Problem Set 7

7.1. Solve the equations:

(a) $x' = x^2/y$, $y' = x$ $(x, y > 0)$
(b) $x' = y^2$, $y' = yz$, $z' = -z^2$ $(y, z > 0)$

7.2. Find a first integral for $x' = x + y$, $y' = x^2 - y^2$.

Hint: Begin with computing $y = y(x)$.

7.3. Consider the equation $x'' + \sin x = 0$.

(a) Find a nontrivial function $V(x, x')$ that is constant along each solution.
(b) Write the equation as a system of 2 first order equations and draw the phase portrait for this system.
(c) Give a formula for the period of periodic solutions as a function of a given amplitude a.

7.4. Let $(x(t), y(t))$ solve a 2-dimensional autonomous system

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)), \end{aligned}$$

in the neighbourhood of t_0 for continuous f and g. Let $f(x(t_0), y(t_0)) \neq 0$ and denote $\tau(x)$ the inverse of $x(t)$ in the neighbourhood of $x(t_0)$. Then function $\hat{y}(x) := y(\tau(x))$ solves

$$\frac{d\hat{y}}{dx}(x) = \frac{g(x, \hat{y}(x))}{f(x, \hat{y}(x))},$$

in the neighbourhood of $x(t_0)$.

7.5. Derive Taylor's formula with the integral form of the remainder

$$x(t) = \sum_{j=0}^{n} \frac{x^{(j)}(t_0)}{j!}(t - t_0)^j + \frac{1}{n!}\int_{t_0}^{t} x^{(n+1)}(s)(t - s)^n \, ds$$

for $x \in C^{n+1}$ from the variation of constants for an autonomous equation of n-th order.

Hint: $x^{(n+1)} = x^{(n+1)}$.

7.6. (*easy and completely optional*) Consider a *companion matrix*

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{n-1} \end{pmatrix}.$$

(a) Show that $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ is the characteristic polynomial of A.
(b) Show that the geometric multiplicity of every eigenvalue of A is one. What does it tell you about its Jordan canonical form? (The geometric multiplicity is dimension of the corresponding eigenspace. It is always smaller than or equal to the algebraic multiplicity.)

7.7. Food for thought: Let x be a C^2 function and $x''(t) + x'(t) + x(t) \to 0$ for $t \to \infty$. Show that then $x(t) \to 0$ for $t \to \infty$.

Hint: Use the foxy hint from 7.5 for inspiration.