Ordinary Differential Equations

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Problem Set 9

9.1. Let I be an interval, q be a continuous function defined in this interval and let y be a non-trivial solution of $y'' + q(t)y = 0$ in I. Consider a number $a > 0$. Show that

(a) if $q(t) \geq a$ in I then the adjacent zero points of y in I are placed at most $\pi/\sqrt{a}$ apart.

(b) if $q(t) \leq a$ in I then the adjacent zero points of y in I are placed at least $\pi/\sqrt{a}$ apart.

9.2. Verify that every solution of $y'' + \dfrac{y}{\sqrt{t}} = 0$ has infinitely many zero points in $(0, \infty)$.

9.3. Let $y'' + p(t)y' + q(t)y = 0$ with continuous p, p' and q. Find a function $u(t)$ such that the substitution $z(t) = u(t)y(t)$ transforms the equation into $z'' + r(t)z = 0$. Express $r(t)$ by means of $p(t)$ and $q(t)$.

9.4. Prove that any non-trivial solution of

(a) $y'' + \sin(t)\, y = 0$ has at most 2 zero points in $[-\pi, \pi]$.

(b) $y'' + 2ty' + 4ty = 0$ has at most 4 zero points in $\mathbb{R}$.

Farewell Gift

Putnam 2/6 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Putnam 2/6 Functions f, g, h are differentiable on some open interval around 0 and satisfy

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad & f(0) &= 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad & g(0) &= 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad & h(0) &= 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Putnam 5/6 Show that there is no strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x.

Putnam 5/6 Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x \to \infty} f(x) = \infty$.