Dalibor Pražák, Josef Žabenský
18. prosince 2017

Problem Set 8

8.1. Let $x : [0, \infty) \to \mathbb{R}$ be twice differentiable such that $x(0) = \frac{1}{2}$, $x'(0) = 1$ and

$$x''(t) - 3x'(t) + 2x(t) \geq 1 \qquad \text{for all } t \geq 0.$$

Prove that then

$$x(t) \geq e^{2t} - e^t + \frac{1}{2} \qquad \text{for all } t \geq 0.$$

8.2. Investigate stability of the origin:

(a) $x' = -2y^3$, $y' = x$
(b) $x' = -x^3 - 2y$, $y' = x - y^3$
(c) $x' = -x - y^2$, $y' = xy - x^2 y$

Hint: For (b) and (c) try $V = ax^2 + by^2$ for appropriate a, b.

8.3. Let $V(x, y) = x^2(x-1)^2 + y^2$. Consider the system

$$\begin{aligned} x' &= -\frac{\partial V}{\partial x} \\ y' &= -\frac{\partial V}{\partial y}. \end{aligned}$$

(a) (voluntary, but practice makes perfect!) Find the critical points of this system and determine their linear stability.
(b) Show that V decreases along every non-stationary solution of the system.
(c) Use the previous point to show that if $z_0 = (x_0, y_0)$ is a global minimum of V then z_0 is a locally asymptotically stable equilibrium.

8.4. Have a continuous function g such that $ug(u) > 0$ if $u \neq 0$. Decide if the origin is a stable equilibrium for

(a) $u'' + g(u') + u = 0$
(b) $u'' + g(u) = 0$

Hint for (b): Find the first integral.

8.5. **Christmas cookie:** Let $f : [0, \infty) \to \mathbb{R}$ be bounded, differentiable and satisfying $f^2(x) f'(x) \geq \sin(x)$ for all $x \geq 0$. Show that $\lim_{x \to \infty} f(x)$ does not exist.

Problem set 8 solutions

8.1

Let $f := x'' - 3x' + 2x \Rightarrow f \geq 1$

$\& \; x'' - 3x' + 2x = f$

<u>variation of constants</u>: $x(t) = x_h(t) + \int_0^t u(t-s) f(s)\, ds, \quad t \geq 0$

where $\begin{cases} x_h'' - 3x_h' + 2x_h = 0 \\ x_h(0) = \frac{1}{2}, \; x_h'(0) = 1 \end{cases}$

$\begin{cases} u'' - 3u' + 2u = 0 \\ u(0) = 0, \; u'(0) = 1 \end{cases}$

$\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) \Rightarrow$ fund system $= \{e^{2t}, e^t\}$

$\Rightarrow x_h = \frac{1}{2} e^{2t}$

$u = e^{2t} - e^t \quad (\geq 0 \quad \forall t \geq 0)$

$$\Rightarrow x(t) = \frac{1}{2} e^{2t} + \int_0^t \underbrace{\underbrace{\left(e^{2(t-s)} - e^{t-s}\right)}_{\geq 0} \underbrace{f(s)}_{\geq 1}}_{\geq \left(e^{2(t-s)} - e^{t-s}\right)} ds$$

$$\geq \frac{1}{2} e^{2t} + e^{2t} \int_0^t e^{-2s}\, ds - e^t \int_0^t e^{-s}\, ds$$

$$= e^{2t} - e^t + \frac{1}{2}$$

8.2 a) $x' = -2y^3$

$y' = x$

$\Rightarrow 2x'x + 4y'y^3 = 0$

$(x^2 + y^4)' = 0$

$\Rightarrow V(x,y) = x^2 + y^4$ is a positive definite 1. integral

$\Rightarrow \exists$ LF $\Rightarrow (0,0)$ is stable

b) $x' = -x^3 - 2y$

$y' = x - y^3$

Let $V(x,y) := x^2 + 2y^2 \Rightarrow$ continuous and pos. def

$\frac{d}{dt} V(x(t), y(t)) = 2xx' + 4yy' = 2x(-x^3 - 2y) + 4y(x - y^3)$

$= -2x^4 - 4y^4 < 0$ outside $(0,0)$

$\Rightarrow (0,0)$ as. stable

c) $x' = -x - y^2$

$y' = xy - x^2 y$

Let $V(x,y) := x^2 + y^2 \Rightarrow$ continuous and pos. def

$\frac{d}{dt} V(x(t), y(t)) = 2xx' + 2yy' = 2x(-x - y^2) + 2y(xy - x^2 y)$

$= -2x^2 - 2x^2 y^2 \leq 0$ (only)

$\Rightarrow (0,0)$ stable

8.3 $\quad V(x,y) = x^2(x-1)^2 + y^2$

$$x' = -2x(x-1)(2x-1)$$
$$y' = -2y$$

a) (critical point = stationary point)

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$Df = \begin{pmatrix} -12x^2+12x-2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$Df(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \Rightarrow \text{as. stable}$$

$$Df(\tfrac{1}{2},0) = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \quad \Rightarrow \text{unstable}$$

$$Df(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \Rightarrow \text{as. stable}$$

b) $$\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x} x' + \frac{\partial V}{\partial y} y' = -\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial y}\right)^2 < 0$$

$\left((x(t), y(t)) \text{ non-stationary} \Leftrightarrow \nabla V \neq 0 \right)$

c) z_0 is a global minimum of $V \Leftrightarrow V(z_0) = 0 \Leftrightarrow z_0 \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

& V is locally positive definite around these points

$\Rightarrow$ V is a Liapunov function strictly decreasing along non-zero solutions

8.4

a) $u'' + g(u') + u = 0$

$\Rightarrow u'' \cdot u' + \underbrace{g(u') \cdot u'}_{\geq 0} + u\,u' = 0$

$\Rightarrow \left(\frac{1}{2}(u')^2\right)' + \frac{1}{2}(u^2)' \leq 0$

$\Rightarrow V(u', u) = (u')^2 + u^2$ is a LF

$\Rightarrow$ 0 is a stable equilibrium

b) $u'' + g(u) = 0$

$\Rightarrow u'' \cdot u' + g(u) \cdot u' = 0$

$\Rightarrow \frac{1}{2}\left((u')^2\right)' + \left(\int_0^u g(s)\,ds\right)' = 0$

$\Rightarrow V(u', u) = \frac{1}{2}(u')^2 + \int_0^u g(s)\,ds$ is a 1. integral

& $\left(s \neq 0 \Rightarrow \operatorname{sgn} g(s) = \operatorname{sgn} s\right)$ (we know $s \cdot g(s) > 0$)

$\Rightarrow \left(u \neq 0 \Rightarrow \int_0^u g(s)\,ds > 0\right)$

$\Rightarrow$ V is pos. def. $\Rightarrow$ V is a LF