

Účlo 8.3 (Ožerece o malým o.)

(1) $f(x) = o(x^n)$, $g(x) = x^m$, $x \rightarrow 0$; kde $m \geq n$.

$$\Rightarrow a f(x) + b g(x) = o(x^n), x \rightarrow 0$$

(2) $f(x) = o(x^n)$, $x \rightarrow x_0$

$$\Rightarrow a x^m f(x) = o(x^{n+m}), x \rightarrow 0$$

(3) $f(x) = o(x^n)$, $g(x) = o(x^m)$, $x \rightarrow 0$

$$\Rightarrow f(x)g(x) = o(x^{n+m}), x \rightarrow 0$$

(4) $f(x) = o(x^n)$, $g(x) \sim x^m$, $x \rightarrow 0$, $m \geq 1$

$$\Rightarrow f(g(x)) = o(x^{nm}), x \rightarrow 0$$

2Q. (1) cíl: $\frac{a f(x) + b g(x)}{x^n} \rightarrow 0, x \rightarrow 0$

... uprave: $a \underbrace{\frac{f(x)}{x^n}}_{\rightarrow 0} + b \underbrace{x^{m-n}}_{(*)} \cdot \underbrace{\frac{g(x)}{x^m}}_{\rightarrow 0}$

(*) : omezení pro $x \rightarrow 0$, totiž $V_{0A}L$
(neboť $m-n \geq 0$) viz. Věta 2.4

$$(2) \text{ cil: } \underbrace{\frac{a x^m f(x)}{x^{m+n}}}_{\text{...}} \rightarrow 0, x \rightarrow 0$$

$$\dots \text{ squeeze: } a \frac{f(x)}{x^n} \rightarrow a \cdot 0$$

$$(3) \frac{f(x) \cdot g(x)}{x^{m+n}} = \frac{f(x)}{x^m} \cdot \frac{g(x)}{x^n} \rightarrow 0 \cdot 0$$

$$(4) \text{ nme: } \frac{f(y)}{y^n} \rightarrow 0, y \rightarrow 0$$

$$\frac{g(x)}{x^m} \rightarrow C, x \rightarrow 0 \text{ where } C \in \mathbb{R} \neq 0$$

$$\text{cil: } \frac{f(g(x))}{x^{mn}} \rightarrow 0, x \rightarrow 0$$

$$\text{squeeze: } \frac{f(g(x))}{(g(x))^n} \cdot \left(\frac{g(x)}{x^m} \right)^n = P_1 \cdot P_2$$

$$P_2 \rightarrow C^m \dots \text{dle VolSF};$$

$$\text{m\u011bj\u0161\u0165: } h(R) = R^m \text{ mo\u017ei\u017ee' } \forall C \in \mathbb{R}$$

$$\text{m\u011bj\u0161\u0165: } g(x) \underset{x^m}{\rightarrow} C$$

$$P_1 \rightarrow 0 \dots \text{op\u011bt dle VolSF}$$

$$\text{m\u011bj\u0161\u0165 fce: } g(x) = \frac{g(x)}{x^m} \cdot x^m \rightarrow 0 \cdot 0$$

(VolA2)

$$\text{m\u011bj\u0161\u0165: } \frac{g(x)}{x^m} \neq 0 \text{ (a tedy } g(x) \neq 0)$$

$$\text{na jist\u011bm } P(0, \delta) \Leftarrow \text{Lemme 2.7}$$

$$\text{m\u011bj\u0161\u0165 fce: } \frac{f(y)}{y^m} \rightarrow 0, y \rightarrow 0.$$

$$\underline{P_2.} \text{ (7)} \quad \frac{x^3}{\sin x \cdot \ln(1+x^2)} \rightarrow 1, x \rightarrow 0$$

$$\dots \text{m\u011bj\u0161\u0165: } \sin x = x + o(x)$$

$$\ln(1+y) = y + o(y)$$

$$\Rightarrow \ln(1+x^2) = x^2 + o(x^2), x \rightarrow 0$$

$$\sin x \cdot \ln(1+x^2) = (x + o(x)) (x^2 + o(x^2)) \\ = x^3 + \underbrace{x o(x^2) + x^2 o(x) + o(x) o(x^2)}_{o(x^3)}, \quad x \rightarrow 0$$

$$f(x) = \frac{x^3}{x^3 + o(x^3)} = \frac{1}{1 + \underbrace{\frac{o(x^3)}{x^3}}_{\rightarrow 0}} \rightarrow \frac{1}{1+0} \quad (\text{VoAL})$$

$$\textcircled{2} \quad \lim_{x \rightarrow +\infty} x \left(\sqrt{1+x^2} - \sqrt[3]{1+x^3} + 2\sqrt[4]{1+x^4} \right) = ?$$

$$= \lim_{y \rightarrow 0+} \frac{1}{y^2} \left(\sqrt{1+y^2} - \sqrt[3]{1+y^3} + 2\sqrt[4]{1+y^4} \right)$$

$$\text{r'me: } (1+x)^a = 1 + ax + o(x), \quad x \rightarrow 0(\pm)$$

$$\Rightarrow \sqrt{1+y^2} = 1 + \frac{1}{2}y^2 + o(y^2)$$

$$\sqrt[3]{1+y^3} = 1 + \frac{1}{3}y^3 + o(y^3)$$

$$\sqrt[4]{1+y^4} = 1 + \frac{1}{4}y^4 + o(y^4)$$

$$\Rightarrow g(y) = \frac{1}{y^2} \left(1 + \frac{1}{2}y^2 + o(y^2) - 3 \left(1 + \frac{1}{3}y^3 + o(y^3) \right) + 2(1 + o(y^2)) \right) \\ = \frac{y^2/2 + o(y^2)}{y^2} \rightarrow \frac{1}{2}, \quad y \rightarrow 0.$$

Def. Pro $f \in C^n(U(x_0), \delta)$ definiuji:

$$R_{n+1, x_0}^f(x) = f(x) - T_{n, x_0}^f(x), \quad x \in U(x_0)$$

Rez. Taylorův zbytek po n -tém členu.

Veře 8.4. Necht $f \in C^{n+1}(U(x_0, \delta))$, necht

$x \in P(x_0, \delta)$ je pevné. Pak $\exists \theta$ mezi x, x_0

1.2.
$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!} (x-x_0)^{n+1}$$

rez. $\exists \theta$ mezi x, x_0 1.2

$$R_{n+1}(x) = \frac{f^{(n+1)}(\theta)}{n!} (x-\theta)^n (x-x_0).$$

2.2. TRIK: pomocné fce $\varphi(t), t \in [x_0, x]$

$$\text{kde } \varphi(t) = f(t) - T_{n, t}^f(x)$$

$$= f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!} (x-t)^k$$

vidíme: $\varphi(x_0) = f(x_0) - T_{n, x_0}^f(x) = R_{n+1}(x)$

$$\varphi(x) = f(x) - T_{n, x}^f(x) = 0$$

pomocný výraz : $\varphi'(t) = \frac{d}{dt} \varphi(t)$

$$= - \left\{ f(t) + \sum_{k=1}^n \frac{1}{k!} f^{(k)}(t) \cdot (x-t)^k \right\}'$$

$$= - f'(t) - \sum_{k=1}^n \frac{1}{k!} \left(f^{(k+1)}(t) \cdot (x-t)^k - f^{(k)}(t) \cdot k(x-t)^{k-1} \right)$$

$$= \underbrace{- f'(t) - \sum_{k=1}^n \frac{f^{(k+1)}(t)}{k!} (x-t)^k}_{\text{...}} + \sum_{k=1}^n \frac{f^{(k)}(t)}{(k-1)!} (x-t)^{k-1}$$

$$= - \sum_{l=1}^{n+1} \frac{f^{(l)}(t)}{(l-1)!} (x-t)^{l-1}$$

$$= - \frac{f^{(n+1)}(t)}{(n+1)!} (x-t)^n$$

.....

Věta 6.8 (Cauchy) $\Rightarrow$ $\boxed{\frac{\varphi(x_0) - \varphi(x)}{\varphi(x_0) - \varphi(x)} = \frac{\varphi'(\theta)}{\varphi'(\theta)}}$

ukládková: (L): $\varphi(t) = (x-t)^{n+1}$

resp. (C) $\varphi(t) = t, \quad t \in [x_0, x].$

$$\underline{ad(L)}: LS = \frac{R_{m+1}(x)}{(x-x_0)^{m+1}}, \quad \psi'(t) = -(m+1)(x-t)^m$$

$$PS = \frac{\frac{-f^{(m+1)}(\theta)}{m!} \cdot (x-\theta)^m}{-(m+1) \cdot (x-\theta)^m} = \frac{f^{(m+1)}(\theta)}{(m+1)!}$$

$$\underline{ad(C)}: LS = \frac{R_{m+1}(x)}{x_0 - x}, \quad \psi'(t) = 1$$

$$PS = -\frac{f^{(m+1)}(\theta)}{m!} (x-\theta)^m$$

Prüfe $f(x) = e^x$, $x_0 = 0$, $x \in \mathbb{R}$ beliebig:

$$R_{m+1}(x) = e^x - \sum_{k=0}^m \frac{x^k}{k!} = \frac{e^\theta}{(m+1)!} x^{m+1}, \quad \text{für } \theta \in [0, x]$$

$$\Rightarrow |R_{m+1}(x)| \leq C \cdot \frac{|x|^{m+1}}{(m+1)!}$$

$$\Rightarrow |R_{m+1}(x)| \rightarrow 0, \quad m \rightarrow \infty \quad (x \text{ beliebig})$$