Exercises on control theory.

1) Show that the problem

$$x' = \cos u, \quad y' = \sin u \tag{1}$$

$$x(0) = y(0) = 0 \tag{2}$$

$$P[u(\cdot)] = \max\{\sqrt{x^2(t) + y^2(t)},\ t \in [0,T]\} \tag{3}$$

does not attain its minimum over admissible controls in $L^\infty(0,T)$.

Interpret geometrically!

2) (General parking problem.) Think of some type of vehicle (car, boat, spacecraft, ...). Describe its motion with suitable state variables X (i.e. position, speed, rudder tilt, wheel angle, ...) and control variables U (engine thrust, breaking force, ...).

The system should be nonlinear, and multidimensional.

Discuss the local (or global) controllability of the system; in particular, can the vehicle be parked (to zero position, zero speed) at a given time $t > 0$?

3) (Weekend house problem.) Consider the problem

$$x' = -kx + u, \qquad x(0) = x_0 \tag{4}$$

$$P[u(\cdot)] = \log x(T) - \int_0^T cu(t)\,dt \tag{5}$$

where x_0, k and $c > 0$ are given. Identify the maxima of P over measurable controls $u(t) : [0,T] \to [0,M]$.

4) (Weekend house problem 2.) Same problem as above, but with measurable controls in $u(t) : [0,T] \to [0,\infty)$.

$$P[u(\cdot)] = \beta x(T) - \int_0^T u(t) + \alpha u^2(t)\,dt$$

5) Let $P[u(\cdot)] = \int_0^T \phi(u(t))\,dt$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function. Assume that ϕ is C^1 and ϕ' is bounded. (i) Prove that if $u_n \overset{*}{\rightharpoonup} u_*$ in $L^\infty(0,T)$ and $P[u_n(\cdot)] \to P_*$, then $P[u_*(\cdot)] \le P_*$.

(ii) Show that the inequality might be strict.

See also the hints / comments on the last page.

1) The infimum is 0: just consider piecewise constant u. However, $P = 0$ implies $x(t) = y(t) = 0$ for all t, hence $x'(t) = y'(t) = 0$, a contradiction.

3) Adjoint equation $p' = kp$, $p(T) = 1/x(T)$; at most one change from $u = 0$ to $u = M$.

4) Adjoint equation $p' = kp$, $p(T) = \beta$ can be solved explicitly; control $u(t)$ can be expressed in terms of $p(t)$ (minimization of a quadratic function – however, beware of the condition $u \geq 0$.)

5) (i) By convexity $\phi(v) \geq \phi(u) + \phi'(v)(u - v)$ for all u, $v \in \mathbb{R}$. Set $v = u_n(t)$, $u = u_*(t)$, integrate $\int_0^T dt$ and ...

   (ii) $u_n(t) = \cos(nt) \stackrel{*}{\rightharpoonup} 0$ (by Riemann-Lebesgue), but $(u_n(t))^2 \stackrel{*}{\rightharpoonup} 1/2$ by the formula $\cos^2 y = (1 + \cos 2y)/2$