.3. Let $\Sigma \subset \Omega$ be a transversal, let $p \in \Omega$. Then $\omega(p) \cap \Sigma$ consists of at most one point.

Corollary. Let $\Sigma \subset \Omega$ be a transversal, let $\Gamma \subset \Omega$ be a periodic orbit. Then $\Gamma \cap \Sigma$ consists of at most one point.

Theorem 15.1. [Poincaré-Bendixson.] Let $p \in \Omega$ be such that $\gamma^+(p)$ is relatively compact in Ω, let furthermore $\omega(p)$ contains no stationary point. Then $\omega(p) = \Gamma$, where Γ is a (non-trivial) periodic orbit.

Theorem 15.2. [Bendixson-Dulac.] Let $\Omega \subset \mathbb{R}^2$ be simply connected and let there exist a C^1 function $B(x) : \Omega \to \mathbb{R}$ such that $\operatorname{div}(Bf)(x) > 0$ a.e. in Ω. Then (1) has no (non-trivial) periodic orbit in Ω.

16. Carathéodory theory.

In this chapter I, J denote arbitrary intervals.

Definition. Function $x : I \to \mathbb{R}^n$ is called *absolutely continuous*, denoted $x \in AC(I)$, provided that for any $\varepsilon > 0$ there is $\delta > 0$ such that for arbitrary *disjoint* intervals $(a_i, b_i) \subset I$ one has

$$\sum_i |a_i - b_i| < \delta \qquad \Longrightarrow \qquad \sum_i |f(a_i) - f(b_i)| < \varepsilon \tag{16.1}$$

Function x is called *locally absolutely continuous*, denoted $x \in AC_{\mathrm{loc}}(I)$, provided that $x \in AC(J)$ for any compact $J \subset I$.

Proposition 1. Let $x \in AC(I)$. Then a finite x' is defined a.e. in I, belongs to $L^1(I)$ and $x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(s)\, ds$ for all t_1, $t_2 \in I$.

Proposition 2. Let $h \in L^1(I)$, and $t_0 \in I$ be fixed. Then the function $x(t) := \int_{t_0}^{t} h(s)\, ds$ belongs to $AC(I)$; furthermore $x' = h$ a.e. in I.

Notation. $\Omega \subset \mathbb{R}^{n+1}$ is an open set of points $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $U = U(x_0, \delta)$ an open ball in $\mathbb{R}^n$, $Q(t_0, x_0) = Q(t_0, x_0; \delta, \Delta)$ is a cylinder $U(t_0, \delta) \times U(x_0, \Delta)$ v $\mathbb{R}^{n+1}$. Given $x = x(t) : I \to \mathbb{R}^n$, we denote $\operatorname{graph} x = \{(t, x(t)); t \in I\} \subset \mathbb{R}^{n+1}$.

Definition. We say that the function $f(t, x) : \Omega \to \mathbb{R}^n$ satisfies *Carathéodory conditions*, writing $f \in \mathrm{CAR}(\Omega)$, if for all $(t_0, x_0) \in \Omega$ there exists a cylinder $Q(t_0, x_0; \delta, \Delta) \subset \Omega$ and a function $m \in L^1(U(t_0, \delta))$ such that

(i) for any $x \in U(x_0, \Delta)$ fixed the function $f(\cdot, x)$ is measurable in $U(t_0, \delta)$

(ii) for almost every $t \in U(t_0, \delta)$ fixed the function $f(t, \cdot)$ is continuous in $U(x_0, \Delta)$

(iii) $|f(t, x)| \le m(t)$ for[3] a.e. t for all x in $Q(t_0, x_0; \delta, \Delta)$

Definition. Let $f \in \mathrm{CAR}(\Omega)$. Function $x : I \to \mathbb{R}^n$ is called a *Carathéodory solution* to

$$x' = f(t, x) \tag{16.1}$$

in Ω, provided that $\operatorname{graph} x \subset \Omega$, $x \in AC_{\mathrm{loc}}(I)$ and one has $x'(t) = f(t, x(t))$ for a.e. $t \in I$.

Lemma 16.1. Let $f \in \mathrm{CAR}(\Omega)$, $x : I \to \mathbb{R}^n$ be continuous and $\operatorname{graph} x \subset \Omega$. Then the function $t \mapsto f(t, x(t))$ belongs to $L^1_{\mathrm{loc}}(I)$.

Lemma 16.2. Let $f \in \mathrm{CAR}(\Omega)$, $x : I \to \mathbb{R}^n$ be a continuous function, and $\operatorname{graph} x \subset \Omega$. Then x is a Carathéodory solution to (16.1), if and only if

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s))\, ds \tag{16.2}$$

[3]The phrase "for almost every t for all …" means: there is a zero measure set N such that for all $t \in N$ and all …

for all t_1, $t_2 \in I$.

Remark. Based on the above integral formulation, one can develope the theory of AC (Carathéodory) solutions, in an analogy to the C^1 (classical) theory: local existence and uniqueness, maximal solutions, continuous dependence on the initial condition ... We will only prove a certain variant of (a generalized) Picard's theorem, which will include even global existence of solutions together with a continuous dependence on the (initial) data.

Theorem 16.1. [Generalized Banach contraction theorem.] Let Λ, X be metric spaces, with X complete, non-empty. Let $\Phi : \Lambda \times X \to X$ be continuous w.r.t. $\lambda \in \Lambda$ for any fixed $x \in X$. Let (the key assumption of *uniform contraction*) there exist $\kappa \in (0,1)$ such that

$$\|\Phi(\lambda, x) - \Phi(\lambda, y)\|_X \leq \kappa \|x - y\|_X \qquad \forall \lambda \in \Lambda,\ x,\ y \in X. \tag{16.3}$$

Then

(i) for any $\lambda \in \Lambda$ there is a unique $x(\lambda) \in X$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$

(ii) the map $\lambda \mapsto x(\lambda)$ is continuous

(iii) $\|y - x(\lambda)\|_X \leq (1-\kappa)^{-1} \|y - \Phi(\lambda, y)\|_X$ for $\forall \lambda \in \Lambda$, $y \in X$

Theorem 16.2. [Generalized Picard theorem.] Let $I = [0, T]$ be an interval, Π a metric space and $f = f(t, x, p) : I \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ satisfies the following:

1. $f(\cdot, \cdot, p) \in \mathrm{CAR}(I \times \mathbb{R}^n)$ for all $p \in \Pi$ fixed
2. there exists $\ell \in L^1(I)$ such that $|f(t, x, p) - f(t, y, p)| \leq \ell(t)|x - y|$ for a.e. $t \in I$ for all x, $y \in \mathbb{R}^n$, $p \in \Pi$
3. the map $p \mapsto \int_0^t f(s, x(s), p)\, ds$ is continuous from Π to $C(I)$, for arbitrary fixed $t_0 \in I$ and $x \in C(I)$

Then: for any $x_0 \in \mathbb{R}^n$ and $p_0 \in \Pi$ there exists a unique $x \in AC(I)$ a (Carathéodory) solution of $x' = f(t, x, p_0)$, with $x(0) = x_0$. This solution depends continuously on x_0 and p_0 in the following sense: $x_{0n} \to x_0$ and $p_{0n} \to p_0$ implies $x_n \rightrightarrows x$ in I, where x_n resp. x are the solutions corresponding to x_{0n}, p_{0n} and x_0, p_0, respectively.

Remark. Second assumption of the above theorem can be called a *generalized Lipschitz continuity* of $f(t, x, p)$ w.r.t. x.

Example. Consider linear equation

$$x' = A(t)x + b(t) \tag{16.3}$$

where $A(t) : [0, T] \to \mathbb{R}^{n \times n}$, $b(t) : [0, T] \to \mathbb{R}^n$ are L^1 functions. Clearly the assumptions of Theorem 16.2. hold (take $\ell(t) = \|A(t)\|$). The right-hand side $b(t)$ is considered as a parameter in $\Pi = L^1(0, T)$. We obtain existence of a global unique solution $x \in AC(I)$ which depends continuously on x_0 and $b(\cdot)$.

18. Optimal control

We will now consider problems of the type

$$x' = f(x,u), \qquad x(0) = x_0 \tag{18.1}$$

where $f(x,u) : \Omega \times U \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, $U \subset \mathbb{R}^m$ and $u = u(t)$ is an *admissible control*, i.e. an element of

$$\mathcal{U} = \{u : [0,T] \to U \text{ measurable}\}$$

Usually $m < n$. Typical tasks to be addressed:

1. for which x_0, $t > 0$ is there $u(\cdot) \in \mathcal{U}$ such that $x(t) = 0$ (controllability)
2. analogous question, but with a minimal time $t > 0$ (time optimal control)
3. more generally: find $u(\cdot) \in \mathcal{U}$ such that the functional

   $$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s))\, ds$$

   has a maximal value. Variants: $T > 0$ arbitrary, but $x(T)$ obeys some ,,final condition" (the problem of Mayer). Alternatively $T > 0$ is fixed, but $x(T)$ can be arbitrary (the problem of Bolza).

18. I. Linear problem – controllability, observability

Consider first the linear problem

$$\begin{gathered} x' = Ax + Bu, \qquad x(0) = x_0, \\ u(\cdot) \in \mathcal{U} = L^\infty(0,T;\mathbb{R}^m). \end{gathered} \tag{18.2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are given matrices. By Carathéodory theory we know that for any $u(t) \in \mathcal{U}$ there is a unique solution $x(t)$, given by the formula (variation of constants)

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} Bu(s)\, ds\,.$$

Definition. We say that control $u(\cdot)$ brings the initial condition x_0 to 0 in t, if $x(t) = 0$ for the corresponding solution. We denote this by $x_0 \xrightarrow[u(\cdot)]{t} 0$. The set

$$\mathcal{R}(t) = \{x_0 \in \mathbb{R}^n;\ \exists u(\cdot) \in \mathcal{U} \text{ such that } x_0 \xrightarrow[u(\cdot)]{t} 0\}$$

is called the *domain of controllability* at time t.

Key observation. By the previous formula we see that for the problem (18.2) we have

$$x_0 \xrightarrow[u(\cdot)]{t} 0 \quad \text{if and only if} \quad x_0 = -\int_0^t e^{-sA} B u(s)\, ds\,. \tag{K.O.}$$

Definition. By *Kalman matrix* of the system (18.2) we understand the $n \times mn$ matrix

$$\mathcal{K}(A, B) = (B, AB, A^2B, \dots, A^{n-1}B)$$

Lemma 18.1. For any integer $l \geq 0$ one has $A^l \in \operatorname{span}\{I, A, A^2, \dots, A^{n-1}\}$.

Theorem 18.1. Given (18.2), for $t > 0$ arbitrary one has $\mathcal{R}(t) = \operatorname{Lin}\{g_1, \dots, g_{mn}\}$, where $\{g_j\}$ are the columns of Kalman matrix $\mathcal{K}(A, B)$.

Corollary. The problem (18.2) is (globally) controllable (i.e. $\mathcal{R}(t) = \mathbb{R}^n$ for any $t > 0$), if and only if $\mathcal{K}(A, B)$ has rank n.

Definition. The problem

$$x' = Ax, \quad x(0) = x_0, \tag{18.3}$$

is called *observable* via the variable $y = Bx$, if there holds: given $x_1(t)$, $x_2(t)$ two solutions such that $Bx_1(t) \equiv Bx_2(t)$ on some non-trivial interval $[0, \tau]$, then necessarily $x_1(0) = x_2(0)$ ($\iff$ one has $x_1(t) \equiv x_2(t)$ for all t).

Theorem 18.2.[4] Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ be given. Then the following are equivalent:

1. problem (18.3) is observable via $y = Bx$
2. problem $x' = A^T x + B^T u$ is globally controllable
3. Kalman matrix $\mathcal{K}(A^T, B^T)$ has rank n

Remark. Previous theorem says that controllability and observability are dual notions. Following theorem, on the other hand, is a typical instance of the linearization principle: smooth, nonlinear problem is locally solvable, provided that the linearized problem is solvable.

Theorem 18.3. [Local controllability.] Let $f(0, 0) = 0$, $f(x, u)$ is C^1 close to $(0, 0)$ and let U (i.e. the set of values of admissible controls) contains a neighborhood of 0. Let the linearized problem, i.e. (18.2) with $A = \nabla_x f(0, 0)$, $B = \nabla_u f(0, 0)$ is globally controllable. Then the problem (18.1) is locally controllable (i.e. for any $t > 0$ fixed the set $\mathcal{R}(t)$ contains a neighborhood of zero).

[4] Proven in exercises.

18. II. Stabilizability via feedback

Can the control be automatic, i.e. in the form of some feedback function $u = F(x)$? Such a system cannot reach the value 0 in finite time (that would contradict the uniqueness). At best, it can be made asymptotically stable.
The answer again depends on the Kalman matrix, and we will again obtain global solution in the linear case, and local solution for the nonlinear case via the linearization principle.

Lemma 18.2.[5] Matrix

$$A = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \beta_0 & \beta_1 & \dots & \beta_{n-2} & \beta_{n-1} \end{pmatrix}$$

has a characteristic polynomial $p(\lambda) = \lambda^n - \sum_{j=0}^{n-1} \beta_j \lambda^j$. In particular: by properly choosing β_j, we can achieve an arbitrary spectrum $\sigma(A)$.

Theorem 18.4.[6] Let the problem $x' = Ax+Bu$ be controllable. Then there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $\sigma(A + BF) = \{\lambda_1, \dots, \lambda_n\}$, where $\lambda_j \in \mathbb{R}$ were chosen arbitrarily. In particular, the problem can be made asymptotically stable via a linear feedback of the form $u = Fx$.

Theorem 18.5. Let the assumptions of Theorem 18.3. hold true. Then there is $F \in \mathbb{R}^{m\times n}$ such that the problem $x' = f(x, Fx)$ is (locally) asymptotically stable at $x = 0$.

18. III. Time optimal control for linear problem.

We will again consider a linear problem, but only with a bounded admissible controls:

$$\begin{gathered} x' = Ax + Bu, \qquad x(0) = x_0, \\ u(\cdot) \in \mathcal{U} = \{u : [0,T] \to [-1,1]^m;\ \text{measurable}\ \} \end{gathered} \tag{18.4}$$

where again $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are given matrices. Symbols $x_0 \xrightarrow[u(\cdot)]{t} 0$ and $\mathcal{R}(t)$ are as in section 18.I above. Recall also the key observation (K.O.) still holds.
Our focus will be the time optimal control. For this we will use some deeper results from functional analysis.

Proposition 1. [Banach-Alaoglu.] The set of admissible controls for (18.4) is $*$-weakly sequentially compact in $L^\infty(0,T;\mathbb{R}^m)$. That is to say, for any sequence $\{u_n\} \subset \mathcal{U}$ there is a subsequence $\{\tilde{u}_n\}$ and a function $u \in \mathcal{U}$ such that $\tilde{u}_n \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;\mathbb{R}^m)$, i.e.

$$\int_0^T M(t)\cdot \tilde{u}_n(t)\,dt \to \int_0^T M(t)\cdot u(t)\,dt$$

[5] Proven in exercises.
[6] Outline of proof only, for $m = 1$.

for an arbitrary fixed function $M(t) \in L^1(0,T;\mathbb{R}^m)$.

Remark. From the variation of constants formula it follows easily that $u_n \overset{*}{\rightharpoonup} u$ implies $x_n(t) \to x(t)$ pointwise in $[0,T]$ for the respective solutions.

Theorem 18.6. Let the problem (18.4) be given. Then for any $t > 0$ fixed the set $\mathcal{R}(t)$ is convex, symmetric and closed. Moreover, $\mathcal{R}(t_1) \subset \mathcal{R}(t_2)$ for $t_1 < t_2$.

Corollary. The set of global controllability $\mathcal{R}_\infty := \bigcup_{t>0} \mathcal{R}(t)$ is convex, symmetric.

Theorem 18.7. Let $\mathcal{K}(A,B)$ has rank n and let $\operatorname{Re}\lambda \le 0$ for any $\lambda \in \sigma(A)$. Then $\mathcal{R}_\infty = \mathbb{R}^n$.

Remark. Under the stronger assumption $\operatorname{Re}\sigma(A) < 0$ follows Theorem 18.7 easily from Theorem 18.3 (on local controllability) and the fact that $x(t) \to 0$ if $u \equiv 0$.

Theorem 18.8. [Existence of time optimal control.] Let $x_0 \in \mathcal{R}_\infty$. Then there exist t^* and $u^*(\cdot) \in \mathcal{U}$ such that $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$, and t^* is the least possible time, i.e. $x_0 \notin \bigcup_{t<t^*} \mathcal{R}(t)$.

Proposition 2. [Krein-Milman.] Let K be a compact, convex, non-empty set in some locally convex topological space X. Then $K = \overline{\operatorname{co}}(\operatorname{ext} K)$, where $\operatorname{ext} K$ are the extremal points of K. In particular, K contains at least one extremal point.
Symbol $\overline{\operatorname{co}}(M)$ denotes closure of the convex hull of M. Point $a \in K$ is called extremal, provided that there exist no x, $y \in K$ such that $x \neq y$ with $a = (x+y)/2$. Equivalently: a is extremal in K if and only if $K \setminus \{a\}$ is convex.

Definition. We say that the control $u(t)$ is *bang-bang*, if $u_i(t) = \pm 1$ for a.e. t for all $i = 1, \dots, n$. In other words, $u(t)$ sits at some vertex of $[-1,1]^m$ for a.a. times.

Theorem 18.9. [Bang-bang principle.] Let $x_0 \in \mathcal{R}(t)$. Then there exists a bang-bang control $u(t) \in \mathcal{U}$ such that $x_0 \xrightarrow[u(\cdot)]{t} 0$.

Remark. In the proof we only used the existence of extremal point. Application of Krein-Milman in full strength implies that any control can be $*$-weak approximated by a convex combination of bang-bang controls. More precisely, any solution is a pointwise limit of solutions driven by convex combinations of bang-bang controls.

Corollary. (of Theorems 18.8. and 18.9.) If $x_0 \in \bigcup_{t>0} \mathcal{R}(t)$, then there exists a bang-bang time optimal control $u^*(\cdot) \in \mathcal{U}$ such that $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$.

Theorem 18.10. [Pontryagin maximum principle.] Let $u^*(\cdot) \in \mathcal{U}$, $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$, where the time t^* is optimal. Then there exist $h \in \mathbb{R}^n \setminus \{0\}$ such that

$$h \cdot e^{-tA} B u^*(t) = \max_{\eta \in [-1,1]^m} h \cdot e^{-tA} B \eta, \qquad \text{for a.e. } t \in [0,t^*]. \tag{18.5}$$

Remark. This is a typical instance of a necessary condition for the occurence of extrema. At first sight it is neither obvious, nor does not look very useful. In particular cases, however, it enables to single out a few candidates, among which it is easy to identify the global extremum (of course, provided we already know that it does exist).

18. IV. Pontryagin principle – the general case.

Finally, we will consider a general problem of optimal control

$$\begin{aligned} x' = f(x,u), \qquad & x(0) = x_0 \\ u(\cdot) \in \mathcal{U} = \{u : [0,T] &\to U \text{ measurable }\} \\ P[u(\cdot)] = g(x(t)) + &\int_0^T r(x(s),u(s))\,ds \end{aligned} \tag{18.6}$$

Our goal is to find $u(\cdot) \in \mathcal{U}$ such that the functional $u(\cdot) \in \mathcal{U}$ has a maximal value. We will consider that $T > 0$ is fixed, but there are no restrictions on $x(T)$ (the so-called problem of Bolza).

Theorem 18.11. [Pontryagin principle – the problem of Bolza.] Let $u^*(t) \in \mathcal{U}$ be a local maximum for (18.6); let $x^*(t)$ be the corresponding solution and finally let $f = f(x,u)$, $r = r(x,u)$ a $g = g(x)$ be C^1 on some neighborhood of the graph of $x^*(t)$, $u^*(t)$.
Then for a.e. $t \in [0,T]$ one has

$$H(x^*(t), p^*(t), u^*(t)) = \max_{\eta \in U} H(x^*(t), p^*(t), \eta) \tag{18.7}$$

where

$$H(x,p,u) = p^T f(x,u) + r(x,u) \tag{18.8}$$

is the so-called *Hamiltonian* and $p^*(t) \in \mathbb{R}^n$ is solution of the *adjoint problem*

$$\big(p^T\big)' = -\nabla_x H(x^*, p, u^*) \tag{18.9}$$

with final condition

$$p^T(T) = \nabla_x g(x^*(T)) \tag{18.10}$$

Remark. Adjoint problem (18.9) in components reads

$$p_i' = -\sum_{j=1}^n p_j \frac{\partial f_j}{\partial x_i}(x^*(t), u^*(t)) - \frac{\partial r}{\partial x_i}(x^*(t), u^*(t)), \qquad p_i(T) = \frac{\partial g}{\partial x_i}(x(T)).$$

It is a linear equation – hence, given $x^*(t)$, $u^*(t)$ there exists a unique (AC) solution $p(t)$.

Lemma 18.3. Let $z' = A(t)z$, let $(p^T)' = -p^T A(t)$. Then $p \cdot z$ is a constant function.

19. Bifurcation theory

Definition. [Bifurcation – ODE version.] A point (x_0, λ_0) is called *regular point* of the equation

$$x' = f(x, \lambda) \tag{19.1}$$

provided there exist $\delta > 0$ and $\mathcal{U}$ a neighborhood of x_0 such that for all $|\lambda - \lambda_0| < \delta$ are the dynamical systems of (19.1) topologically conjugate in $\mathcal{U}$.

A point (x_0, λ_0) is called a *point of bifurcation*, provided that it is not a regular point.

Remark. Here $\lambda \in \mathbb{R}$ is called a bifurcation parameter. Typically "bifurcation theorem" describes the behavior near the bifurcation point in a more precise way (e.g. the curve(s) of stationary points and their stability).

Remarks. A non-stationary point of (19.1) is always regular (by Theorem 13.3). A hyperbolic stationary point is also regular (a long proof using IFT, Rouché, Hartman-Grobman theorem).
Hence, a necessary condition for bifurcation is presence of a non-hyperbolic stationary point, i.e. $f(x_0) = 0$ and $\nabla f(x_0)$ with at least one purely imaginary eigenvalue.

Lemma 19.1. [Division lemma.] Let $h(x, \lambda)$ be C^k, where $k \geq 1$, and let $h(0, \lambda) = 0$ on some neighborhood of 0. Then there exists $H(x, \lambda)$ of class C^{k-1} such that $h(x, \lambda) = xH(x, \lambda)$ on some neighborhood of $(0, 0)$. Moreover, one has

$$H(0,0) = \partial_x h(0,0), \qquad \partial_x H(0,0) = \frac{1}{2}\partial^2_{xx} h(0,0),$$
$$\partial_\lambda H(0,0) = \partial^2_{x\lambda} h(0,0), \qquad \partial^2_{xx} H(0,0) = \frac{1}{3}\partial^3_{xxx} h(0,0)$$

Theorem 19.1. [Saddle-node in 1d.] Let $f(x, \mu)$ be C^2 close to $(0, 0) \in \mathbb{R}^2$. Let $f(0, 0) = 0$, $\partial_x f(0, 0) = 0$, let $\partial_\mu f(0, 0) \neq 0$, $\partial^2_{xx} f(0, 0) \neq 0$. Then the equation

$$x' = f(x, \mu) \tag{19.2}$$

has a saddle-node bifurcation in $(0, 0)$.

Theorem 19.2. [Transcritical in 1d.] Let $f(x, \mu)$ be C^2 close to $(0, 0) \in \mathbb{R}^2$. Let $f(0, 0) = 0$, $\partial_x f(0, 0) = 0$; let moreover $f(0, \mu) = 0$ (hence also $\partial_\mu f(0, 0) = 0$) for μ close to 0. Let $\partial^2_{\mu x} f(0, 0) \neq 0$, $\partial^2_{xx} f(0, 0) \neq 0$. Then the equation (19.2) has a transcritical bifurcation in $(0, 0)$.

Theorem 19.3.[7] [Pitchfork in 1d.] Let $f(x, \mu)$ be C^3 close $(0, 0) \in \mathbb{R}^2$. Let $f(0, 0) = 0$, $\partial_x f(0, 0) = 0$; let moreover $f(0, \mu) = 0$ (hence also $\partial_\mu f(0, 0) = 0$) for μ close to 0 and let $\partial^2_{xx} f(0, 0) = 0$. Let finally $\partial^2_{\mu x} f(0, 0) \neq 0$ and $\partial^3_{xxx} f(0, 0) \neq 0$. Then the equation (19.2) has a pitchfork bifurcation in $(0, 0)$.

Theorem 19.4. [Hopf bifurcation in 2d.] Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_\mu \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y, \mu) \\ g(x, y, \mu) \end{pmatrix} \tag{19.3}$$

where f, g, $\nabla_{x,y} f$, $\nabla_{x,y} g$ are smooth and equal to zero at $(0, 0, \mu)$. Let A_μ be a real 2×2 matrix smoothly depending on μ such that (key assumption)

$$\sigma(A_\mu) = \big\{\alpha(\mu) \pm i\omega(\mu)\big\}$$
$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \omega(0) \neq 0$$

[7] Not proven.

Then there is a family of (non-trivial) periodic solutions close to $(x, y, \mu) = (0, 0, 0)$.

Theorem 19.5. [Normal form of Hopf bifurcation.] Let A_μ, $f(x, y, \mu)$, $g(x, y, \mu)$ be as in Theorem 19.4. Let moreover

$$A_0 = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}$$

Then the system (19.3) is near $(x, y, \mu) = (0, 0, 0)$ topologically conjugate to the system (in polar coordinates)

$$r' = r(\alpha_1 \mu + ar^2), \qquad \varphi' = 1 \tag{19.4}$$

where $\alpha_1 = \alpha'(0)$ and

$$\begin{aligned} 16a &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\omega_0}\Big(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy} \Big)\Big|_{(0,0,0)} \end{aligned} \tag{19.5}$$

20. Invariant manifolds.

Problem. We will start with auxiliary problem

$$\begin{aligned} x' &= Ax + f(x, y) \\ y' &= By + g(x, y) \end{aligned} \tag{20.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $f = g = 0$ at $(x, y) = (0, 0)$.

Assumptions. For certain positive ε, c_0, β, ρ and σ it holds:

- $\operatorname{Re} \sigma(A) \geq -\varepsilon$ ($\implies x \cdot Ax \geq -\varepsilon |x|^2$)
- $\operatorname{Re} \sigma(B) < -\beta$ ($\iff \|e^{tB}\| \leq c_0 e^{-t\beta}$, $t \geq 0$)
- $|f|$, $|g| \leq \rho$ in $\mathbb{R}^{n+m}$
- $\operatorname{Lip} f$, $\operatorname{Lip} g \leq \sigma$ in $\mathbb{R}^{n+m}$

Typically ρ, σ are small, hence f and g are small perturbations of the linear problem.

Goal. We want to construct an invariant manifold, i.e. a function $\Phi \in \mathcal{X}$, where

$$\mathcal{X} = \Big\{ \Phi : \mathbb{R}^n \to \mathbb{R}^m;\ \Phi(0) = 0,\ |\Phi| \leq b,\ \operatorname{Lip} \Phi \leq \ell \ \Big\}$$

with the property

$$(x(t), y(t)) \text{ solve (20.1)},\ y(0) = \Phi(x(0)) \quad \implies \quad y(t) = \Phi(x(t)),\ \forall t \in \mathbb{R} \tag{INV}$$

Equivalently, $\operatorname{graph} \Phi = \{(x, y) \in \mathbb{R}^{n+m},\ y = \Phi(x)\}$ is invariant under (20.1).

Notation. The equation

$$p' = Ap + f(p, \Phi(p)) \tag{20.2}$$

kde $\Phi \in \mathcal{X}$, will be called a *reduced equation.* Observe that due to the (global) boundedness and Lipschitz continuity of f, g, Φ, for any initial condition $x(0)$, $y(0)$ or $p(0)$, there is a global (i.e., defined for all $t \in \mathbb{R}$) solution to (20.1) or (20.2).

Lemma 20.1. $\Phi \in \mathcal{X}$ has the property (INV), iff it has the property

$$p(t) \text{ solves (20.2)} \qquad \Longrightarrow \qquad (x(t), y(t)) := (p(t), \Phi(p(t))) \text{ solve (20.1)} \tag{RED}$$

Remark. Intuitively speaking, (RED) means that on the manifold, the dynamics of the second (stable) variable y is reduced to a functional relation $y = \Phi(x)$.

Theorem 20.1.[8] Let the constants of problem (20.1) satisfy certain assumptions on smallness of ρ and σ ...

$$\frac{c_0\rho}{\beta} \le b \qquad \frac{c_0\sigma(1+\ell)}{\beta - \varepsilon - \sigma(1+\ell)} \le \ell \qquad c_0\sigma\left(\frac{1}{\beta} + \frac{1+\ell}{\beta - \varepsilon - \sigma(2+\ell)}\right) < 1$$

Then there exists a unique $\Phi \in \mathcal{X}$, satisfying (INV).
Moreover, if $\nabla g(0,0) = 0$, then $\nabla\Phi(0) = 0$, i.e. the manifold is *tangent* to the plane $y = 0$ at origin. Finally, if f and g are C^k, then Φ is also C^k.

Remark. The above conditions are typically satisfied as follows: constants β, c_0 and ε are determined by the spectrum of A, B. Constants b, ℓ, defining the space $\mathcal{X}$, can be taken arbitrary. Based on that, ρ, σ (controlling the nonlinearities f, g) have to be chosen small enough.

Application 1. Consider the system

$$X' = MX + F(X) \tag{20.3}$$

in some neighborhood of $X = 0$, where $F(0) = 0$, $\nabla F(0) = 0$. Assume that $\sigma(M)$ lies partly in $\{\mathrm{Re} < 0\}$, partly on the imaginary axis $\{\mathrm{Re} = 0\}$. Then a suitable linear transfromation brings (20.3) to (20.1), with $X = (x, y)$. At the same time, nonlinearities can be redefined to keep the same value close to $(0, 0)$, while being globally small.
Applying Theorem 20.1, we obtain (locally) invariant so-called *center manifold*, tangent to the central space x.

Application 2. The same procedure can be applied if $\mathrm{Re}\,\sigma(B) < 0$, but $\mathrm{Re}\,\sigma(A) > 0$. Then we obtain the (local) *unstable manifold.* We can also reverse the time and apply Theorem 20.1 to get the (local) *stable manifold*, i.e. tangent to the stable directions y.

In what follows, we will focus on the situation as in Application 1 above. Note that $X = 0$ is a non-hyperbolic stationary point, and its stability cannot be determined by linearization.

[8] Without proof.

We will firstly show that the stability of the full system is equivalent to the stability of the reduced equation. Second, we will develop a method of approximation of the centre manifold, which in applications enables to investigate the behavior (in particular, the stability) of the reduced equation.

Notation. For some fixed $\mu > 0$ we define *cone* and its *shadow* (i.e. closure of complement)

$$\mathcal{K} = \big\{X = (x,y) \in \mathbb{R}^{n+m};\ |y| \le \mu|x|\big\}$$
$$\mathcal{V} = \big\{X = (x,y) \in \mathbb{R}^{n+m};\ |y| \ge \mu|x|\big\}$$

More generally, cone and shadow with center at X_0 is defined as $\mathcal{K}(X_0) = \{\tilde{X};\ \tilde{X}-X_0 \in \mathcal{K}\}$, $\mathcal{V}(X_0) = \{\tilde{X};\ \tilde{X} - X_0 \in \mathcal{V}\}$.

Lemma 20.4. For a suitable choice of $\mu > 0$ and the constants of the system (20.1) there holds:

1. cone is positively invariant: if X_1, X_2 solve (20.1) and $X_1(0) \in \mathcal{K}(X_2(0))$, then $X_1(t) \in \mathcal{K}(X_2(t))$ for all $t \ge 0$.
2. shadow is exponentially stable: if X_1, X_2 solve (20.1) and $X_1(t) \in \mathcal{V}(X_2(t))$ on some interval I, then $|X_1(t) - X_2(t)| \le e^{-\gamma(t-s)}|X_1(s) - X_2(s)|$ for all $t \in I$.

Theorem 20.2. [Tracking property of center manifold.] Let the assumptions of Lemma 20.4 hold true; let moreover $\mu > \ell$. Then the centre manifold has the following property: for an arbitrary fixed $X(t)$ a solution to (20.1), there exists a $P(t)$ solution on c.m. such that $X(t) - P(t)$ decays exponentially.
Moreover: if $X(0)$ is close to origin, then $P(0)$ can also be chosen close to origin.

Corollary. [Principle of reduced stability.] Let 0 be stable (asymptotically stable, unstable) for the reduced equation (20.2). Then $(0,0)$ has the same property for (20.1).

Observation. Lemmas 20.1 and 20.3 assert two equivalent formulation of the invariance property (INV) of the manifold Φ. For $\Phi \in C^1$, one can verify that (INV) is equivalent to $M\Phi(x) = 0$ for all $x \in \mathbb{R}^n$, where

$$M\Phi(x) = \nabla\Phi(x)\big[Ax + f(x,\Phi(x))\big] - B\Phi(x) - g(x,\Phi(x)) \tag{DE}$$

Remark. In fact, (DE) is nothing else then the orbital derivative of $y - \Phi(x)$ along the solutions of (20.1). Finding (a smooth) centre manifold is thus equivalent to solving a certain (partial) differential equation. This task is hopeless in general (typically, the equation is degenerate in view of presence of zero eigenvalues in A). However, from the point of view of applications, the following approximation principle is enough.

Theorem 20.3.[9] [Approximation of c.m.] Let the assumptions of Theorem 20.1 hold true and let $\Phi(x) \in \mathcal{X}$ be the corresponding centre manifold. Let moreover $\Psi(x) : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function, satisfying $\Psi(0) = 0$, $\nabla\Psi(0) = 0$ and

$$M\Psi(x) = \mathcal{O}(|x|^q), \qquad |x| \to 0, \tag{20.4}$$

[9] Without proof.

with some $q > 1$. Then $\Phi(x) = \Psi(x) + \mathcal{O}(|x|^q)$, $|x| \to 0$.

Corollary. [Asymptotic uniqueness of c.m.] If $\Phi(x)$, $\tilde{\Phi}(x)$ be (local) c.m., then $\Phi(x) - \tilde{\Phi}(x) = \mathcal{O}(|x|^q)$, $|x| \to 0$ holds for *arbitrary* $q > 1$.

Remark. In practice, we use trial and error to find Ψ so that (20.4) holds with large enough q. This yields a good enough approximation to analyse the reduced equation. In view of Theorem 20.2, the result then transfers readily to the original system (20.1).