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Problem Set 8

8.1. Let $x : [0, \infty) \to \mathbb{R}$ be twice differentiable such that $x(0) = \frac{1}{2}$, $x'(0) = 1$ and

$$x''(t) - 3x'(t) + 2x(t) \geq 1 \qquad \text{for all } t \geq 0.$$

Prove that then

$$x(t) \geq e^{2t} - e^t + \frac{1}{2} \qquad \text{for all } t \geq 0.$$

8.2. Investigate stability of the origin:

(a) $x' = -2y^3$, $y' = x$
(b) $x' = -x^3 - 2y$, $y' = x - y^3$
(c) $x' = -x - y^2$, $y' = xy - x^2y$

Hint: For (b) and (c) try $V = ax^2 + by^2$ for appropriate a, b.

8.3. Let $V(x, y) = x^2(x - 1)^2 + y^2$. Consider the system

$$x' = -\frac{\partial V}{\partial x}$$
$$y' = -\frac{\partial V}{\partial y}.$$

(a) (voluntary, but practice makes perfect!) Find the critical points of this system and determine their linear stability.
(b) Show that V decreases along every non-stationary solution of the system.
(c) Use the previous point to show that if $z_0 = (x_0, y_0)$ is a global minimum of V then z_0 is a locally asymptotically stable equilibrium.

8.4. Have a continuous function g such that $ug(u) > 0$ if $u \neq 0$. Decide if the origin is a stable equilibrium for

(a) $u'' + g(u') + u = 0$
(b) $u'' + g(u) = 0$

Hint for (b): Find the first integral.

8.5. Christmas cookie: Let $f : [0, \infty) \to \mathbb{R}$ be bounded, differentiable and satisfying $f^2(x)f'(x) \geq \sin(x)$ for all $x \geq 0$. Show that $\lim_{x\to\infty} f(x)$ does not exist.