Věta 11.4. Nechť $F(z)=\sum_{k=0}^{\infty} c_k (z-z_0)^k$ má poloměr konvergence $R>0$. Označme $f(z)=\sum_{k=1}^{\infty} k\, c_k (z-z_0)^{k}$. Potom $F'(z)=f(z)$

pro $\forall z\in\mathbb{C}$, $|z-z_0|<R$.

Dk. BÚNO $z_0=0$, $|z|<R$ pevné

cíl: $\varphi(h)\to 0$, pro $h\to 0$

kde $\varphi(h)=\dfrac{F(z+h)-F(z)}{h}-f(z)$

$$F(z+h)= c_0+c_1(z+h)+\sum_{k=2}^{\infty} c_k (z+h)^k$$

$$F(z)= c_0+c_1 z+ \quad \dots \quad z^k$$

$$f(z)= \qquad c_1 + \sum_{k=2}^{\infty} k\, c_k z^{k-1}$$

$$\Rightarrow \varphi(h)=\sum_{k=2}^{\infty} c_k\left[\frac{(z+h)^k-z^k}{h}-k z^{k-1}\right]$$

Binomická věta:

$$(R+h)^r = R^r + rR^{r-1}h + \sum_{j=2}^{r}\binom{r}{j}R^{r-j}h^j$$

CELKEM:

$$\varphi(h) = \sum_{r=2}^{\infty} c_r \left[\sum_{j=2}^{r}\binom{r}{j}R^{r-j}h^{j-1}\right]$$

$$= h\sum_{r=2}^{\infty} c_r \sum_{j=2}^{r}\binom{r}{j}R^{r-j}h^{j-2}$$

$$= h\sum_{r=2}^{\infty} c_r \underbrace{\sum_{l=0}^{r-2}\binom{r}{l+2}R^{(r-2)-l}h^{l}}_{S_r}$$

$$\binom{r}{l+2} = \frac{r!}{(l+2)!(r-l-2)!} = \frac{r(r-1)(r-2)!}{l(l+1)l!((r-2)-l)!}$$

$$\leq r(r-1)\frac{(r-2)!}{l!((r-2)-l)!} = r(r-1)\binom{r-2}{l}$$

ODSUD: *) viz dole

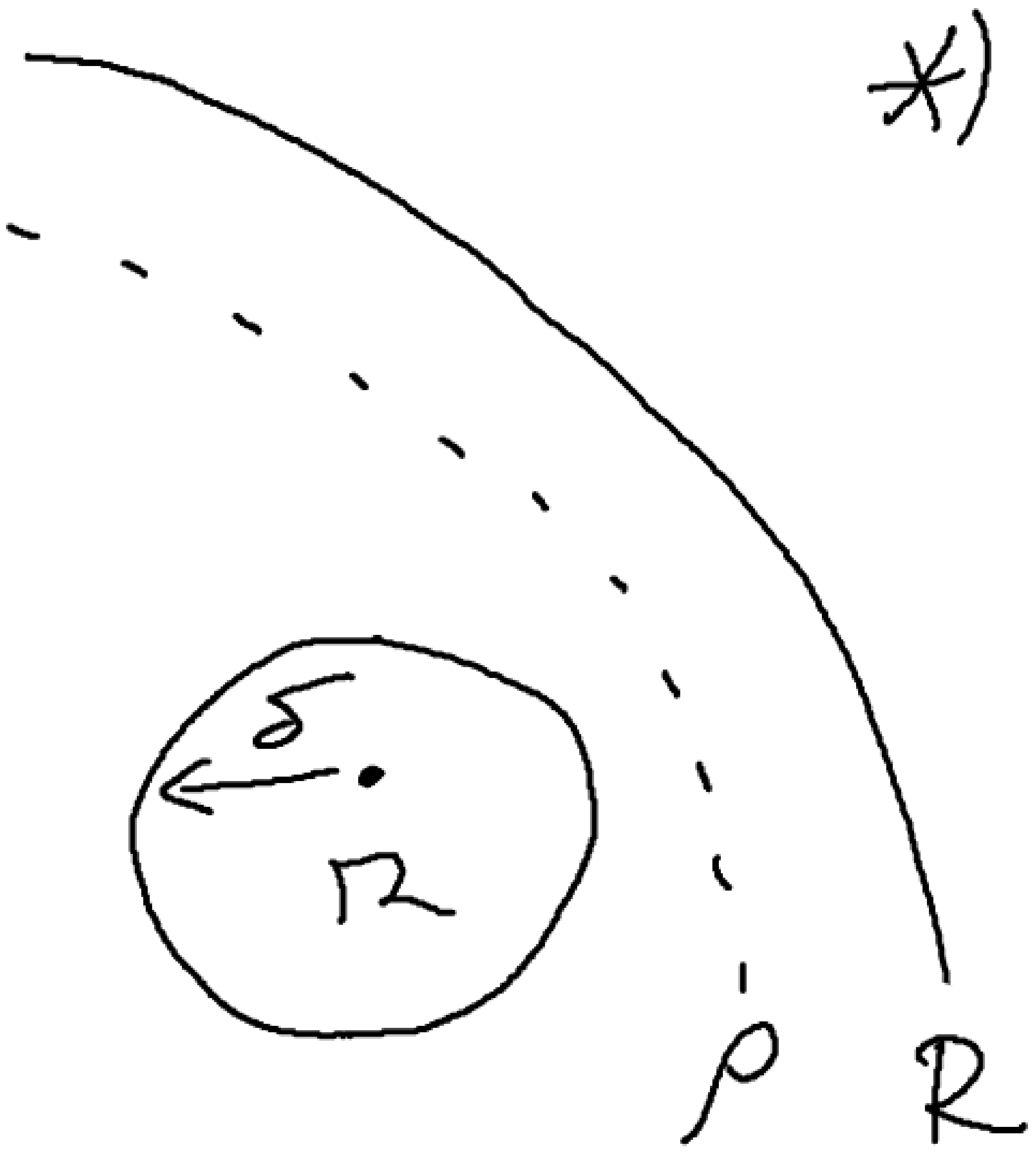
$$|S_k| \le k(k-1)\sum_{l=0}^{k-2}\binom{k-2}{l}|z|^{(k-2)-l}\delta^l$$

$$= k(k-1)(|z|+\delta)^{k-2} = k(k-1)\rho^{k-2}$$

a tedy: $|\varphi(h)| \le |h| \cdot K$

kde $$K = \sum_{k=2}^{\infty} k(k-1)|a_k|\rho^{k-2} < +\infty$$

(viz Lemma 11.1, řada pro $F''(z)$)



*) BÚNO: $|h| < \delta$, kde $\delta > 0$ je t.ž. $|z| + \delta = \rho < R$

<u>Věta 11.5</u> Nechť $F(x)=\sum\limits_{k=0}^{\infty} c_k (x-x_0)^k$ má poloměr konvergence $R>0$. Potom

$$F^{(k)}(x_0)=k!\,c_k\,, \quad \text{pro } \forall k\geq 0.$$

<u>Dk.:</u> $$F(x)=\sum_{k=0}^{\infty} c_k\, k!\, Q_{k,x_0}(x)\,,$$

$$\text{kde } Q_{k,x_0}(x)=\frac{(x-x_0)^k}{k!}$$

Věta 11.4 (užiji k-krát)

$$\Rightarrow F^{(l)}(x)=\sum_{k=0}^{\infty} c_k\, k!\, Q^{(l)}_{k,x_0}(x)$$

$$\text{pro } \forall\, |x-x_0|<R$$

$$F^{(l)}(x_0)=\sum_{k=0}^{\infty} c_k\, k!\, \underbrace{Q^{(l)}_{k,x_0}(x_0)}_{1,\ k=l \qquad 0,\ k\neq l} = c_l\cdot l!$$

(Lemma 8.1)

Věta 11.6 Nechť $F(z)=\sum_{k=0}^{\infty} c_k (z-z_0)^k$ a $\tilde{F}(z)=\sum_{k=0}^{\infty} \tilde{c}_k (z-z_0)^k$ mají poloměry konvergence $R, \tilde{R}>0$. Nechť $F(z)=\tilde{F}(z)$ na jistém $U(z_0,\delta)$.

Pak $c_k=\tilde{c}_k$ pro $\forall k\geq 0$.

Dk. $F\equiv\tilde{F}$ na $U(z_0,\delta)$

$$\Rightarrow F^{(l)}\equiv\tilde{F}^{(l)} \text{ na } U(z_0,\delta)$$

speciálně: $\underbrace{F^{(l)}(z_0)}_{l!\,c_l}=\underbrace{\tilde{F}^{(l)}(z_0)}_{l!\,\tilde{c}_l}$

V. 11.5 $\Rightarrow$

Lemma 11.3 Pro $\forall x,y\in\mathbb{C}$ platí:

$$\exp(x+iy)=e^x(\cos y+i\sin y)$$

$$\sin x=\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)$$

$$\cos x=\frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

__Dk.__ __1.__ $e^{iy} = \cos y + i \sin y$

pro $l \geq 0$ platí: $(i)^{2l} = (i^2)^l = (-1)^l$

$$(i)^{2l+1} = \cdots (-1)^l i$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{1}{k!} (iy)^k$$

$$= \sum_{l=0}^{\infty} \frac{1}{(2l)!} (iy)^{2l} + \sum_{l=0}^{\infty} \frac{1}{(2l+1)!} (iy)^{2l+1}$$

$$= \underbrace{\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} y^{2l}}_{\cos y} + i \underbrace{\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!} y^{2l+1}}_{\sin y}$$

__2.__ $e^{x+iy} = e^x \cdot e^{iy}$

__3.__ $e^{ix} + e^{-ix} = 2 \cos x$

$$e^{ix} - e^{-ix} = 2i \sin x$$