# Mathematics II

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- Functions of several variables

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- Antiderivative and the Riemann Integral

# V.1. $\mathbb{R}^n$ as a linear and metric space

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## Definition

The set $\mathbb{R}^n$, $n \in \mathbb{N}$, is the set of all ordered $n$-tuples of real numbers, i.e.

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For $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ we set

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n), \qquad \alpha\boldsymbol{x} = (\alpha x_1, \dots, \alpha x_n).$$

Further, we denote $\boldsymbol{o} = (0, \dots, 0)$ – the origin.

## Properties of $+, \cdot$

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ (commutativity)
2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ (associativity)
3. $\exists \boldsymbol{o} \in \mathbb{R}^n \, \forall x \in \mathbb{R}^n : \quad \boldsymbol{x} + \boldsymbol{o} = \boldsymbol{x}$ (existence of neutral element)
4. $\forall x \in \mathbb{R}^n \, \exists (-x) \in \mathbb{R}^n : \quad x + (-x) = \boldsymbol{o}$ (existence of opposite element)
5. $\alpha(\beta \boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$, for any $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^n$

## Comment

Properties 1- 4 tells that $\mathbb{R}^n$ is a group with respect to the operation $+$

## Definition

The Euclidean metric (distance) on $\mathbb{R}^n$ is the function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\ .$$

The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called the distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$.

## Theorem 1 (properties of the Euclidean metric)

*The Euclidean metric $\rho$ has the following properties:*

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

## Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

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(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

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(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}\colon \rho(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

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(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n \colon \rho(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

## Remark

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## Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if

$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

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## Remark

The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x} \in \mathbb{R}^n$ if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists j_0 \in \mathbb{N} \ \forall j \in \mathbb{N}, j \geq j_0 \colon \boldsymbol{x}^j \in U(\boldsymbol{x}, \varepsilon).$$

## Theorem 2 (convergence is coordinatewise)

*Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number $x_i$.*

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Theorem 2 says that the convergence in the space $\mathbb{R}^n$ is the same as the “coordinatewise” convergence. It follows that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ has at most one limit. If it exists, then we denote it by $\lim_{j\to\infty} \boldsymbol{x}^j$. Sometimes we also write simply $\boldsymbol{x}^j \to \boldsymbol{x}$ instead of $\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

## Definition

Let $\boldsymbol{x} \in \mathbb{R}^n,\ r \in \mathbb{R}, r > 0$. The set $U(\boldsymbol{x}, r)$ defined by

$$U(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called an open ball with radius $r$ centred at $\boldsymbol{x}$ or the $r$-neighbourhood of $\boldsymbol{x}$.

Another notations for $U(\boldsymbol{x}, r)$: $U_r(\boldsymbol{x}), B(\boldsymbol{x}, r)$.

## Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of $M$, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

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Exterior of $M$ is $\operatorname{Ext} M = \operatorname{Int}(M^C)$, where $M^C = \mathbb{R}^n \setminus M$ is the complement of $M$.

## Theorem 3 (properties of open sets)

(i) *The empty set and* $\mathbb{R}^n$ *are open in* $\mathbb{R}^n$*.*

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(ii) *Let $G_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be open in $\mathbb{R}^n$. Then $\bigcup_{\alpha \in A} G_\alpha$ is open in $\mathbb{R}^n$.*

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(iii) *Let* $G_i \subset \mathbb{R}^n$*,* $i = 1, \ldots, m$*, be open in* $\mathbb{R}^n$*. Then* $\bigcap_{i=1}^m G_i$ *is open in* $\mathbb{R}^n$*.*

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(ii) *A union of an arbitrary system of open sets is an open set.*
(iii) *An intersection of a finitely many open sets is an open set.*

## Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of $M$ if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

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A set $M \subset \mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$ if it contains all its boundary points, i.e. if $\operatorname{bd} M \subset M$, or in other words if $\overline{M} = M$.

## Proposition

$x \in \operatorname{bd} M$ iff $x \notin \operatorname{Int} M$ and $x \notin \operatorname{Ext} M$.

## Consequence

$\mathbb{R}^n = \operatorname{Int} M \sqcup \operatorname{Ext} M \sqcup \operatorname{bd} M$ (here $A \sqcup B = A \cup B, A \cap B = \emptyset$ is disjoint union).

$\operatorname{bd} M = \operatorname{bd} M^C$.

## Theorem 4 (characterisation of closed sets)

*Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:*

(i) *$M$ is closed in $\mathbb{R}^n$.*

(ii) *$\mathbb{R}^n \setminus M$ is open in $\mathbb{R}^n$.*

(iii) *Any $\boldsymbol{x} \in \mathbb{R}^n$ which is a limit of a sequence from $M$ belongs to $M$.*

## Theorem 5 (properties of closed sets)

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## Remark

(ii) *An intersection of an arbitrary system of closed sets is closed.*

(iii) *A union of finitely many closed sets is closed.*

## Lemma 6

(i) *For any set $M$ we have* $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.

(ii) *If $M \subset \mathbb{R}^n$ is closed, then* $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.

## Theorem 7

*Let $M \subset \mathbb{R}^n$. Then the following holds:*

(i) *The set* $\operatorname{Int} M$ *is open in* $\mathbb{R}^n$ *(i.e.* $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$*).*

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We have $\operatorname{Int} M \subset M \subset \overline{M}$. Moreover, the set $\operatorname{Int} M$ is the largest open set contained in $M$ in the following sense: If $G$ is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$.

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## Lemma 8 (monotonicity of taking interior and closure)

*If* $A \subset B \subset \mathbb{R}^n$, *then*

(i) $\operatorname{Int} A \subset \operatorname{Int} B$,

(ii) $\overline{A} \subset \overline{B}$.

## Exercise 1

Let $A \sqcup B = \mathbb{R}^n$. Then

(a) $\operatorname{Int} A = \mathbb{R}^n \setminus \overline{B}$, (*b*) $\overline{A} = \mathbb{R}^n \setminus \operatorname{Int} B$.

## Exercise 2

Let $A, B \subset \mathbb{R}^2$. Prove that (a) $\overline{\operatorname{bd} A} = \operatorname{bd} A$;
(b) $\overline{A} = \operatorname{Int} A \Longrightarrow ((A = \mathbb{R}^2) \vee (A = \emptyset))$; (c) disprove the equality $\overline{A \cap B} = \overline{A} \cap \overline{B}$, change the equality sign to a correct inclusion and prove the latter.