ODE2 – 1st term – 19 Jan 2017

Problem 1 [7 pts] Show that the system

$$\begin{aligned} x' &= -y \\ y' &= x + 2\mu y - 24y^3 \end{aligned}$$

has a Hopf bifurcation for the parameter value $\mu = 0$ in the origin $(x, y) = (0, 0)$. Determine the stability of the origin and the periodic solutions (in case these exist). Draw the bifurcation diagram in the half-plane (μ, r), where $r = \sqrt{x^2 + y^2}$.

Problem 2 [5 pts] Consider the system

$$\begin{aligned} x' &= x^2 + \sin(\alpha y - u) \\ y' &= 2x + \beta y \\ z' &= \frac{1}{1 - \gamma y} - \frac{1}{1 + u} + z^2 \end{aligned}$$

where α, β, γ are real parameters. Under which conditions is the system locally controllable in the neighborhood of $(x, y, z) = (0, 0, 0)$? – Admissible controls are of the form $u : [0, \infty) \to (-\delta, \delta)$ and measurable, with some small $\delta > 0$ fixed.

Problem 3 [8 pts] Show that the system (with a real parameter a)

$$\begin{aligned} x' &= ax^3 + x^2 y \\ y' &= -y + y^2 + xy - x^3 \end{aligned}$$

has a centre manifold of the form $y = \phi(x)$ in some neighborhood of $(x, y) = (0, 0)$. Find a suitable approximation of $\phi(x)$ to determine the stability of the origin.