Ordinary Differential Equations

ZS 17/18

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Dalibor Pražák, Josef Žabenský

Problem Set 4

4.1. Solve the systems corresponding to the following matrices:

(a) $A = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}$

(b) $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

4.2. Solve

$$\begin{aligned} x' &= -y - t, \qquad & x(0) &= 1, \\ y' &= x + t, \qquad & y(0) &= 0. \end{aligned}$$

Calculating the integral is completely voluntary.

4.3. Find a 2×2 matrix such that $(x(t), y(t)) = (\sinh(t), e^t)$ is a solution.

4.4. Which of the following functions $(x(t), y(t))$

(a) $(3e^t + e^{-t}, e^{2t})$

(b) $(3e^t + e^{-t}, e^t)$

(c) $(3e^t + e^{-t}, te^t)$

(d) $(3e^t, t^2e^t)$

(e) $(e^t + 2e^{-t}, e^t + 2e^{-t})$

can be solutions of a first-order autonomous homogeneous system?

4.5. Function $u \colon \mathbb{R} \to \mathbb{R}$ fulfills $u(0) = 0$, $u'(0) = 1$ and $u''(t) \geq -u(t)$ for all $t \in [0, \pi]$. Show that $u(t) \geq \sin(t)$ for all $t \in [0, \pi]$.

Hint: Rewrite the given inequality as $u''(t) + u(t) = f(t)$ with $f \geq 0$ and solve this ODE by means of variation of constants. Recall that you have to work with a first-order ODE in the first place.

4.6. Is there a real matrix A such that

$$\exp(A) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad \alpha, \beta > 0?$$

What if α and β are distinct?

Hint: If desperate, recall $\sigma(\exp(A)) = \exp(\sigma(A))$. If still desperate, try $A = \begin{pmatrix} a & \pi \\ -\pi & a \end{pmatrix}$.

4.7. By means of the matrix exponential show $A^2 = -I \Rightarrow \sigma(A) = \{\pm i\}$ for any complex square matrix A.

Hint: In other words $A^2 = -I \Rightarrow \sigma(\pi A) = \{\pm \pi i\}$.

4.8. Food for thought: There are 25 mechanical horses and a single racetrack. Each horse completes the track in a pre-programmed time, and the horses all have different finishing times, unknown to you. You can race 5 horses at a time. After a race is over, you get a list with the order the horses finished, but not the finishing times of the horses. What is the minimum number of races you need to identify the fastest 3 horses?