Ordinary Differential Equations

ZS 17/18

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Problem Set 3

3.1. Let $x(t) = \varphi(t, t_0, x_0)$ solve $x' = f(t, x)$ with the initial condition $x(t_0) = x_0$. Find $\frac{\partial}{\partial x_0}\varphi(t, t_0, x_0)$ for given t_0 and x_0 if

(a) $f = 2x + t^2x^2 - x^3$, $x(0) = 0$

(b) $f = \ln(1 - x) - x^2 - t^2x^2$, $x(0) = 0$

(c) $f = t(1 - x^2)$, $x(t_0) = 1$, $t_0 \in \mathbb{R}$

3.2. Let $x(t) = \varphi(t, t_0, x_0)$ solve $x' = f(t, x)$ with the initial condition $x(t_0) = x_0 \in \mathbb{R}$ and $f \in C^2(\mathbb{R}^2)$. Derive the equation for $\frac{\partial^2}{\partial x_0^2}\varphi(t, t_0, x_0)$ and compute $\frac{\partial^2}{\partial x_0^2}\varphi(t, 0, 1/2)$ for the equation presented by me, i.e. $f = e^{2x} - e$. You can consider all information I have deduced as already known. Expand then $\varphi(t, 0, h + 1/2)$ for small h up to the second-order term.

3.3. Let $x(t) = \varphi(t, t_0, x_0, \lambda)$ be the solution to $x' = f(t, x, \lambda)$, $x(t_0) = x_0 \in \mathbb{R}$ with λ being a real parameter. Consider a beautifully smooth $f \in C^1(\mathbb{R}^3)$. Find the equation for $\frac{\partial\varphi}{\partial\lambda}$ and apply your discovery to compute $\frac{\partial\varphi}{\partial\lambda}(t, 1, 0, 1)$ for $x' = \lambda\cos(\lambda\pi)x + t\lambda$.

3.4. Let I be an open interval, $f = f(t, x, x') \in C^1(I \times \mathbb{R}^2)$ and $x(t) = \varphi(t, t_0, x_0^1, x_0^2)$ solve

$$\begin{aligned} x'' &= f(t, x, x'), \\ x(t_0) &= x_0^1, \\ x'(t_0) &= x_0^2. \end{aligned}$$

Derive equations for $u(t) = \frac{\partial}{\partial x_0^1}\varphi(t, t_0, x_0^1, x_0^2)$ and $v(t) = \frac{\partial}{\partial x_0^2}\varphi(t, t_0, x_0^1, x_0^2)$. Finally, apply the result to $f = 4x' + 21x - 3$, $t_0 = 0$ and compute the respective partial derivatives.

3.5. Food for thought: Bobek the rabbit is hiding in one of five hats that are lined up in a row. The hats are numbered 1 to 5. Each night Bobek hops into an adjacent hat, exactly one number away. Each morning you can peek into a single hat to test whether Bobek is inside. Can you think up a strategy to find him?