(A1) $f(x,y) = x\sqrt[3]{y}$ [6b]

(a) f je součin spojitých funkcí $(x,y) \longmapsto x$
(v celém $\mathbb{R}^2$) $(x,y) \longmapsto \sqrt[3]{y}$. [1]

(b) $$\lim_{t\to 0} \frac{1}{t}\left[f(2+t,1+2t) - f(2,1)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[(2+t)\sqrt[3]{1+2t} - 2\right] \quad \text{l'Hosp. } „\frac{0}{0}"$$

$$= \lim_{t\to 0} \frac{1}{1}\cdot\Big[\underbrace{\sqrt[3]{1+2t}}_{\to 1} + \underbrace{(2+t)}_{\to 2}\cdot \underbrace{\frac{1}{3}\left(\sqrt[3]{(1+2t)^2}\right)^{-1}\cdot 2}_{\to \frac{2}{3}}\Big] = \frac{7}{3}.$$

[2]

(c) $$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0}\frac{1}{t}\big[\underbrace{f(t,0) - f(0,0)}_{\equiv 0}\big] = 0;$$

$$\frac{\partial f}{\partial y}(0,0) = 0.$$ [1]

Kandidát na $df(0,0)$: $L:(u,v) \longmapsto 0$.

ověření: $$\lim_{(u,v)\to(0,0)} \frac{f(u,v) - f(0,0) - L(u,v)}{\sqrt{u^2+v^2}} = 0 \quad ??$$

$$\frac{f(u,v) - f(0,0) - L(u,v)}{\sqrt{u^2+v^2}} = \frac{u\sqrt[3]{v}}{\sqrt{u^2+v^2}} = \underbrace{\frac{u}{\sqrt{u^2+v^2}}}_{|\ | \leq 1}\cdot\underbrace{\sqrt[3]{v}}_{\to 0}.$$

ANO; $df(0,0) = 0$.

[2]

$\frac{\partial f}{\partial y} = x\cdot\frac{1}{3\sqrt[3]{y^2}} \leftarrow$ nespojitá v $(0,0)$.

(A2) $F_1(x,y,z) = 2x + y^2 z + z^3$ [6b]

$F_2(x,y,z) = x^3 + 2xy + 3z$.

předpoklady VIF: $F_1(1,1,-1) = 2-1-1=0$

$F_2(1,1,-1) = 1+2-3=0$

[1]

$F_1, F_2 \in C^\infty(\mathbb{R}^3)$.

klíčový předpoklad: $\begin{pmatrix} \frac{\partial F_1}{\partial y}, & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y}, & \frac{\partial F_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 2yz, & y^2+3z^2 \\ 2x, & 3 \end{pmatrix}$

pro $(x,y,z) = (1,1,-1)$: $\begin{pmatrix} -2, & 4 \\ 2, & 3 \end{pmatrix}$ .. regulární. [2]

$\Rightarrow \exists\ C^\infty$ funkce: $Y(x), Z(x)$ tak, že na okolí

$(1,1,-1)$ platí $F_1(x,y,z)=0$, $F_2(x,y,z)=0$ $\iff$ $y = Y(x)$, $z = Z(x)$.

$x\ y\ z$

víme tedy, že $Y(1) = 1$; $Z(1) = -1$. [1]

výpočet derivací: platí identity $(x \in \mathcal{U}(1))$.

$$2x + Y^2(x)Z(x) + Z^3(x) = 0$$
$$x^3 + 2xY(x) + 3Z(x) = 0 \quad \Big/ \ \frac{d}{dx}$$

$$2 + 2YY'Z + Y^2Z' + 3Z^2Z' = 0$$
$$3x^2 + 2Y + 2xY' + 3Z' = 0$$

dosadíme: $x=1$; $Y(1)=1$, $Z(1)=-1$.

[2]

(B1) $f(x,y) = \dfrac{\ln(1+x^2+y^2)}{\underbrace{2x^2+Ay^2}_{\text{spoj.};\ >0\ \text{v}\ \mathbb{R}^2\setminus\{(0,0)\}}} - \sin x\ ;\qquad A>0.$

$\sin x$: spoj. v $\mathbb{R}^2$

[1]

(b) $\lim\limits_{t\to 0} f(t,0) = \lim\limits_{t\to 0} \dfrac{\ln(1+t^2)}{2t^2} - \sin t = \dfrac{1}{2}$

$\lim\limits_{t\to 0} f(0,t) = \lim\limits_{t\to 0} \dfrac{\ln(1+t^2)}{At^2} = \dfrac{1}{A}.$ nutně $A=2$.

[1]

polož $A=2$: $f(x,y) = \dfrac{\ln(1+x^2+y^2)}{2(x^2+y^2)} - \sin x$

polární souřadnice: $f(r,\varphi) = \dfrac{\ln(1+r^2)}{2r^2} - \sin(r\cos\varphi) \to \dfrac{1}{2}$ [2]

(c) dodefinuji: $f(0,0) = \frac{1}{2}.$; $\Rightarrow$ f je spojitá v $\mathbb{R}^2$.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{1}{t}\left[f(t,0) - f(0,0)\right]$$

$$= \frac{1}{t}\left[\frac{\ln(1+t^2)}{2t^2} - \frac{1}{2} + \sin t\right]$$

$$= \underbrace{\frac{\ln(1+t^2) - t^2}{2t^3}}_{\to 0;} - \underbrace{\frac{\sin t}{t}}_{\to -1} \to \underline{-1}.$$

neboť $\ln(1+t^2) = t^2 + \mathcal{O}(t^4)$

podobně $\dfrac{\partial f}{\partial y}(0,0) = \underline{0}.$ [2]

(B2) $F(x,y,z) = xyz - xy + 2yz - 4xz$;

$F(1,2,1) = 2 - 2 + 4 - 4 = 0$ [1]

$F \in C^\infty$

$\frac{\partial F}{\partial z} = xy + 2y - 4x \Big|_{(1,2,1)} = 2 + 4 - 4 = \underline{2} \neq 0$ [2]

$\Rightarrow$ $\exists$ lokálně $Z = Z(x,y)$ ~~tak~~, že pro (x,y,z) v okolí $(1,2,1)$ platí $F(x,y,z) = 0$ $\Leftrightarrow$ $z = Z(x,y)$.

také je $Z(1,2) = 1$.

? derivace: identita $xy\,Z(x,y) - xy + 2y\,Z(x,y) - 4x\,Z(x,y) = 0$

$\frac{\partial}{\partial x}$: $yZ + xy\frac{\partial Z}{\partial x} - y + 2y\frac{\partial Z}{\partial x} - 4Z - 4x\frac{\partial Z}{\partial x} = 0$

$\frac{\partial}{\partial y}$: $xZ + xy\frac{\partial Z}{\partial y} - x + 2Z + 2y\frac{\partial Z}{\partial y} - 4x\frac{\partial Z}{\partial y} = 0$ [1]

dosadí: $x = 1,\ y = 2;\ Z = (1,2) = 1$:

1. rce $\Rightarrow$ $2\,\frac{\partial Z}{\partial x}(1,2) - 4 = 0$ $\Rightarrow$ $\frac{\partial Z}{\partial x}(1,2) = 2$

2. rce $\Rightarrow$ $2\,\frac{\partial Z}{\partial y}(1,2) + 2 = 0$; $\Rightarrow$ $\frac{\partial Z}{\partial y}(1,2) = -1$.

[2]