Proof of Hölder-Mañé theorem
(following Robinson, p. 407)

Theorem. Let X be separable Hilbert space, $A \subset X$ a compact set with finite fractal dimension. Let $m \in \mathbb{N}$ be such that $m > 2d_f(A) + 1$.
Then there exists a bounded linear map $L : X \to \mathbb{R}^m$ such that $L|_A$ is injective.
Proof. We will show that the set

$$\big\{L \in \mathcal{L}(X, \mathbb{R}^m);\ L|_A \text{ is not injective}\big\}$$

is first category, hence the set of injective projections is dense by Baire category theorem. Here $\mathcal{L}(X, \mathbb{R}^m)$ are bounded, linear maps $L : X \to \mathbb{R}^m$, i.e. a Banach space with the norm

$$\|L\|_{\mathcal{L}} = \sup\big\{|Lu|;\ x \in X,\ \|x\| \le 1\big\}.$$

STEP 1. Set

$$B = \big\{u - v;\ u,\, v \in A\big\}.$$

It follows that B is compact and $d_f(B) \le 2d_f(A) < m - 1$. Observe that L is injective on A if and only if $Ly \neq 0$ for any nonzero $y \in B$.

STEP 2. Define the sets

$$B_r = \big\{u \in B;\ |u| \ge 1/r\big\}$$
$$B_{r,j,n} = \big\{u \in B_r;\ |(e_j, u)| \ge 1/n\big\}$$

with the indices r, j, $n \in \mathbb{N}$ and $\{e_j\}_j$ is the Hilbert basis of X.
It is easy to verify that

$$\bigcup_r B_r = \bigcup_{r,j,n} B_{r,j,n} = \big\{u \in B;\ u \neq 0\big\}.$$

STEP 3. We further define

$$\mathcal{L}_{r,j,n} = \big\{L \in \mathcal{L}(X, \mathbb{R}^m);\ \ Lu \neq 0 \text{ for } \forall u \in B_{r,j,n}\ \big\}$$

It follows from the above that

$$\begin{aligned}\bigcap_{r,j,n} \mathcal{L}_{r,j,n} &= \big\{L \in \mathcal{L}(X, \mathbb{R}^m);\ \ Lu \neq 0 \text{ for } \forall u \in \bigcup_{r,j,n} B_{r,j,n}\big\}\\ &= \big\{L \in \mathcal{L}(X, \mathbb{R}^m);\ \ Lu \neq 0 \text{ for } \forall u \in B,\ u \neq 0\ \big\}\\ &= \big\{L \in \mathcal{L}(X, \mathbb{R}^m);\ L \text{ is injective on } A\big\}.\end{aligned}$$

Hence to finish the proof, we need to show that each of the sets $\mathcal{L}_{r,j,n}$ is open and dense in $\mathcal{L}(X,\mathbb{R}^m)$.

STEP 4. To show that $\mathcal{L}_{r,j,n}$ is open, fix $L \in \mathcal{L}_{r,j,n}$. We first note that $|Lu| > 0$, hence $|Lu| \geq \eta > 0$ on a compact set $B_{r,j,n}$. Further, let $R > 0$ be such that $|u| \leq R$ for $\forall u \in B_{r,j,n}$. We claim that $\mathcal{L}_{r,j,n}$ contains ε-neighborhood of L, where $\varepsilon = \eta/2R$.
Indeed, if $\|\tilde{L} - L\|_{\mathcal{L}} < \varepsilon$, then

$$|\tilde{L}u| \geq |Lu| - |(L-\tilde{L})u| \geq \eta - \varepsilon R > \varepsilon/2\,,$$

for any $u \in B_{r,j,n}$, hence $\tilde{L} \in \mathcal{L}_{r,j,n}$.

STEP 5. **Key observation:** for given $L_0 \in \mathcal{L}(X,\mathbb{R}^m)$ and $r \in \mathbb{N}$, there exists $z \in \mathbb{R}^m$, with $|z| = 1$ such that L_0u is NOT a non-zero multiple of z for any $u \in B_r$. In short

$$\big\{\lambda z;\ \lambda \in \mathbb{R}\setminus\{0\}\big\} \cap L_0B_r = \emptyset \tag{1}$$

Geometrically: we can find z on the unit sphere such that the line passing through z never intersects L_0B_r with the exception of the origin. The argument (which we postpone to the next step) is based on the fact that the dimension of L_0B_r is strictly smaller than the dimension of the unit sphere.

Once the key observation is proved, the density of $\mathcal{L}_{r,j,n}$ follows easily. Given L_0 and $\varepsilon > 0$, we define $L \in \mathcal{L}(X,\mathbb{R}^m)$ by

$$Lu = L_0u + \varepsilon(u,e_j)z\,.$$

It is clear that $\|L - L_0\|_{\mathcal{L}} \leq \varepsilon$ and we claim that $L \in \mathcal{L}_{r,j,n}$. If not, there exists $u \in B_{r,j,n}$ such that $Lu = 0$, i.e.

$$L_0y = -\varepsilon(u,e_j)z\,.$$

However, $(u,e_j) \neq 0$ as $u \in B_{r,j,n}$; this contradicts (1).

STEP 6. It remains to prove the "key observation" from previous step. This is equivalent to showing that

$$\phi(L_0B_r \setminus \{0\}) \neq S_{m-1}\,,$$

where S_m is the unit sphere

$$S_m = \big\{z \in \mathbb{R}^m;\ |z| = 1\big\}$$

and $\phi : \mathbb{R}^m \setminus \{0\} \to S_{m-1}$ is the "projection" $\phi : x \mapsto x/|x|$.

We set $M = L_0 B_r \setminus \{0\}$. Clearly $d_f(M) \le d_f(B) < m-1$. If ϕ would be Lipschitz (which is "almost true"), we would have $d_f(\phi(M)) < m-1$, hence $\phi(M)$ cannot equal to S_{m-1} which has dimension $m-1$.
However, ϕ is not Lipschitz close to the origin, and we have to elaborate here a bit. We set

$$M_k = \left\{ x \in M;\ |x| \ge 1/k \right\}.$$

Clearly $\phi(M) = \phi(\cup_k M_k) = \cup_k \phi(M_k)$. Now ϕ is Lipschitz on M_k, and M_k is compact. Hence $\phi(M_k)$ is compact subset of S_{m-1} with $d_f(\phi(M_k)) < m-1$. It follows that $\phi(M_k)$ has empty interior (with respect to S_{m-1}); hence $\phi(M) = \cup_k \phi(M_k)$ cannot fill S_m by Baire category theorem.

Remark. The end of the proof can simplified, using the properties of Hausdorff dimension d_h.
Indeed, $d_h(\phi(M_k)) \le d_f(\phi(M_k)) \le d_f(M)$. Hence, as d_h is countably subaditive (unlike d_f),

$$d_h(\phi(M)) = \sup_k d_h(\phi(M_k)) \le d_f(M) < m-1\,.$$

Hence $\phi(M) \ne S_{m-1}$ as required.