ZS 17/18
16. října 2017

Problem Set 2

2.1. Perform qualitative analysis to the following equations (especially with respect to uniqueness and possible blow-ups)

(a) $x' = e^x - 1$,

(b) $x' = \sqrt[3]{1 - x^2}$.

Note: Let $\sqrt[3]{-1} = -1$. Describe here in addition all solutions satisfying $x(0) = 1$.

2.2. Consider a *predator-prey system*

$$x' = x(1 - x) - xy,$$
$$y' = -2y + xy,$$

where $x(t)$ represents the number of animals at time t being hunted by a population of $y(t)$ predators

(a) Show that the solutions cannot leave the first quadrant.

(b) Find stationary points and sketch the course of solutions in the first quadrant, using elementary arguments.

(c) What happens for $t \to \pm\infty$?

2.3. Prove the differential forms of the Gronwall lemma (without resorting to the integral one; that is why you have (a) and (b) here – to guide you towards (c)): Let $-\infty < a < b < \infty$, $u \in C^1([a, b])$ and assume

$$u'(t) \leq \beta(t)u(t) + \alpha(t) \quad \text{for every } t \in [a, b]$$

with functions $\alpha, \beta : [a, b] \to \mathbb{R}$ specified below.

(a) If $\alpha \equiv 0$ and $\beta \in \mathbb{R}$ then

$$u(t) \leq u(a)e^{(t-a)\beta} \quad \text{for every } t \in [a, b].$$

Hint: Rewrite the starting inequality in the form $F'(t) \leq 0$ for some F.

(b) If $\alpha \equiv 0$ and $\beta \in C([a, b])$ then

$$u(t) \leq u(a)e^{\int_a^t \beta(s)\,ds} \quad \text{for every } t \in [a, b].$$

(c) If $\alpha, \beta \in C([a, b])$ then

$$u(t) \leq \left(u(a) + \int_a^t \alpha(s)e^{-\int_a^s \beta(r)\,dr}\,ds\right)e^{\int_a^t \beta(s)\,ds} \quad \text{for every } t \in [a, b].$$

Usually the assumptions read only $\alpha, \beta \in L^1(a, b)$ but then one requires in addition that β be non-negative a.e. in (a, b). Why?

2.4. Food for thought: Kurděj has a garden with 100 poisonous flowers that he wants to eradicate. He can destroy exactly 3, 5, 14, or 17 at a time, but if at least one flower survives, then the flowers also grow back based on how many were destroyed (3 die $\to$ 12 grow back, 5 die $\to$ 17 grow back, 14 die $\to$ 8 grow back, 17 die $\to$ 2 grow back). If the number of flowers is exactly 0, then the flowers never grow back. Can Kurděj ever get rid of all the flowers in his garden?