Problems on center manifolds.

For given problems, find a suitable approximation of centre manifold. Investigate stability of the reduced equation.

2)

$$\begin{aligned} x' &= -x^3 + 3xy^2z \\ y' &= -y^3 - 2x^2yz \\ z' &= -z + 10(x^2 + y^2) \end{aligned}$$

3)

$$\begin{aligned} x' &= -z^k \qquad k \geq 2 \\ y' &= -y + x^2 \\ z' &= -2z - x^2 \end{aligned}$$

4)

$$\begin{aligned} x' &= x(y - z) \\ y' &= -2y + z + z^2 - x^2 \\ z' &= y - 3z + xyz \end{aligned}$$

See also solutions on page 2.

2) C.m. $z = \phi(x, y)$, and $\psi = 0$ or $\psi = 10(x^2 + y^2)$ are approximations of 2nd or 4th order. Reduced equation is asymptotically stable (using Lyapunov function $V = p^2 + q^2$).

3) C.m. $y = \phi_1(x)$, $z = \phi_2(x)$. Approximations $\psi_1 = x^2$, $\psi_2 = -x^2/2$ are of order $2k + 1 \geq 5$. Reduced equation is unstable.

3) C.m. $y = \phi_1(x)$, $z = \phi_2(x)$. Approximations $\psi_1 = -\frac{3}{5}x^2$, $\psi_2 = -\frac{1}{5}x^2/2$ are of 4th order. Reduced equation is asymptotically stable.

③ apply Thm 20.1 (local version, i.e. Application 1)

$m = n = 1$, $A = (0)$, $B = (-1)$, $f = x^2 y$

$g = y^2 + xy - x^3$

$\Rightarrow \exists$ c.m. $y = \phi(x)$; reduced equation is

$$\boxed{\begin{aligned} x' &= ax^3 + x^2 y \\ y' &= -y + y^2 + xy - x^3 \end{aligned}}$$

$$p' = ap^3 + p^2 \phi(p)$$

approximation: apply Thm 20.3. (local version)

$$M\phi(x) = \phi'(x)\left(ax^3 + x^2\phi(x)\right) + \phi(x) - \phi^2(x) - x\phi(x) + x^3$$

i) try $\psi(x) = 0$ .. $M\psi(x) = x^3 = O(|x|^3)$, $x \to 0$

hence $\phi(x) = 0 + O(|x|^3)$; $x \to 0$

reduced eq: $p' = ap^3 + p^2 O(|p|^3) = p^3\left(a + O(|p|^2)\right)$

$$\boxed{\begin{aligned} \Rightarrow \quad & a > 0 : \text{ asympt. stable} \\ & a < 0 : \text{ unstable.} \end{aligned}}$$

if $a = 0$, this approximation is not good enough ($p' = p^2 O(|p|^2)$ has no sign information)

ii) try $\psi(x) = cx^2$ .. $M\psi(x) = O(|x|^2)$ ..

hence $\phi(x) = cx^2 + O(|x|^2)$ .. still no info about the sign

iii) try $\psi(x) = cx^3$ .. $M\psi(x) = \underbrace{cx^3 + x^3}_{= 0 \text{ by taking } c = -1} + O(|x|^4)$

$\Rightarrow \phi(x) = -x^3 + O(|x|^4)$

$$p' = p^2\left(-p^3 + O(|p|^4)\right) = -p^5 + O(|p|^6) \Rightarrow \boxed{\text{asympt. stable.}}$$

3 zř. (15.1.2014)

$$x' = -x^3 + 3xy^2 z$$
$$y' = -y^3 - 2x^2 y z$$
$$z' = -z + 10(x^2+y^2)$$

(x,y) .. centrální, $m=2$

z .. stabilní $m=1$

C.V. dotazy

$$\phi(x,y) - z = 0 \quad \frac{d}{dt}$$

$$\frac{\partial\phi}{\partial x}x' + \frac{\partial\phi}{\partial y}y' - z' = \frac{\partial\phi}{\partial x}\left(-x^3 + 3xy^2\phi\right) + \frac{\partial\phi}{\partial y}\left(-y^3 - 2x^2y\phi\right)$$
$$- \left(-\phi + 10(x^2+y^2)\right)$$

$$M\phi = \frac{\partial\phi}{\partial x}\left(-x^3+3xy^2\phi\right) + \frac{\partial\phi}{\partial y}\left(-y^3-2x^2y\phi\right) + \phi - 10(x^2+y^2)$$

(i) $\psi = 0$: $M\psi = -10(x^2+y^2) = O(x^2+y^2)$,

$\Rightarrow \phi(x,y) = O(x^2+y^2)$.

(ii) $\psi = 10(x^2+y^2)$: $M\psi = 20x\cdot\left(-x^3 + 3xy^2\cdot 10(x^2+y^2)\right)$
$$+ 20y\left(-y^3 - 2x^2y\cdot 10(x^2+y^2)\right)$$
$$= O\left(x^4+y^4\right)$$

$$|x^3y^2| \le x^2y^2 \le \frac{1}{2}(x^4+y^4) \ldots$$

$$\Rightarrow \phi(x,y) = 10(x^2+y^2) + O\left((x^2+y^2)^2\right).$$

? stab.:

$$x' = -x^3 + 3xy^2\, O(x^2+y^2) \quad \Big| \; 2x$$
$$y' = -y^3 - 2x^2y\, O(x^2+y^2) \quad \Big| \; 2y$$

$$\frac{d}{dt}(x^2+y^2) = -2(x^4+y^4) + O\left(x^2y^2(x^2+y^2)\right)$$

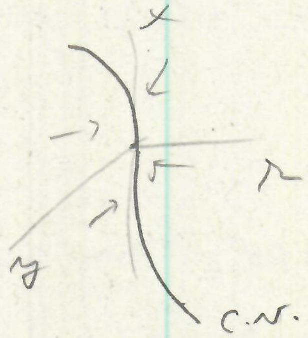
32) $x' = -z^{\ell} \quad (\ell \geq 2)$ $\quad A = (0)$, $m = 1$

$y' = -y + x^2$ $\quad B = \begin{pmatrix} -1, & 0 \\ 0, & -2 \end{pmatrix}$, $m = 2$

$z' = -2z - x^2$

C.N. $\quad y = \phi_1(x)$

$\quad z = \phi_2(x)$



$$\left.\begin{aligned} \phi_1(x) - y &= 0 \\ \phi_2(x) - z &= 0 \end{aligned}\right| \frac{d}{dt}$$

$$\phi_1' \cdot x' - y' = \phi_1' \cdot (-\phi_2^{\ell}) - (-\phi_1 + x^2) = 0$$

$$\phi_2' \cdot x' - z' = \phi_2' (-\phi_2^{\ell}) - (-2\phi_2 - x^2) = 0.$$

$$\Pi_1(\underset{\sim}{\phi}) = -\phi_1' \phi_2^{\ell} + \phi_1 - x^2 = 0$$

$$\Pi_2(\underset{\sim}{\phi}) = -\phi_2' \phi_2^{\ell} + 2\phi_2 + x^2 = 0$$

(i) $\phi_1 = x^2, \phi_2 = -\frac{1}{2}x^2 : \Pi = O(x^{1+2\ell}) = O(x^5)$

$$\Rightarrow \quad x' = -\left(-\frac{1}{2}x^2 + O(x^5)\right)^{\ell}$$

$$x' = -\left(-\frac{1}{2}\right)^{\ell} x^{2\ell} + O(x^{3\ell})$$

$\vdots$

vždy nestabilní

3 př.:

$$x' = x(y - z)$$
$$y' = -2y + z + z^2 - x^2$$
$$z' = y - 3z + xyz$$

$A = (0)$, $m = 1$

$$B = \begin{pmatrix} -2, & 1 \\ 1, & -3 \end{pmatrix}.$$

$y = \phi_1(x)$: $n_1 : y' - \phi_1' x'$

$z = \phi_2(x)$: $n_2 : z' - \phi_2' x'$

$$n_1 = \underbrace{-2\phi_1 + \phi_2 + \phi_2^2 - x^2} - \phi_1' x(\phi_1 - \phi_2)$$
$$n_2 = \underbrace{\phi_1 - 3\phi_2} + x\phi_1\phi_2 - \phi_2' x(\phi_1 - \phi_2).$$

? kvadratický: $\phi_1 = ax^2$, $\phi_2 = bx^2$

$$-2a + b - 1 = 0 \quad + O(x^4)$$
$$a - 3b \quad = 0 \quad \text{—''—}$$

$\Rightarrow b = -1/5$, $a = -3/5$

$$x' = x(\underbrace{ax^2 - bx^2}_{-\frac{2}{5}x^2} + O(x^4)) \quad \Rightarrow \text{as. stab.}$$