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SHARP ESTIMATES OF THE DIMENSION OF INERTIAL MANIFOLDS FOR NONLINEAR PARABOLIC EQUATIONS

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ABSTRACT. Sufficient conditions are obtained for the existence of a k-dimensional invariant manifold that attracts as $t \to \infty$ all solutions $u(t)$ of the evolution equation $\dot{u} = -Au + F(u)$ in a Hilbert space, where A is a linear selfadjoint operator, semibounded from below, with compact resolvent, and F is a uniformly Lipschitz (in suitable norms) nonlinearity; these conditions sharpen previously known conditions and cannot be improved.

§1. INTRODUCTION

Semilinear parabolic equations (SPE) have recently attracted a lot of attention, primarily in connection with their role in the description of various selforganization phenomena in nature, often combined in the term "synergetics".

A very interesting effect which is important for applications is the *asymptotic finite-dimensionality* of a SPE, which in the study of steady-state regimes allows us to pass from, say, partial differential equations to ordinary differential equations (ODE) in $\mathbf{R}^k$. Thus, one can reduce (in some sense) an infinite-dimensional physical system to a system with a finite number of degrees of freedom and use known properties of ODEs in $\mathbf{R}^k$ for the qualitative analysis of the behavior of solutions of the original equation for large time.

In this paper for SPEs in a Hilbert space E we obtain constructive conditions that ensure the existence and (under some conditions) the uniqueness of a k-dimensional (i.e., homeomorphic to $\mathbf{R}^k$) invariant manifold $H \subset E$, which attracts as $t \to +\infty$ all the solutions of the equations; such manifolds are now customarily termed *inertial manifolds*. These conditions improve similar results (see [2]–[5], [7], [8] and also [11]–[15], [17], [21]) and are sharp in a certain sense. Here, in contrast to the majority of the above-mentioned papers, the property possessed by H to attract the phase space is established in a form that allows us to *really* reduce the description of stable limit regimes of a SPE to the analogous problem for its restriction to H, i.e., actually for some ODE in $\mathbf{R}^k$. In particular, the presence of a k-dimensional inertial manifold allows us to obtain results about the stability of stationary and periodic solutions of the SPE (especially for $k = 1$ and $k = 2$).

The construction of the inertial manifold used here is based on a nontrivial generalization of an approach originally developed for ODEs by Smith [1].

A natural class of objects on which the effect of asymptotic finite-dimensionality manifests itself is the class of nonlinear diffusion equations (NDE); a corresponding example with concrete conclusions about the stationary and periodic solutions of an

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equation is given in §4. The results of this paper turn out to be meaningful also when we deal with an ODE from the start; then we talk about the possibility of reducing a dynamical system in $\mathbf{R}^n$ to a dynamical system in $\mathbf{R}^k$ with $k < n$.

The main results of the paper (in a somewhat less final form) were announced in [6].

All the information about semilinear parabolic equations that are used below can be found in the monograph [11].

In §2 we give definitions and statements of results. The proof of theorems on the inertial manifold is given in §3. Finally, §4 applies these results to NDEs.

§2. Statements of results

We consider equations of the form

$$\dot{u} = -Au + F(u), \qquad u(0) = u_0, \tag{1}$$

in a real separable Hilbert space E with norm $\|\cdot\|$ and scalar product $(\cdot, \cdot)$. Here $u = u(t) \in E$, $\dot{u}(t)$ is the derivative with respect to $t \in \mathbf{R}$, and A and F are a linear and a nonlinear operator in E, respectively.

We assume that A is selfadjoint and semibounded from below, and that it has a compact resolvent and eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ (including multiplicities). If $\lambda_1 > 0$, then for $\vartheta > 0$ we can define the power A^ϑ with a dense domain of definition E_ϑ in E; here E_ϑ is a Hilbert space with scalar product $(u, u)_\vartheta = (A^\vartheta u, A^\vartheta u)$ and norm $\|u\|_\vartheta = \|A^\vartheta u\|$ ($E_0 = E$, $E_\alpha \supset E_\vartheta$ for $\alpha < \vartheta$). We assume that for some $\vartheta \in [0, 1)$ the nonlinearity $F\colon E_\vartheta \to E$ satisfies the following uniform Lipschitz condition:

$$\|F(u) - F(u')\| \leq L\|u - u'\|_\vartheta \tag{2}$$

for $u, u' \in E_\vartheta$. For $\vartheta = 0$ it is possible that $\lambda_1 \leq 0$.

Under these conditions solutions $u(t)$ of the SPE (1) exist for $t \geq 0$ for all $u_0 \in E_\vartheta$. Moreover, for $t > 0$ we have $u(t) \in E_1 \subset E_\vartheta$, and $\dot{u}(t) \in E_\alpha$ with an arbitrary $\alpha < 1$ is a *smoothing action* of the parabolic equation.

In the case $\dim E = n < \infty$ equation (1) is an ODE in $\mathbf{R}^n$.

In the phase space E_ϑ equation (1) generates a *completely continuous semiflow* $\{\Phi_t\}_{t\geq 0}\colon \Phi_t(u_0) = u(t)$, and the evolution operators Φ_t are completely continuous in E_ϑ for $t > 0$.

We distinguish the stationary $(u(t) \equiv u_0)$ and periodic $(u(t+p) = u(t))$ solutions of (1).

We assume further that *equation* (1) *has at least one stationary solution* $\bar{u} \in E_\vartheta$.

We say that a set $\mathfrak{M} \subset E_\vartheta$ is invariant if $\Phi_t\mathfrak{M} = \mathfrak{M}$ for $t \geq 0$, and that $\mathfrak{M}$ is positively invariant if $\Phi_t\mathfrak{M} \subset \mathfrak{M}$ for $t \geq 0$. A standard object in studying steady-state regimes of dynamical systems is *compact invariant* (CI) *sets* in phase space. We say that a CI set $\mathfrak{M} \subset E_\vartheta$ is *stable* if there exists an open set $\mathfrak{D}$ in E_ϑ, $\mathfrak{D} \supset \mathfrak{M}$, such that $\operatorname{dist}(\mathfrak{M}, u(t)) \to 0$ as $t \to +\infty$ for $u_0 \in \mathfrak{D}$.

It is known that CI sets of equation (1) have finite Hausdorff dimension. In case $F \subset C^1$ this follows from the (by now classical) result of J. Mallet-Paret on nonlinear completely continuous mappings of Hilbert space (see [16]). Recently various estimates of the dimension of CI sets have been obtained for a wide class of evolution equations (see the survey [18] and references therein).

We proceed to give precise statements.

In what follows for an estimate of solutions of equation (1), in addition to the phase space norm $\|\cdot\|_\vartheta$ we shall also use the norm $\|\cdot\|_\alpha$ with $\alpha = \vartheta/2$. We note that $\|u\|_\vartheta \geq \lambda_1^\alpha \|u\|_\alpha$ for $u \in E_\vartheta$.

Definition 1. For $\gamma \in \mathbf{R}$ we denote by $\mathscr{Z}(\gamma)$ a set in E_ϑ consisting of trajectories of solutions $u(t)$, existing for $t \in \mathbf{R}$ and such that

$$\|u(t)\|_\vartheta = O(e^{-\gamma t}) \quad \text{as } t \to -\infty.$$

We start from the orthogonal sum decomposition $E = X \oplus Y$ of E, where X and Y are subspaces of E and $X \neq \{0\}$, $Y \subset E_\vartheta$, $\dim Y = k$ $(1 \leq k < \infty)$. Such a decomposition is called an *orthopair* (X, Y), and we always denote by P and Q the orthogonal projections in E onto Y and X, $x = Qu$, $y = Pu$ for $u \in E$. In particular, we can take $Y = Y_k$, $X = X_k = Y_k^\perp$, where Y_k is the eigensubspace of the linear operator A corresponding to the part $\{\lambda_1, \dots, \lambda_k\}$ of its spectrum.

If $\dim E = \infty$, we restrict ourselves to the case $Y = Y_k$; if $\dim E < \infty$, we assume $\vartheta + 0$ (except for the counterexample in Lemma 3).

Suppose also that $\xi, \xi' > 0$ and that $\sigma \colon Y \to X \cap E_\vartheta$ is a mapping such that

$$\|\sigma(y) - \sigma(y')\|_\alpha \leq \xi\|y - y'\|_\alpha, \tag{3a}$$

$$\|\sigma(y) - \sigma(y')\|_\vartheta \leq \xi'\|y - y'\|_\vartheta \tag{3b}$$

for $y, y' \in Y$.

Definition 2. The set

$$H = \{u \in E_\vartheta : u = y + \sigma(y), y \in Y\} \tag{4}$$

is called a *k-dimensional inertial manifold* of equation (1) in the orthopair (X, Y), $\dim Y = k$, if it is invariant, attracts E_ϑ and consists of trajectories of solutions with bounded exponential growth as $t \to -\infty$:

(A) for $u_0 \in H$ there exist solutions $u(t)$ for $t \in \mathbf{R}$ and $u(t) \in H$;

(B) for every $u_0 \in E_\vartheta$ there is a $\bar{u}_0 \in H$ such that

$$\|u(t) - \bar{u}(t)\|_\vartheta \leq C\|u_0 - \bar{u}_0\|_\alpha e^{-\gamma t} \tag{5a}$$

for $t \geq 1$, and

$$\|u_0 - \bar{u}_0\|_\alpha \leq C_1\|x_0 - \sigma(y_0)\|_\alpha,$$

where the constants C, C_1, and $\gamma > 0$ do not depend on u_0 or $\bar{u}_0$;

(C) $H = \mathscr{Z}(\gamma')$ with $\gamma' < \gamma$.

Remark. Property (C) turns out to be important in establishing the uniqueness of an inertial manifold.

The inertial manifold H is homeomorphic to $\mathbf{R}^k$. The mapping $\psi(y) = y + \sigma(y) \colon Y \to E_\vartheta$ is called the *generating function* of H. We have $\psi(Y) = H$, $P(H) = Y$ and $u - \psi(y) = x - \sigma(y)$ for $u = x + y \in E$. It follows from the invariance of H and the smoothing action of the parabolic equation that $H \in E_1$.

A consequence of the definition of inertial manifold is the convergence of the solutions $u(t)$ to their natural projection $\psi(y(t))$ on H. Indeed, $u(t) - \psi(y(t)) = x(t) - \sigma(y(t)) = x(t) - \bar{x}(t) + \bar{x}(t) - \sigma(y(t))$. By (3b) we have $\|\bar{x}(t) - \sigma(y(t))\|_\vartheta \leq \xi'\|\bar{y}(t) - y(t)\|_\vartheta$, and from (5) we deduce (for $t \geq 1$):

$$\begin{aligned}\|u(t) - \psi(y(t))\|_\vartheta &\leq \|x(t) - \bar{x}(t)\|_\vartheta + \xi'\|y(t) - \bar{y}(t)\|_\vartheta \\ &\leq (1 + \xi')\|u(t) - \bar{u}(t)\|_\vartheta \leq C_2\|u_0 - \psi(y_0)\|_\alpha e^{-\gamma t}.\end{aligned}$$

Definition 3. Equation (1) is said to be *asymptotically k-dimensional* if there exists a k-dimensional inertial manifold in E_ϑ.

Let $\mathfrak{N}$ be a CI set of equation (1), $u \in \mathfrak{N}$, and $t > 0$; then $u = \Phi_t(u_0)$ for $u_0 \in \mathfrak{N}$. Since $\mathfrak{N}$ is bounded in E_ϑ, and, a fortiori, in E_α, by property (B) of the inertial manifold H the quantity $\|u - \Phi_t(\bar{u}_0)\|_\vartheta$ $(\bar{u}_0 \in H)$ becomes arbitrarily small as t grows, i.e., $u \in H$. Thus, the inertial manifold contains all the CI sets of equation (1).

If H is a k-dimensional inertial manifold and $u_0 \in H$, then $u(t) = \psi(y(t))$, and $y(t) = Pu(t)$ satisfies the following ODE in $Y \simeq \mathbf{R}^k$:

$$\dot{y} = -PA\psi(y) + PF(\psi(y)), \qquad y(0) = y_0 = Pu_0. \tag{6}$$

Under our hypotheses the right-hand side of (6) is uniformly Lipschitz. Indeed, for $PF\psi$ this follows from conditions (2), (3b) and the equivalence of the different norms in Y. The mapping $PA\psi$ is uniformly Lipschitz in the case $\dim E < \infty$, and $PA\psi = PA$ if $\dim E = \infty$, where PA is a bounded linear operator in Y.

Actually equation (6) is the restriction of the SPE (1) to H; we denote by $\mathfrak{n}$ CI sets of (6). The mapping ψ establishes a one-to-one correspondence between the CI sets of equations (6) and (1): $\psi(\mathfrak{n}) = \mathfrak{N}$, $P\mathfrak{N} = \mathfrak{n}$, and the rather strong nature of the attraction of the phase space E_ϑ by H easily allows us to reduce the problem of describing the stable CI sets of the SPE (1) to the analogous problem for the ODE (6) in $\mathbf{R}^k$.

Lemma 1 (reduction principle). *If there exists a k-dimensional inertial manifold H in the orthopair (X, Y), then the CI sets $\mathfrak{N}$ of equation* (1) *and the CI sets $\mathfrak{n} = P\mathfrak{N}$ of the ODE* (6) *are simultaneously stable.*

Proof. The stability of $\mathfrak{n}$ clearly follows from the stability of $\mathfrak{N}$. Now suppose that $\mathfrak{n}$ is stable. This means that there exists an open set $\mathfrak{d} \supset \mathfrak{n}$ in Y such that for $y_0 \in \mathfrak{d}$ the solutions $y(t)$ of the ODE (6) tend to $\mathfrak{n}$ for large time. We shall assume that the E_α-norm is defined in $Y \simeq \mathbf{R}^k$. We set

$$\mathfrak{D} = \{u \in E_\vartheta : u = x + y,\, y \in \mathfrak{d},\, \|u - \psi(y)\|_\vartheta < \lambda_1^\alpha C_1^{-1} \rho(y)\},$$

where $\rho(y)$ is the distance from $y \in \mathfrak{d}$ to the boundary of $\mathfrak{d}$, and C_1 is the constant from (5b). The open set $\mathfrak{D} \supset \mathfrak{N} = \psi(\mathfrak{n})$ and, by property (5a) of inertial manifolds, for $u_0 \in \mathfrak{D}$ we have $\|u(t) - \bar{u}(t)\| \to 0$ as $t \to +\infty$ with $\bar{u}_0 \in H$. Using (5b), we find that

$$\|y_0 - \bar{y}_0\|_\alpha \le \|u_0 - \bar{u}_0\|_\alpha \le C_1 \|u_0 - \psi(y_0)\|_\alpha \le \lambda_1^{-\alpha} C_1 \|u_0 - \psi(y_0)\|_\vartheta < \rho(y_0);$$

hence, $\bar{y}_0 \in \mathfrak{d}$ and $\bar{y}(t) \to \mathfrak{n}$ as $t \to +\infty$. Since $\bar{u}(t) = \psi(\bar{y}(t))$, it follows that $\bar{u}(t) \to \mathfrak{N}$ and $\operatorname{dist}(\mathfrak{N}, u(t)) \to 0$ as $t \to +\infty$. This proves the lemma.

The main results of this paper, the existence and uniqueness of inertial manifolds, are stated in the following way.

Theorem 1 (existence). *Let $k \ge 1$ and suppose that the conditions*

$$\lambda_{k+1} - \lambda_k > 2L, \qquad \lambda_{k+1} > L \tag{7a}$$

hold for $\vartheta = 0$, or the conditions

$$\lambda_{k+1} - \lambda_k > (\lambda_{k+1}^\vartheta - \lambda_k^\vartheta)L, \qquad \lambda_1 > 0 \tag{7b}$$

hold for $\vartheta > 0$. Then there exists a k-dimensional inertial manifold of equation (1) *in the orthopair (X_k, Y_k).*

Theorem 2 (uniqueness). *For each $k \ge 1$ there exists at most one k-dimensional inertial manifold in E_ϑ.*

The meaning of the relations (7) is that the nonlinearity F must be "sufficiently small" in comparison with the gap $\lambda_{k+1} - \lambda_k$ between adjacent eigenvalues of the operator A. Various results of this kind, containing undetermined constants, can be found in [11]–[15], [17], [21]; these are, for example, conditions of the type $\lambda_{k+1}-\lambda_k > M$, $\lambda_1 > 0$ ($\vartheta = 0$, and $M > 0$ does not depend on k), guaranteeing the existence of a finite-dimensional inertial manifold for the SPE (1), when $\sup \lambda_{k+1} - \lambda_k = \infty$ ($k \geq 1$). If it is necessary to establish the existence of an inertial manifold of a *given* dimension, then sufficient conditions are needed that depend explicitly on the data of the problem, say, on the parameters ϑ, L, $\{\lambda_k\}$.

The first constructive conditions of the type (7) were obtained in [2]. A very general (but not very concrete) theorem on inertial manifolds is given in [11], Chapter 6; by modifying the proof from [11] somewhat, one can obtain (see [3]) sufficient conditions for the existence of a k-dimensional inertial manifold for (1) for $\vartheta = 0$ in the form

$$\lambda_{k+1} - \lambda_k \geq 4L, \qquad \lambda_1 \geq 0.$$

Finally, in [4], [5] (see also [7]) these conditions (also for $\vartheta = 0$) were improved to

$$\lambda_{k+1} - \lambda_k \geq 2\sqrt{2}L, \qquad \lambda_{k+1} > 2L,$$

and it was shown that the constant factors of L cannot be less than 2 and 1, respectively.

We note that in many papers (including [4] and [5]) the attracting property of the inertial manifold H is formulated in a weaker form than condition (5):

$$\lim_{t\to+\infty} \operatorname{dist}(H, u(t)) = 0$$

for all $u_0 \in E_\vartheta$; in general this does not allow us to establish the reduction principle (Lemma 1), which in turn makes it difficult to use properties of ODEs in $\mathbf{R}^k$ to study steady-state regimes of equation (1).

In [8] results that partially overlap with the conclusions of Theorem 1 are given. Here, however, there are differences both in the restrictions on the nonlinearity F (significantly more special in comparison with (2)), and in the very definition of an inertial manifold. Namely, in [8] the inertial manifold H is constructed as a local (contained in some ball) positively invariant (but not invariant) manifold, and, moreover, the attracting property of H is defined in the weak form (8).

Various aspects of the theory of invariant manifolds for ODEs with a small parameter were considered in [10], but the question of sufficient conditions in the form (7) for the existence of such manifolds was not explicitly raised there.

In what follows the existence theorem for inertial manifolds will be stated and proved in more general terms. Thus, the following proposition allows us to explicitly express the constants that characterize the inertial manifold, in terms of the parameters ϑ, L, $\{\lambda_k\}$.

Theorem 3. *Let $k \geq 1$, $0 < h < 1$, $\omega(h) = (h^2 + h^{-2})/2$, and suppose that the conditions*

$$\text{(9a)} \qquad \lambda_{k+1} - \lambda_k > (1+\omega(h))_L, \qquad \lambda_{k+1} > L,$$

$$\text{(9b)} \qquad \lambda_{k+1} - \lambda_k > (\lambda_{k+1}^\vartheta + \omega(h)\lambda_k^\vartheta)L, \qquad \lambda_1 > 0$$

hold for $\vartheta = 0$ or $\vartheta > 0$, respectively. Then there exists a k-dimensional inertial manifold H in the orthopair (X_k, Y_k) with constants

$$\xi = h, \;\; \gamma = \lambda_{k+1} - \lambda_{k+1}^\vartheta L, \;\; \gamma' = \lambda_k + \lambda_k^\vartheta L.$$

Remark. The constant ξ' from (3b) is expressed in a more complicated way.

Of course, Theorem 1 is a special case of Theorem 3 (with the same constants γ, γ' and with $\xi < 1$).

We also note that, by replacing A by $A + T$ and F by $F + T$ in equation (1), where T is a bounded linear selfadjoint operator in E (for example, a scalar operator), one can sometimes make conditions (7), (9) hold if they do not hold with $T = 0$. In particular, this approach allows us to treat the case $\vartheta > 0$, $\lambda_1 \le 0$, not considered here, if we set $T = bI$ with $b > -\lambda_1$ (I is the identity operator in E).

It turns out that conditions of the type (7) for the existence of a k-dimensional inertial manifold are already the best possible in a certain sense (for more about this see also [4], [5]).

Lemma 2. *If $\vartheta = 0$ and $c_1, c_2 > 0$ are constants such that under the conditions*

$$\lambda_{k+1} - \lambda_k > c_1 L, \qquad \lambda_{k+1} > c_2 L$$

there always exists a k-dimensional inertial manifold for the SPE (1), then $c_1 \ge 2$ and $c_2 \ge 1$.

Proof. Let $E = \mathbf{R}^2$, $u = (u_1, u_2)$, $Au = (\lambda_1 u_1, \lambda_2 u_2)$ with $\lambda_1 < \lambda_2$, and $F(u) = (u_2, -u_1)$. Then $L = 1$, and if $\lambda_2 > c_2$ but $c_1 < \lambda_2 - \lambda_1 < 2$, then the stationary point $u = 0$ is a focus, and hence, there does not exist a one-dimensional inertial manifold for this ODE.

Now suppose $F(u) = (u_1, u_2)$. Then $L = 1$, and if $\lambda_2 - \lambda_1 > c_1$ but $c_2 < \lambda_2 < 1$, then the point $u + 0$ is an unstable node and in this case there does not exist a one-dimensional inertial manifold either. This proves the lemma.

Lemma 3. *If $\vartheta > 0$, and $c_1, c_2 > 0$ are constants such that under the conditions*

$$\lambda_{k+1} - \lambda_k > (c_1 \lambda_{k+1}^{\vartheta} + c_2 \lambda_k^{\vartheta}) L, \qquad \lambda_1 > 0$$

there always exists a k-dimensional inertial manifold for the SPE (1), then $c_1 \ge 1$ and $c_1 + c_2 \ge 2$.

Proof. Let $E = \mathbf{R}^2$, $u = (u_1, u_2)$, $Au = (\lambda_1 u_1, \lambda_2 u_2)$ with $0 < \lambda_1 < \lambda_2$, $F(u) = (u_2, -u_1)$. Then $L = \lambda_1^{-\vartheta}$, $(c_1 \lambda_2^{\vartheta} + c_2 \lambda_1^{\vartheta}) L = c_1 \kappa + c_2$, where $\kappa = (\lambda_2/\lambda_1)^{\vartheta} > 1$, and if κ is close to 1, but $c_1 + c_2 < c_1 \kappa + c_2 < \lambda_2 - \lambda_1 < 2$, then the stationary point $u = 0$ is a focus, and hence, there does not exist a one-dimensional inertial manifold for this ODE.

Now let $F(u) = (\lambda_1^{\vartheta} u_1, \lambda_2^{\vartheta} u_2)$ and $\lambda_2 < 1$; then $L = 1$. Assume that $c_1 < 1$ and that c_2 is arbitrarily large. If λ_1 is sufficiently close to 0 and λ_2 is sufficiently close to 1, then $(c_1 \lambda_2^{\vartheta} + c_2 \lambda_1^{\vartheta}) L < \lambda_2 - \lambda_1$, but $u = 0$ is an unstable node, and in this case there does not exist a one-dimensional inertial manifold either. This proves the lemma.

In the majority of the cited papers the mapping $\sigma \colon Y \to X$ whose graph is the inertial manifold H, has been constructed as a fixed point of a certain transformation in the metric space of Lipschitz mappings from Y into X, most often with the use of the classical Krylov-Bogolyubov scheme (see [9], [10] for ODEs and [11] for SPEs). Here we have used an essentially different approach (different from the one presented in [7], [8] as well), which develops and generalizes the techniques for lowering the dimension of dynamical systems in $\mathbf{R}^n$, proposed in [1], to the problem (1), (2) in an infinite-dimensional phase space.

In the orthopair (X, Y) equation (1) is written in the form

$$\dot{x} = -A_2 x + QF(x + y), \qquad \dot{y} = -A_1 y + PF(x + y), \tag{10}$$

where $A_1 = PA$, $A_2 = QA$. For $\xi > 0$ on E_ϑ we define an indefinite quadratic form

$$V_\xi(u) = (x, x)_\alpha - \xi^2(y, y)_\alpha \qquad (\alpha = \vartheta/2)$$

in the orthopair (X, Y). We denote the corresponding negative and positive cones by

$$\mathscr{V}_\xi^- = \{u \in E_\vartheta : V_\xi(u) \le 0\}, \qquad \mathscr{V}_\xi^+ = \{u \in E_\vartheta : V_\xi(u) \ge 0\}.$$

The following result (with the natural change of statement) is, in particular, proved in [1].

Theorem 4. *Let $E = \mathbf{R}^n$, and let $F: \mathbf{R}^n \to \mathbf{R}^n$ be a locally Lipschitz mapping. Suppose that all solutions of equation* (1) *exist for $t \in \mathbf{R}$, and there is a stationary solution $\bar{u} \in E$. Then if in some orthopair (X, Y), $\dim Y = k \ge 1$, the inequality*

$$\frac{d}{dt}(e^{2\gamma t} V_\xi(u(t) - u'(t))) \le -\varepsilon e^{2\gamma t} \|u(t) - u'(t)\|_\alpha^2 \tag{11}$$

holds with $\vartheta = 0$, $\xi = 1$, $\gamma > 0$, and $\varepsilon > 0$, for solutions of (1), *then in E there exists a k-dimensional invariant manifold H of the form* (4) *with Lipschitz constant* 1 *in* (3), *containing all the CI sets of the ODE* (1).

Later on, relation (11) will be used in the somewhat weaker form:

$$V_\xi(u(t) - u'(t)) \le V_\xi(u_0 - u_0')e^{-2\gamma t} \tag{12}$$

or even (in case $u' = 0$ is a solution of (1))

$$V_\xi(u(t)) \le V_\xi(u_0)e^{-2\gamma t} \tag{12a}$$

for $t \ge 0$.

We shall show that in our situation inequality (12) is a consequence of conditions (9) in Theorem 3. We note that $(u, u)_\alpha = (u, A^\vartheta u)$ for $u \in E_\vartheta$.

Lemma 4. *Suppose conditions* (9) *hold with $k \ge 1$ and $h < 1$, $\omega(h) = (h^2 + h^{-2})/2$, and the form V_ξ is defined in the orthopair (X_k, Y_k). The inequality* (12) *holds for the solutions of* (1) *for $\xi \in [h, h^{-1}]$ and $\gamma \in [\gamma_0, \gamma_1]$ with*

$$\gamma_0 = \lambda_k + \omega(h)\lambda_k^\vartheta L, \qquad \gamma_1 = \lambda_{k+1} - \lambda_{k+1}^\vartheta L. \tag{13}$$

Remark. If (9) holds, we have $\gamma_0 < \gamma_1$ and $\gamma_1 > 0$. For $\vartheta > 0$ we always have $\gamma_0 > 0$; for $\vartheta = 0$ we may possibly have $\gamma_0 \le 0$.

Proof. We set $X = X_k$, $Y = Y_k$, and for $u = x + y$, $u' = x' + y' \in E_\vartheta$ we set

$$J = (F(x+y) - F(x'+y'), A^\vartheta(x - x' - \xi^2(y - y'))).$$

We write equation (1) for $u(t)$ and $u'(t)$ in the form (10). Then

$$\dot{x} - \dot{x}' = -A_2(x - x') + Q(F(x+y) - F(x'+y')),$$
$$\dot{y} - \dot{y}' = -A_1(y - y') + P(F(x+y) - F(x'+y')).$$

Taking the scalar multiple in E of the first of these relations by $A^\vartheta(x - x')$ and of the second by $A^\vartheta(y - y')$, and denoting the left-hand side of (11) divided by $2e^{2\gamma t}$ by $Z(t)$, we have (omitting the dependence on t):

$$\begin{aligned} Z &= \gamma V_\xi(u - u') + (\dot{x} - \dot{x}', x - x')_\alpha - \xi^2(\dot{y} - \dot{y}', y - y')_\alpha \\ &= \gamma V_\xi(u - u') - (A_2(x - x'), x - x')_\alpha + \xi^2(A_1(y - y'), y - y')_\alpha + J. \end{aligned}$$

From condition (2) on the nonlinearity F we find that

$$|J| \le L\|x - x' + y - y'\|_\vartheta \cdot \|x - x' - \xi^2(y - y')\|_\vartheta.$$

Using the orthogonality of X and Y in E_ϑ and the elementary inequality $((a+b) \times (a+cb))^{1/2} \leq a + c_1 b$ with $c_1 = (1+c)/2$ (and $a = \|x-x'\|_\vartheta^2$, $b = \|y-y'\|_\vartheta^2$, $c = \xi^4$), we obtain

$$|J| \leq L(x-x', x-x')_\vartheta + L\xi_1(y-y', y-y')_\vartheta$$

with $\xi_1 = (1+\xi^4)/2$. Furthermore, for $u \in E_\vartheta$ we have $(A^\vartheta u, u)_\alpha = (u, u)_\vartheta$ and $Ax = A_2 x$, $Ay = A_1 y$, so that

$$\begin{aligned} Z \leq \gamma(x-x', x-x')_\alpha &- \gamma\xi^2(y-y', y-y')_\alpha \\ &+ (T_2(x-x'), x-x')_\alpha + (T_1(y-y'), y-y')_\alpha, \end{aligned}$$

where $T_2 = LA^\vartheta - A$ and $T_1 = L\xi_1 A^\vartheta + \xi^2 A$ $(A^0 = I)$ are linear self-adjoint operators. We see that $\mu = L\xi_1\lambda_k^\vartheta + \xi^2\lambda_k$ is the maximal eigenvalue of T_1 in Y, so that $(T_1 y, y)_\alpha \leq \mu(y, y)_\alpha$ for $y \in Y$. Moreover, let $\mu_j = L\lambda_j^\vartheta - \lambda_j$ $(j \geq k+1)$ be the eigenvalues of T_2 in X, and let $\kappa_j = \lambda_j \lambda_{k+1}^{-1}$. Then $\mu_j = L\lambda_{k+1}^\vartheta \kappa_j^\vartheta - \lambda_{k+1}\kappa_j$, and since $\lambda_{k+1} > 0$ and $\mu_{k+1} < 0$ in (9), it follows that $\kappa_j \geq 1$, $\mu_j \leq \mu_{k+1}\kappa_j^\vartheta \leq \mu_{k+1}$, and hence, $(T_2 x, x)_\alpha \leq \mu_{k+1}(x, x)_\alpha$ in X.

Finally we conclude that

$$Z \leq (x-x', x-x')_\alpha(\gamma + L\lambda_{k+1}^\vartheta - \lambda_{k+1}) + (y-y', y-y')_\alpha(-\gamma\xi^2 + L\xi_1\lambda_k^\vartheta + \xi^2\lambda_k) \leq 0$$

in view of the estimates (13) for γ and the condition $\xi \in [h, h^{-1}]$ (from which it follows that $\xi_1\xi^{-2} = \omega(\xi) \leq \omega(h)$). The inequality $Z(t) \leq 0$ is equivalent to (12), and the lemma is proved.

The constructions of [1] make significant use of the finite-dimensionality of the phase space. In the proof given below, this technique is essentially revised, which allows us to cover the case $\dim E = \infty$, to obtain sufficient conditions for the existence of the inertial manifold H in the form (7) suitable for applications, and to establish the attracting property for H in its strongest form (5) (including the case $\dim E < \infty$).

§3. Existence and uniqueness of the inertial manifold

Theorems 1 and 3 start from a *given* representation of the right-hand side of (1) as a sum of a linear operator and a nonlinear one, and also from a *prescribed* orthogonal decomposition of $E = X \oplus Y$ with $X = X_k$ and $Y = Y_k$. Here we prove a more general assertion.

Theorem 5. *Suppose $0 < h < 1$, $\gamma_0 < \gamma_1$, $\gamma_1 > 0$, and suppose that inequality (12) holds in some orthopair (X, Y), $\dim Y = k \geq 1$, for the solutions of (1), with $\xi \in \{h, h^{-1}\}$, $\gamma = \gamma_1$, and $\xi = 1$, $\gamma = \gamma_0$. Then there exists a k-dimensional inertial manifold H in the orthopair (X, Y) with constants $\xi = h$, $\gamma = \gamma_1$, $\gamma' = \gamma_0$.*

Remark. The parameter ξ^2 is included linearly in relation (12), and thus in the hypotheses of the theorem inequality (12) also holds for all values of (ξ, γ_1) with $\xi \in [h, h^{-1}]$, including $\xi = 1$, $\gamma = \gamma_1$. In the sequel, we use the notation $q = h^{-1}$.

Theorem 3 is a consequence of Theorem 5 and Lemma 4. Indeed, under the conditions of Theorem 3 inequality (12) holds (by Lemma 4) for $\xi \in \{h, h^{-1}\}$, $\gamma_1 = \lambda_{k+1} - \lambda_{k+1}^\vartheta L$ and for $\xi = 1$, $\gamma_0 = \lambda_k + \lambda_k^\vartheta L$, and hence by Theorem 5 there exists a k-dimensional inertial manifold with constants $\xi = h$, $\gamma = \gamma_1$, $\gamma' = \gamma_0$.

Without loss of generality we assume that equation (1) has the stationary solution $u = 0$ (i.e., $F(0) = 0$).

Before passing directly to the proof of Theorem 5, we establish some estimates. For $\xi \in (0, 1)$ we set $K_\xi = (1-\xi^2)^{-1}$.

Lemma 5. *Let* (X, Y) *be an orthopair,* $0 < \xi < 1$, $\eta = \xi^{-1}$, $u_1, u_2, u_3 \in E_\vartheta$, *and* $y_1 = y_2$, $u_1 - u_3 \in \mathscr{V}_\eta^+$, $u_2 - u_3 \in \mathscr{V}_\xi^-$. *Then*

$$\|x_1 - x_3\|_\alpha \le K_\xi \|x_1 - x_2\|_\alpha.$$

Proof. We have

$$\begin{aligned}\|x_1 - x_3\|_\alpha &\le \|x_1 - x_2\|_\alpha + \|x_2 - x_3\|_\alpha \\ &\le \|x_1 - x_2\|_\alpha + \xi\|y_2 - y_3\|_\alpha \le \|x_1 - x_2\|_\alpha + \xi^2\|x_1 - x_3\|_\alpha\end{aligned}$$

(since $y_1 = y_2$), from which the desired estimate follows.

Lemma 6. *Let* (X, Y) *be an orthopair,* $0 < \xi < \eta$, *and* $u \in \mathscr{V}_\eta^+$. *Then*

$$\|u\|_\alpha^2 \le \frac{1+\eta^2}{\eta^2 - \xi^2} V_\xi(u);$$

in particular, for $\xi < 1$ *and* $\eta = \xi^{-1}$ *we have* $\|u\|_\alpha^2 \le K_\xi V_\xi(u)$.

In fact, for $u \in \mathscr{V}_\eta^+$ we immediately deduce

$$\|u\|_\alpha^2 \le (1 + \eta^{-2})\|x\|_\alpha^2 \le \frac{1+\eta^2}{\eta^2 - \xi^2} V_\xi(u),$$

and the lemma is proved.

The following assertion reflects the compactness property of the semiflow $\{\Phi_t\}$ and, apparently, is well known (although we have not been able to find an exact reference).

Lemma 7. *For* $u_0, u_0' \in E_\vartheta$, $\alpha = \vartheta/2$, *and* $t > 0$ *we have*

$$\|u(t) - u'(t)\|_\vartheta \le M\|u_0 - u_0'\|_\alpha,$$

where $M = M(t, \vartheta)$.

Proof. Let $\delta = \lambda_1$. For $\beta \in [0, 1)$ and $t > 0$ the operator $A^\beta e^{-At}$ is bounded in E and $\|A^\beta e^{-At}\| = \|e^{-At}\|_\beta \le M_\beta t^{-\beta} e^{-\delta t}$ (see [11], p. 27). From this, for $u \in E_\vartheta$,

$$\|e^{-At}u\|_\vartheta = \|A^\alpha e^{-At} A^\alpha u\| \le M_\alpha t^{-\alpha} e^{-\delta t}\|u\|_\alpha.$$

We set $b = \|u_0 - u_0'\|_\alpha$, $\varphi(t) = e^{\delta t}\|u(t) - u'(t)\|_\vartheta$. Writing the SPE (1) as an integral equation, we have

$$u(t) - u'(t) = e^{-At}(u_0 - u_0') + \int_e^t e^{-A(t-s)}(F(u(s)) - F(u'(s)))\, ds.$$

Multiplying this equation by $e^{\delta t}$, we obtain, using (2),

$$\varphi(t) \le bM_\alpha t^{-\alpha} + LM_\vartheta \int_0^t (t-s)^{-\vartheta}\varphi(s)\, ds.$$

Now applying the generalized Gronwall inequality (see [11], p. 188), we find that $\varphi(t) \le bM'(t, \vartheta)$, and hence, the assertion of the lemma is also true with $M = M'e^{-\delta t}$.

The proof of Theorem 5 is divided into four steps:

construction of the manifold H in the form (4) (§3.1);
invariance and Lipschitz property of H (§3.2);
attraction of the phase space E_ϑ by H, in the form (5) (§3.3);
$H = \mathscr{Z}(\gamma_0)$ (§3.4).

3.1. **Construction of the manifold H in the form (4).** Following [1], for $\tau \geq 0$ and $y \in Y$ we set

$$g_\tau(y) = P\Phi_\tau(y),$$

and we show that under certain conditions the mapping g_τ is a homeomorphism of Y onto itself.

Lemma 8. *If in some orthopair (X, Y) inequality (12) holds for solutions of equation (1) with $\xi > 0$ and $\gamma \in \mathbf{R}$, then the mapping $g_\tau: Y \to Y$ is a homeomorphism for all $\tau \geq 0$.*

Proof. We have $\xi^2\|y\|_\alpha^2 = -V_\xi(y)$ for $y \in Y$ and $\xi^2\|y\|_\alpha^2 \geq -V_\xi(u)$ for $u \in E_\vartheta$. From this and inequality (12) (multiplied by -1) we find that

$$\|g_\tau(y) - g_\tau(y')\|_\alpha \geq e^{-\gamma\tau}\|y - y'\|_\alpha \tag{14}$$

for $y, y' \in Y$. The mapping g_τ of $Y \simeq \mathbf{R}^k$ into itself is continuous, since the semiflow $\{\Phi_t\}$ is continuous in E_ϑ, and from (14) we see that g_τ is one-to-one and the inverse map g_τ^{-1} is uniformly Lipschitz. Hence, using Brouwer's invariance of domain theorem we deduce that g_τ^{-1} is bounded on all of Y, and the lemma is proved.

We may assume that inequality (12) for solutions of (1) holds with $\xi \in \{h, 1, q\}$, $\gamma = \gamma_1$ and with $\xi = 1$, $\gamma = \gamma_0$. Here it follows from $u_0 - u_0' \in \mathscr{V}_\xi^-$ that $u(t) - u'(t) \in \mathscr{V}_\xi^-$ for $t > 0$, and, conversely, $u(t) - u'(t) \in \mathscr{V}_\xi^+$ for some $t > 0$ implies that $u_0 - u_0' \in \mathscr{V}_\xi^+$.

For $\tau > 0$, $t \in \mathbf{R}$, and $y \in Y$ we set $z(\tau, y) = g_\tau^{-1}(y) \in Y$,

$$\chi(t, y) = \lim_{\tau \to +\infty} \Phi_{\tau+t}(z(\tau, y)), \tag{15}$$

where the existence of the limit is still to be established. We note that by the definition of g_τ, $P\Phi_\tau(z(\tau, y)) \equiv y$.

Lemma 9. *For $t \in \mathbf{R}$ and $y \in Y$ the limit (15) exists in the E_ϑ norm.*

Proof. Passing to Cauchy sequences and using Lemma 7, we find that the convergence of (15) for $t = t_0$ in the E_α norm implies the existence of the limit for $t > t_0$ in the E_ϑ norm as well. Thus, it suffices to prove that for $t \leq 0$ and $\bar{y} \in Y$ the mapping $\tau \to \Phi_{\tau+t}(z(\tau, \bar{y}))$ satisfies the Cauchy criterion in E_α for $\tau \to +\infty$. Let $\tau_2 > \tau_1 \geq \tau > -t$, and let $u = x + y$, $u' + x' + y'$ be solutions of (1) with

$$u_0 = z(\tau_1, \bar{y}), \qquad u_0' = \Phi_{\tau_2-\tau_1}(z(\tau_2, \bar{y})).$$

We have $y(\tau_1) = y'(\tau_1) = \bar{y}$, $x_0 = 0$, and for $t \leq 0$

$$u_1(\tau_1 + t) - u'(\tau_1 + t) = \Phi_{\tau_1+t}(z(\tau_1, \bar{y})) - \Phi_{\tau_2+t}(z(\tau_2, \bar{y})) \in \mathscr{V}_q^+,$$

since $u(\tau_1) - u'(\tau_1) \in X \subset \mathscr{V}_q^+$. Applying (12) with $\xi = 1$ and $\gamma = \gamma_1$, we obtain

$$V_1(u(\tau_1 + t) - u'(\tau_1 + t)) \leq e^{-2\gamma_1(\tau_1+t)}(\|x_0'\|_\alpha^2 - \|y_0 - y_0'\|_\alpha^2). \tag{16}$$

Now setting $\xi = 1$, $\gamma = \gamma_0$ in (12a), we see that

$$e^{2\gamma_0\tau_1}V_1(u(\tau_1)) \leq V_1(u_0),$$

and since $V_1(u_0) = -\|y_0\|_\alpha^2$ and $V_1(u(\tau_1)) \geq -\|\bar{y}\|_\alpha^2$, it follows that

$$\|y_0\|_\alpha^2 \leq e^{2\gamma_0\tau_1}\|\bar{y}\|_\alpha^2. \tag{17}$$

Using the elementary Hilbert space relation $\|y_0-y_0'\|_\alpha^2 \geq (\|y_0\|_\alpha - \|y_0'\|_\alpha)^2$ and the fact that $u_0' \in \mathscr{V}_h^-$, since $z(\tau_2, \bar{y}) \in Y \subset \mathscr{V}_h^-$, in the right-hand side of (16) we have

$$\|x_0'\|_\alpha^2 - \|y_0 - y_0'\|_\alpha^2 \leq h^2\|y_0'\|_\alpha^2 - (\|y_0\|_\alpha - \|y_0'\|_\alpha)^2 \leq \kappa\|y_0\|_\alpha^2 \tag{18}$$

if $\kappa = \kappa(h) > 0$ is so large that the quadratic trinomial $(1-h^2)\zeta^2 - 2\zeta + \kappa + 1$ is positive definite (we write $\zeta = \|y_0'\|_\alpha/\|y_0\|_\alpha$; for $y_0 = 0$ inequality (18) holds trivially). Since $\varphi = u(\tau_1+t) - u'(\tau_1+t) \in \mathscr{V}_q^+$, by Lemma 6 with $\xi = 1$ and $\eta = q$ we have $\|\varphi\|_\alpha^2 \leq M'(q)V_1(\varphi)$.

Finally, combining relations (16)–(18), we obtain

$$\|u(\tau_1+t) - u'(\tau_1+t)\|_\alpha^2 \leq Me^{-2(\gamma_1-\gamma_0)\tau_1}\|\bar{y}\|_\alpha^2$$

with $M > 0$ independent of τ_1, τ_2. Moreover, this inequality is true with τ_1 replaced by $\tau \leq \tau_1$ on the right-hand side, and thus the mapping $\Phi_{\tau+t}(z(\tau, \bar{y})) \colon \mathbf{R}^+ \to E_\alpha$ actually satisfies the Cauchy criterion as $\tau \to +\infty$. This proves Lemma 9.

Now we set $\psi(y) = \chi(0, y)$, $\sigma(y) = Q\psi(y)$ for $y \in Y$, and from (15) it follows that $P\psi(y) = y$. We denote by H the manifold of the form (4) with defining function $\psi(y) = y + \sigma(y)$, and the first step of the proof of Theorem 5 is complete.

3.2. **Invariance and Lipschitz property of H.**

Lemma 10. *For $t \in \mathbf{R}$ and $y \in Y$ we have $\chi(t, y) \in H$.*

Proof. We set $\chi(t, y) = u'$, $Pu' = y'$; then $u' - \psi(y') \in X$. Further, $\psi(y') = \chi(0, y')$, and we can replace τ by $\tau + t$ in the definition (15) of $\chi(0, y')$. Hence

$$u' - \psi(y') = \lim_{\tau\to+\infty}(\Phi_{\tau+t}(z(\tau, y)) - \Phi_{\tau+t}(z(\tau+t, y'))) \in \mathscr{V}_h^-,$$

because $z(\tau, y) - z(\tau+t, y') \in Y \subset \mathscr{V}_h^-$; since $X \cap \mathscr{V}_h^- = 0$, we have $u' - \psi(y') = 0$, and the lemma is proved.

Now let $u_0 = x_0 + y_0 \in H$, $u_0 = \chi(0, y_0)$, and $t > 0$. Using the continuity of the evolution operators Φ_t in E_ϑ, we find from (15) that $\Phi_t(u_0) = \chi(t, y_0) \in H$. In exactly the same way, $u_0 = \Phi_t(\chi(-t, y_0))$, where $\chi(-t, y_0) \in H$, and we have proved the invariance of H.

For $y, y' \in Y$ we have

$$\psi(y) - \psi(y') = \lim_{\tau\to+\infty}(\Phi_\tau(z(\tau, y)) - \Phi_\tau(z(\tau, y'))).$$

Since $z(\tau, y) - z(\tau, y') \in Y \subset \mathscr{V}_h^-$, we also have $\Phi_\tau(z(\tau, y)) - \Phi_\tau(z(\tau, y')) \in \mathscr{V}_h^-$, from which $\psi(y) - \psi(y') \in \mathscr{V}_h^-$, i.e., $\|\sigma(y) - \sigma(y')\|_\alpha \leq h\|y - y'\|_\alpha$, and we have obtained the estimate (3a). For $u, u' \in H$ and $t = 1$ we now set $u = \Phi_t(u_0)$ and $u' = \Phi_t(u_0')$ with $u_0, u_0' \in H$. Applying inequality (12) with $\xi = 1$ and $\gamma = \gamma_0$, we find that

$$\|x - x'\|_\alpha^2 - \|y - y'\|_\alpha^2 \leq e^{-2\gamma_0}(\|x_0 - x_0'\|_\alpha^2 - \|y_0 - y_0'\|_\alpha^2),$$

from which, in view of (3a),

$$(1-h^2)e^{-2\gamma_0}\|y_0 - y_0'\|_\alpha^2 \leq \|y - y'\|_\alpha^2.$$

Using this estimate and Lemma 7 with $t = 1$, $M = M(\vartheta)$ and inequality (3a) again, for $u, u' \in H$ we have

$$\begin{aligned}\|x - x'\|_\vartheta &\leq \|u - u'\|_\vartheta \leq M(\vartheta)\|u_0 - u_0'\|_\alpha \\ &\leq M(\vartheta)(1+h^2)^{1/2}\|y_0 - y_0'\|_\alpha \leq \xi'\|y - y'\|_\vartheta,\end{aligned}$$

where ξ' does not depend on y_0, y_0' (we have also used the equivalence of norms in Y). Thus, the Lipschitz conditions (3) have been completely established.

Remark. In particular, we have proved that $u - u' \in \mathscr{V}_h^-$ for $u, u' \in H$, and since $u' = 0$ is a solution of (1), $H \subset \mathscr{V}_h^-$.

We note that on the invariant manifold H the semiflow $\{\Phi_t\}_{t\geq 0}$ restricts to a flow $\{\Phi_t\}_{t\in\mathbf{R}}$; $q = h^{-1}$ for $h \in (0, 1)$.

3.3. **Attraction of the phase space by** H. We recall that in the hypotheses of Theorem 5 we set $\alpha = \vartheta/2$ and $K_h = (1-h^2)^{-1}$ for $h \in (0, 1)$ and $q = h^{-1}$.

Let $u(t) = x(t)+y(t)$ be an arbitrary solution of equation (1) for $t \geq 0$ with $u_0 = x_0 + y_0 \in E_\vartheta$. Since H is an invariant manifold, we may set $\bar{u}_0(t) = \Phi_{-t}(\psi(y(t)))$, and $\bar{u}_0(t) \in H$. Since $u(t) - \psi(y(t)) \in X \subset \mathscr{V}_q^+$, we also have $u_0 - \bar{u}_0(t) \in \mathscr{V}_q^+$. Now we can apply Lemma 5 with $u_1 = u_0$, $u_2 = \psi(y_0)$, $u_3 = \bar{u}_0(t)$. Indeed, $u_1 - u_3 \in \mathscr{V}_q^+$, $y_1 = y_2 = y_0$, and $u_2 - u_3 \in \mathscr{V}_h^-$, since $u_2, u_3 \in H$. By the lemma,

$$\|x_0 - \bar{x}_0(t)\|_\alpha \leq K_h\|x_0 - \sigma(y_0)\|_\alpha. \tag{19}$$

We need to prove the convergence of an arbitrary solution $u(t)$ as $t \to +\infty$ to a certain solution $\bar{u}(t) \in H$. Since $u_0 - \bar{u}_0(t) \in \mathscr{V}_q^+$, the uniform boundedness of $\|x_0 - \bar{x}_0(t)\|_\alpha$ for $t \geq 0$, established above, implies the uniform boundedness of the set $\{\bar{y}_0(t)\}_{t\geq 0}$ in the finite-dimensional subspace Y. Following a well-known scheme (see [11], pp. 150–151), we single out a sequence $\bar{y}_0(t_\nu) \to \bar{y}_0$ as $t_\nu \to +\infty$; since $\bar{u}_0(t) \in H \subset \mathscr{V}_h^-$, we also have $\bar{x}_0(t_\nu) \to \bar{x}_0$, $\bar{u}_0(t_\nu) \to \bar{u}_0 \in H$ in the E_α norm. We denote by $\bar{u}(t)$ a solution of (1) with $\bar{u}(0) = \bar{u}_0$. From the invariance property of H we see that $\bar{u}(t) \in H$ for $t \geq 0$, and, using Lemma 7, we find that, for $t > 0$,

$$\bar{u}(t) = \lim_{t_\nu \to +\infty} \Phi_t(\bar{u}_0(t_\nu)) \tag{20}$$

in the E_ϑ norm. Furthermore, for $t_\nu > t$ we have

$$\Phi_{t_\nu}(u_0) - \Phi_{t_\nu}(\bar{u}_0(t_\nu)) = u(t_\nu) - \psi(y(t_\nu)) \in X \subset \mathscr{V}_q^+,$$

and hence, $u(t) - \Phi_t(\bar{u}_0(t_\nu)) \in \mathscr{V}_q^+$. It now follows from (20) that $u(t) - \bar{u}(t) \in \mathscr{V}_q^+$ for $t \geq 0$. According to (12) with $\xi = h$ and $\gamma = \gamma_1$

$$V_h(u(t) - \bar{u}(t)) \leq V_h(u_0 - \bar{u}_0)e^{-2\gamma_1 t}$$

for $t \geq 0$. By Lemma 6

$$\|u(t) - \bar{u}(t)\|_\alpha^2 \leq K_h V_h(u(t) - \bar{u}(t)),$$

and since $V_h(u_0 - \bar{u}_0) \leq \|u_0 - \bar{u}_0\|_\alpha^2$,

$$\|u(t) - \bar{u}(t)\|_\alpha \leq K_h^{1/2}\|u_0 - \bar{u}_0\|_\alpha e^{-\gamma_1 t}$$

for $t \geq 0$. By Lemma 7,

$$\|u(t) - \bar{u}(t)\|_\vartheta \leq M\|u(t-1) - \bar{u}(t-1)\|_\alpha$$

for $t \geq 1$ with $M = M(\vartheta)$, and property (5a) is proved.

Now passing to the limit in (19) as $t = t_\nu \to +\infty$ and $\bar{x}_0(t_\nu) \to \bar{x}_0$, we see that $\|x_0 - \bar{x}_0\|_\alpha \leq K_h\|x_0 - \sigma(y_0)\|_\alpha$. Since $u_0 - \bar{u}_0 \in \mathscr{V}_q^+$, it follows that

$$\|u_0 - u_0'\|_\alpha^2 \leq (1+h^2)\|x_0 - \bar{x}_0\|_\alpha^2 \leq K_h^2(1+h^2)\|x_0 - \sigma(y_0)\|_\alpha^2,$$

i.e., we have obtained estimate (5b), which completes the third step of the proof of Theorem 5.

In particular, it has been shown that we always have $u(t) - \bar{u}(t) \in \mathscr{V}_q^+$ for $t \geq 0$ for $u_0 \in E_\vartheta$.

3.4. $H = \mathscr{Z}(y_0)$.

Lemma 11 (see also [1, p. 345]). *Let* $0 < h < 1$, $\gamma_0 < \gamma_1$, *and suppose that inequality* (12) *with* $\xi = h$, $\gamma = \gamma_1$ *and* $\xi = 1$, $\gamma = \gamma_0$, *holds in the orthopair* (X, Y) *for solutions of* (1). *Then for any two solutions* $u(\tau)$ *and* $u'(\tau)$, *existing for* $\tau \in \mathbf{R}$,

$$\|u(\tau) - u'(\tau)\|_\vartheta = O(e^{-\gamma_0\tau}), \qquad \tau \to -\infty, \tag{21}$$

if and only if $u(\tau) - u'(\tau) \in \mathscr{V}_h^-$ *for* $\tau \le 0$.

Proof. From Lemma 7 and the estimate $\|\cdot\|_\vartheta \ge \lambda_1^\alpha\|\cdot\|_\alpha$ we see that the E_ϑ-norm can be replaced by the E_α-norm in (21). For $u(\tau) - u'(\tau)$ with $\tau \le 0$ relation (12) gives

$$V_\xi(u_0 - u_0') \le e^{2\gamma\tau} V_\xi(u(\tau) - u'(\tau)). \tag{22}$$

If $u(\tau) - u'(\tau) \in \mathscr{V}_h^-$, then, setting $\xi = 1$ and $\gamma = \gamma_0$, we find from (22) that

$$e^{2\gamma_0\tau}(1 - h^2)\|y(\tau) - y'(\tau)\|_\alpha^2 \le \|y_0 - y_0'\|_\alpha^2,$$
$$\frac{1 - h^2}{1 + h^2}\|u(\tau) - u'(\tau)\|_\alpha^2 \le e^{-2\gamma_0\tau}\|y_0 - y_0'\|_\alpha^2.$$

Conversely, let $\|u(\tau) - u'(\tau)\|_\alpha \le ce^{-\gamma_0\tau}$ for $\tau \le 0$. Setting $\xi = h$, $\gamma = \gamma_1$ in (22) and using the fact that $V_\xi(u) \le \|u\|_\alpha^2$ for $u \in E_\vartheta$ we have

$$V_h(u_0 - u_0') \le c^2 e^{2(\gamma_1 - \gamma_0)\tau}$$

for $\tau \le 0$, from which $V_h(u_0 - u_0') \le 0$; making a shift in time, we obtain $V_h(u(\tau) - u'(\tau)) \le 0$ for $\tau \le 0$ in the same way, and the lemma is proved.

Corollary. *Since* $u' = 0$ *is a solution of* (1), *under the hypotheses of the lemma* $u(t) \in \mathscr{Z}(\gamma_0)$ *is equivalent to* $u(t) \in \mathscr{V}_h^-$ *for every solution* $u(t)$ *that exists for* $t \in \mathbf{R}$.

We continue the proof of Theorem 5. It has already been shown that $H \subset \mathscr{V}_h^-$, and according to the corollary, $H \subset \mathscr{Z}(\gamma)$ with $\gamma = \gamma_0$. Now suppose $u_0 \in \mathscr{Z}(\gamma)$ and $u_0' = \psi(Pu_0) \in H$; then $u_0 - u_0' \in X$. On the other hand, it follows from $u_0 \in \mathscr{Z}(\gamma)$ and $u_0' \in \mathscr{Z}(\gamma)$ that $u(t) - u'(t)$ satisfies the estimate (21) for $t \le 0$, and by Lemma 11, $u_0 - u_0' \in \mathscr{V}_h^-$. Hence $u_0 - u_0' = 0$, i.e., $H = \mathscr{Z}(\gamma_0)$, and Theorem 5 is completely proved.

It remains to establish the uniqueness of an inertial manifold of a given dimension.

Proof of Theorem 2. Let H and H' be k-dimensional ($k \ge 1$) inertial manifolds in orthopairs (X, Y) and (X', Y') with defining functions ψ and ψ', $H = \mathscr{Z}(\gamma)$ and $H' = \mathscr{Z}(\gamma')$, and let P and P' be the orthogonal projections onto Y and Y' in E. If $\gamma = \gamma'$, then $H = H'$. Let $\gamma' < \gamma$; then $\mathscr{Z}(\gamma') \subset \mathscr{Z}(\gamma)$, and hence $H' \subset H$. The mapping $P\psi' \colon Y' \to Y$ is continuous; we show that it is one-to-one. If $y_1, y_2 \in Y'$ and $y_1 \ne y_2$, then $\psi'(y_1), \psi'(y_2) \in H' \subset H$ and $\psi'(y_1) \ne \psi'(y_2)$, $P\psi'(y_1) \ne P\psi'(y_2)$. Actually $P\psi'$ is a mapping from $\mathbf{R}^k$ to $\mathbf{R}^k$, and by Brouwer's theorem its image is an open set. For $u \in H' \subset H$ we have $\psi(Pu) = u$, and hence $P'\psi \colon Y \to Y'$ is a left inverse for $P\psi'$, and since $P'\psi$ is uniformly Lipschitz, $(P\psi')(Y') = Y$, $PH' = Y$, and $H' = H$. The case $\gamma' > \gamma$ is examined analogously. This completes the proof of Theorem 2.

§4. Some supplements and applications

We assume that the above-described apparatus can be used for the qualitative analysis of steady-state regimes of partial differential equations as well as for ODEs.

In connection with this it is necessary to make the standard remarks concerning the global nature of condition (2) on the nonlinearity F, which does not seem to be a realistic condition for specific problems. We assume that the Lipschitz condition (2) holds only locally (on bounded sets in E_ϑ), but there exists a closed convex bounded positively invariant set Ω in E_ϑ, on which (2) holds with the constant $L(\Omega)$. We note that for SPEs describing real physical processes without peaking regimes (i.e., without solutions that go to ∞ in finite time), such a set Ω usually exists. We extend F from Ω to E_ϑ to a uniformly Lipschitz mapping F_1 (ideally with the same constant $L(\Omega)$). Then equation (1) is equivalent in Ω to the equation $\dot{u} = -Au + F_1(u)$, where F_1 already satisfies a Lipschitz condition in the whole space E_ϑ, and the theorems on inertial manifolds can be used in the study of steady-state regimes of the semiflow $\{\Phi_t\}$ in Ω. In particular, one can speak about the asymptotic k-dimensionality of equation (1) on the positively invariant set $\Omega \subset E_\vartheta$.

We now consider applications of our approach to nonlinear diffusion equations (NDEs), restricting ourselves to the case $\vartheta = 0$ to start with. Consider the problem

$$\frac{\partial U}{\partial t} = d\Delta U + BU + f(U), \qquad U(0, x) = u_0(x) \tag{23a}$$

in a bounded domain $G \subset \mathbf{R}^N$ $(N \geq 1)$ with a sufficiently smooth boundary Γ. Here $U = U(t, x) = (U_1, \dots, U_m)$, $m \geq 1$; $t > 0$ and $x \in G$; B is a symmetric scalar $(m \times m)$ matrix, $f\colon \mathbf{R}^m \to \mathbf{R}^m$ is a locally Lipschitz mapping, and $f(0) = 0$; d is a diagonal scalar matrix with elements $d_i > 0$ on the diagonal, and Δ is the Laplace operator. The boundary condition is

$$(1 - \beta(x))\frac{\partial U}{\partial \mathbf{n}} + \beta(x)U|_\Gamma = 0, \tag{23b}$$

where $\mathbf{n}$ is the exterior normal to Γ, $\beta(x)$ a sufficiently smooth function on Γ, and $0 \leq \beta(x) \leq 1$.

We let $\{\mu_j\}_{j\geq 1}$ denote the eigenvalues of the operator $(-\Delta)$ with boundary condition (23b).

Usually in the study of NDEs one attempts to establish the existence of stationary or periodic solutions $U(t, x)$ which are stable as $t \to +\infty$ and inhomogeneous for $x \in G$. Here it is natural to consider the convergence of solutions in C-norms, not in L^2-norms.

Let $\Gamma_1 = \{x \in \Gamma : p(x) = 1\}$, and let

$$E = L^2(G; \mathbf{R}^m), \qquad C_0 = \{u \in C(\overline{G}; \mathbf{R}^m) : u|_{\Gamma_1} = 0\}$$

be spaces of vector-valued functions. The problem (23) can be written (formally) as an equation of type (1) in the Hilbert space E:

$$\dot{u} = (d\Delta + B)u + F(u), \qquad u(0) = u_0, \tag{24}$$

with $u(t) = U(t, \cdot)$ and $F(u)(x) = f(u(x))$. Mora [3], [19] showed that equation (24) generates a local semiflow $S_t\colon u_0 \to u(t)$, $0 \leq t < \omega(u_0) \leq \infty$ in the Banach space $C_0 \subset E$. Here, if u_0 belongs to Ω, a bounded domain in C_0, positively invariant for $\{S_t\}$, then the solutions $u(t)$ exist for $t \geq 0$, the semiflow $\{S_t\}_{t\geq 0}$ is completely continuous in Ω, and the corresponding semitrajectories are relatively compact in C_0.

If there exists an invariant manifold $H \subset E$ for equation (24) in the Hilbert space E, then $H \subset C_0$, which is a consequence of the smoothing action of the parabolic equation and the appropriate embedding theorems (see [11], Example 3.6, for an instance of similar arguments).

We now assume that the domain

$$\mathscr{D} = \{v \in \mathbf{R}^m : a_i < v_i < b_i , \ 1 \le i \le m\}$$

is positively invariant for the ODE $\dot{v} = Bv + f(v)$ in $\mathbf{R}^m$, and $O \in \mathscr{D}$. The usual arguments of maximum principle type for parabolic equations show (see also [20]) that the domain

$$\Omega = \{u \in C_0 : u(x) \in \mathscr{D} , \ x \in G\}$$

is positively invariant for the semiflow $\{S_t\}$. Let $\{\lambda_k\}_{k\ge 1}$ denote the eigenvalues of the family of matrices $(d\mu_j - B)$ $(j \ge 1)$, in increasing order, and let

$$L(\Omega) = \sup_{v\in\mathscr{D}} |f'(v)|_{\mathrm{sp}} , \tag{25}$$

where f' is the Jacobian matrix of f, and $|\cdot|_{\mathrm{sp}}$ is the spectral norm of matrices.

Let f_1 be a uniformly Lipschitz extension of $f(v)$ from $\mathscr{D}$ to $\mathbf{R}^m$ with the same Lipschitz constant $L(\Omega)$ (it is not hard to construct such an extension via reflections). Here $L(\Omega)$ is a Lipschitz constant for the nonlinear mapping $F_1(u)(x) = f_1(u(x))$ in the Hilbert space E. Now if there exists a k-dimensional inertial manifold H_1 for the SPE $\dot{u} = (d\Delta + B)u + F_1(u)$ in E, then $H_1 \subset C_0$ and $H_1 \ni 0$. In this case we call the local manifold $H = H_1 \cap \Omega \subset C_0$ $(H \ni 0$, since $\Omega \ni 0)$ an inertial manifold of the problem (23) in the domain $\Omega \subset C_0$, and we say that the NDE (23) is asymptotically k-dimensional in Ω. Simple arguments, based on the relative compactness of semitrajectories of the semiflow $\{S_t\}$ in the positively invariant domain Ω, show that the manifold H attracts Ω not only in L^2, but also in the C-norm.

From Theorem 1 we now get an assertion that strengthens similar results in [3], [4].

Theorem 6. *For the problem* (23) *suppose that* $k \ge 1$ *and*

$$\lambda_{k+1} - \lambda_k > 2L(\Omega) , \qquad \lambda_{k+1} > L(\Omega).$$

Then in Ω *there exists a* k*-dimensional inertial manifold attracting* Ω *in the* C*-norm.*

We exhibit these results using the example of the system of equations (see also [4])

$$\begin{aligned} \frac{\partial U_1}{\partial t} &= d_1\Delta U_1 + U_1 - (1+\delta)U_1^3 + \delta U_1 U_2^2 , \\ \frac{\partial U_2}{\partial t} &= d_2\Delta U_2 + U_2 - (1+\delta)U_2^3 - \delta U_1^2 U_2. \end{aligned} \tag{26}$$

Here

$$G = (0, \pi) , \qquad \delta = 1/12 , \qquad U_x(t, 0) = U(t, \pi) = 0 ,$$
$$\mu_j = (j - 1/2)^2 , \qquad \beta(0) = 0 , \qquad \beta(\pi) = 1 , \qquad \Gamma_1 = \{\pi\} ,$$
$$\Omega = \{u \in C_0 : |u_1(x)| < 1 , \ |u_2(x)| < 1\}.$$

In Ω there is one homogeneous stationary solution $u = 0$. We write this system in the form of a SPE (24) with $B = \kappa I$, where I is the (2×2) identity matrix, and κ is a real parameter chosen in order to minimize the value of $L(\Omega)$ in (25). For $\kappa = -15/24$, $L(\Omega) \le 15/8$. We also set $d_- = \min(d_1 , d_2)$, $d_+ = \max(d_1 , d_2)$, $r = 15$, $s = r/8$.

Applying Theorem 6, we find that the NDE (26) is asymptotically one-dimensional in the domain Ω for $d_- > s$, $d_+ - d_- > r$ and asymptotically two-dimensional in Ω for $d_- > s$, $d_+ + r < 9d_-$ and for $d_- > s/2$, $d_+ - r > 9d_-$.

Now using the elementary properties of ODEs in $\mathbf{R}^1$ and $\mathbf{R}^2$, and also the fact that the homogeneous stationary solution $u = 0$ is unstable for $d_- < 4$, we come to the following deductions about limit regimes of the system (26):

1) Let $s < d_- < 4$ and $d_+ - d_- > r$. Then in Ω there are no periodic solutions and there are at least two stable nonhomogeneous stationary solutions; moreover, the number of such solutions is even.

2) Let $s < d_- < 4$, $d_+ + r < 9d_-$, $d_+ - d_- \leq r$ or $s/2 < d_- \leq s$, $d_+ - r > 9d_-$. Then in Ω there is at least one stable nonhomogeneous stationary or periodic solution.

We point out that these deductions are *generically* true in the parameters (d_1, d_2).

§5. Conclusion

Conditions (7) and (9) for the asymptotic k-dimensionality of semilinear parabolic equations proposed in this paper appear to be rather constructive in the sense that all the parameters $(\vartheta, L, \{\lambda_k\})$ appearing in them are computable in principle or, in any case, lend themselves to estimates. Results about nonhomogeneous stationary or periodic solutions of NDEs, obtained via the construction of inertial manifolds of small dimension, can be complemented (but not in general overlaped) using the traditional methods for studying SPEs: bifurcation theory, rotations of completely continuous vector fields, et al.

We also note that the method of inertial manifolds can be used not only for the qualitative study of steady-state regimes, but also for the numerical solution of nonlinear parabolic equations (see [21] and references there).

Shortly after this paper was submitted for publication, the author learned of the paper [22], in which a construction of an inertial manifold was given that is similar to ours. However, the sufficient conditions presented in [22] for the existence of a k-dimensional inertial manifold with an "asymptotic completeness" property (of type (5a)) turn out to be more rigid in comparison with our conditions (7).

Very recently an interesting paper by Miklavčič [23] appeared, in which a result analogous to Theorem 1 is obtained (by an absolutely different method).

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