Mathematics II

Mathematics II

- Functions of several variables

Mathematics II

- Functions of several variables
- Matrix calculus

Mathematics II

- Functions of several variables
- Matrix calculus
- Antiderivative and the Riemann Integral

V.1. $\mathbb{R}^n$ as a linear and metric space

V.1. $\mathbb{R}^n$ as a linear and metric space

Definition

The set $\mathbb{R}^n$, $n \in \mathbb{N}$, is the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \{(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}.$$

V.1. $\mathbb{R}^n$ as a linear and metric space

Definition

The set $\mathbb{R}^n$, $n \in \mathbb{N}$, is the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}.$$

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ we set

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n), \qquad \alpha\boldsymbol{x} = (\alpha x_1, \ldots, \alpha x_n).$$

Further, we denote $\boldsymbol{o} = (0, \ldots, 0)$ – the origin.

Properties of $+, \cdot$

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ (commutativity)
2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ (associativity)
3. $\exists \boldsymbol{o} \in \mathbb{R}^n \, \forall x \in \mathbb{R}^n : \quad \boldsymbol{x} + \boldsymbol{o} = \boldsymbol{x}$ (existence of neutral element)
4. $\forall x \in \mathbb{R}^n \, \exists (-x) \in \mathbb{R}^n : \quad x + (-x) = \boldsymbol{o}$ (existence of opposite element)
5. $\alpha(\beta \boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$, for any $\alpha, \beta \in \mathbb{R}$ and $x \in \mathbb{R}^n$

Comment

Properties 1- 4 tells that $\mathbb{R}^n$ is a group with respect to the operation $+$

Definition

The Euclidean metric (distance) on $\mathbb{R}^n$ is the function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\ .$$

The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called the distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n \colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R} \colon \rho(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) \leq \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}\colon \rho(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}) = |\lambda|\rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}+\boldsymbol{z}, \boldsymbol{y}+\boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

Remark

Properties I-III make the pair $(\mathbb{R}^n, \rho)$ a metric space.

Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if

$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

The vector $\boldsymbol{x}$ is called the limit of the sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$.

Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if

$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

The vector $\boldsymbol{x}$ is called the limit of the sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$. The sequence $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ of points in $\mathbb{R}^n$ is called convergent if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{y}$.

Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ **converges to $\boldsymbol{x}$**, if

$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

The vector $\boldsymbol{x}$ is called the **limit of the sequence** $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$. The sequence $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ of points in $\mathbb{R}^n$ is called **convergent** if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{y}$.

Remark

The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x} \in \mathbb{R}^n$ if and only if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists j_0 \in \mathbb{N}\ \forall j \in \mathbb{N}, j \geq j_0 \colon \boldsymbol{x}^j \in U(\boldsymbol{x}, \varepsilon).$$

Theorem 2 (convergence is coordinatewise)

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

Theorem 2 (convergence is coordinatewise)

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

Remark

Theorem 2 says that the convergence in the space $\mathbb{R}^n$ is the same as the "coordinatewise" convergence.

Theorem 2 (convergence is coordinatewise)

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^{\infty}$ converges to the real number x_i.

Remark

Theorem 2 says that the convergence in the space $\mathbb{R}^n$ is the same as the “coordinatewise” convergence. It follows that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ has at most one limit. If it exists, then we denote it by $\lim_{j\to\infty} \boldsymbol{x}^j$. Sometimes we also write simply $\boldsymbol{x}^j \to \boldsymbol{x}$ instead of $\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

Definition

Let $\boldsymbol{x} \in \mathbb{R}^n,\ r \in \mathbb{R}, r > 0$. The set $U(\boldsymbol{x}, r)$ defined by

$$U(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called an open ball with radius r centred at $\boldsymbol{x}$ or the r-neighbourhood of $\boldsymbol{x}$.

Another notations for $U(\boldsymbol{x}, r)$: $U_r(\boldsymbol{x}), B(\boldsymbol{x}, r)$.

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

The set of all interior points of M is called the interior of M and is denoted by $\operatorname{Int} M$.

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

The set of all interior points of M is called the interior of M and is denoted by $\operatorname{Int} M$.

The set $M \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$, if each point of M is an interior point of M, i.e. if $M = \operatorname{Int} M$.

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

The set of all interior points of M is called the interior of M and is denoted by $\operatorname{Int} M$.

The set $M \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$, if each point of M is an interior point of M, i.e. if $M = \operatorname{Int} M$.

The system of all open sets are denoted $\mathfrak{G}(\mathbb{R}^n)$.

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $U(\boldsymbol{x}, r) \subset M$.

The set of all interior points of M is called the interior of M and is denoted by $\operatorname{Int} M$.

The set $M \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$, if each point of M is an interior point of M, i.e. if $M = \operatorname{Int} M$.

The system of all open sets are denoted $\mathfrak{G}(\mathbb{R}^n)$.

Exterior of M is $\operatorname{Ext} M = \operatorname{Int}(M^C)$, where $M^C = \mathbb{R}^n \setminus M$ is the complement of M.

Theorem 3 (properties of open sets)

(i) *The empty set and $\mathbb{R}^n$ are open in $\mathbb{R}^n$.*

Remark

Theorem 3 (properties of open sets)

(i) *The empty set and $\mathbb{R}^n$ are open in $\mathbb{R}^n$.*

(ii) *Let $G_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be open in $\mathbb{R}^n$. Then $\bigcup_{\alpha \in A} G_\alpha$ is open in $\mathbb{R}^n$.*

Remark

(ii) *A union of an arbitrary system of open sets is an open set.*

Theorem 3 (properties of open sets)

(i) *The empty set and $\mathbb{R}^n$ are open in $\mathbb{R}^n$.*

(ii) *Let $G_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be open in $\mathbb{R}^n$. Then $\bigcup_{\alpha\in A} G_\alpha$ is open in $\mathbb{R}^n$.*

(iii) *Let $G_i \subset \mathbb{R}^n$, $i = 1, \dots, m$, be open in $\mathbb{R}^n$. Then $\bigcap_{i=1}^m G_i$ is open in $\mathbb{R}^n$.*

Remark

(ii) *A union of an arbitrary system of open sets is an open set.*
(iii) *An intersection of a finitely many open sets is an open set.*

Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

The boundary of M is the set of all boundary points of M (notation $F_r(M)$ or $H(M)$ or $\operatorname{bd} M$).

Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

The boundary of M is the set of all boundary points of M (notation $F_r(M)$ or $H(M)$ or $\operatorname{bd} M$).

The closure of M is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$).

Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$U(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad U(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

The boundary of M is the set of all boundary points of M (notation $F_r(M)$ or $H(M)$ or $\operatorname{bd} M$).

The closure of M is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$).

A set $M \subset \mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$ if it contains all its boundary points, i.e. if $\operatorname{bd} M \subset M$, or in other words if $\overline{M} = M$.

Proposition

$x \in \operatorname{bd} M$ iff $x \notin \operatorname{Int} M$ and $x \notin \operatorname{Ext} M$.

Consequence

$\mathbb{R}^n = \operatorname{Int} M \sqcup \operatorname{Ext} M \sqcup \operatorname{bd} M$ (here $A \sqcup B = A \cup B, A \cap B = \emptyset$ is disjoint union).

$\operatorname{bd} M = \operatorname{bd} M^C$.

Theorem 4 (characterisation of closed sets)

Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:

(i) *M is closed in $\mathbb{R}^n$.*

(ii) *$\mathbb{R}^n \setminus M$ is open in $\mathbb{R}^n$.*

(iii) *Any $\boldsymbol{x} \in \mathbb{R}^n$ which is a limit of a sequence from M belongs to M.*

Theorem 5 (properties of closed sets)

(i) *The empty set and the whole space $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.*

Remark

Theorem 5 (properties of closed sets)

(i) *The empty set and the whole space $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.*

(ii) *Let $F_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be closed in $\mathbb{R}^n$. Then $\bigcap_{\alpha \in A} F_\alpha$ is closed in $\mathbb{R}^n$.*

Remark

(ii) *An intersection of an arbitrary system of closed sets is closed.*

Theorem 5 (properties of closed sets)

(i) *The empty set and the whole space $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.*

(ii) *Let $F_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be closed in $\mathbb{R}^n$. Then $\bigcap_{\alpha \in A} F_\alpha$ is closed in $\mathbb{R}^n$.*

(iii) *Let $F_i \subset \mathbb{R}^n$, $i = 1, \dots, m$, be closed in $\mathbb{R}^n$. Then $\bigcup_{i=1}^m F_i$ is closed in $\mathbb{R}^n$.*

Remark

(ii) *An intersection of an arbitrary system of closed sets is closed.*

(iii) *A union of finitely many closed sets is closed.*

Lemma 6

(i) *For any set M we have* $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.

(ii) *If $M \subset \mathbb{R}^n$ is closed, then* $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.

Theorem 7

Let $M \subset \mathbb{R}^n$. Then the following holds:

(i) *The set* $\operatorname{Int} M$ *is open in* $\mathbb{R}^n$ *(i.e.* $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$*).*

(ii) *The set* $\overline{M}$ *is closed in* $\mathbb{R}^n$ *(i.e.* $\overline{\overline{M}} = \overline{M}$*).*

Lemma 6

(i) *For any set M we have $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.*

(ii) *If $M \subset \mathbb{R}^n$ is closed, then $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.*

Theorem 7

Let $M \subset \mathbb{R}^n$. Then the following holds:

(i) *The set $\operatorname{Int} M$ is open in $\mathbb{R}^n$ (i.e. $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$).*

(ii) *The set $\overline{M}$ is closed in $\mathbb{R}^n$ (i.e. $\overline{\overline{M}} = \overline{M}$).*

Remark

We have $\operatorname{Int} M \subset M \subset \overline{M}$. Moreover, the set $\operatorname{Int} M$ is the largest open set contained in M in the following sense: If G is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$.

Lemma 6

(i) *For any set M we have* $\overline{M} = \operatorname{Int} M \sqcup \operatorname{bd} M$.

(ii) *If* $M \subset \mathbb{R}^n$ *is closed, then* $M = \operatorname{Int} M \sqcup \operatorname{bd} M$.

Theorem 7

Let $M \subset \mathbb{R}^n$. *Then the following holds:*

(i) *The set* $\operatorname{Int} M$ *is open in* $\mathbb{R}^n$ *(i.e.* $\operatorname{Int}(\operatorname{Int} M) = \operatorname{Int} M$*).*

(ii) *The set* $\overline{M}$ *is closed in* $\mathbb{R}^n$ *(i.e.* $\overline{\overline{M}} = \overline{M}$*).*

Remark

We have $\operatorname{Int} M \subset M \subset \overline{M}$. Moreover, the set $\operatorname{Int} M$ is the largest open set contained in M in the following sense: If G is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$. Similarly $\overline{M}$ is the smallest closed set containing M.

Lemma 8 (monotonicity of taking interior and closure)

If $A \subset B \subset \mathbb{R}^n$, then

(i) $\operatorname{Int} A \subset \operatorname{Int} B$,

(ii) $\overline{A} \subset \overline{B}$.

Exercise 1

Let $A \sqcup B = \mathbb{R}^n$. Then

(a) $\operatorname{Int} A = \mathbb{R}^n \setminus \overline{B}$, (b) $\overline{A} = \mathbb{R}^n \setminus \operatorname{Int} B$.

Exercise 2

Let $A, B \subset \mathbb{R}^2$. Prove that (a) $\overline{\operatorname{bd} A} = \operatorname{bd} A$;
(b) $\overline{A} = \operatorname{Int} A \Longrightarrow ((A = \mathbb{R}^2) \vee (A = \emptyset))$; (c) disprove the equality $\overline{A \cap B} = \overline{A} \cap \overline{B}$, change the equality sign to a correct inclusion and prove the latter.

Definition

We say that the set $M \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $M \subset U(\boldsymbol{o}, r)$.

Definition

We say that the set $M \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $M \subset U(\boldsymbol{o}, r)$. A sequence of points in $\mathbb{R}^n$ is bounded if the set of its members is bounded.

Definition

We say that the set $M \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $M \subset U(\boldsymbol{o}, r)$. A sequence of points in $\mathbb{R}^n$ is bounded if the set of its members is bounded.

Theorem 9

A set $M \subset \mathbb{R}^n$ is bounded if and only if its closure $\overline{M}$ is bounded.

Definition

We say that a point $\boldsymbol{a} \in \mathbb{R}^n$ is an accumulation point (or condensation point) of a function $f : D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}^n$, if every neighborhood of $\boldsymbol{a}$ contains at least one point of D_f, not equal to $\boldsymbol{a}$. The point $\boldsymbol{a}$ might be or might not be a point of D_f.

Definition

We say that a function f of n variables has a limit at a point $\boldsymbol{a} \in \mathbb{R}^n$ equal to $A \in \mathbb{R}^*$ if $\boldsymbol{a}$ is an accumulation point of D_f and

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \boldsymbol{x} \in U(\boldsymbol{a}, \delta) \cap D_f \setminus \{\boldsymbol{a}\} \colon f(\boldsymbol{x}) \in U(A, \varepsilon).$$

Remark

- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.

Remark

- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.
- For limits of functions of several variables one can prove similar theorems as for limits of functions of one variable (arithmetics, the sandwich theorem, ...).

Theorem 10

Let $r, s \in \mathbb{N}$, $\boldsymbol{a} \in \mathbb{R}^s$, *and let* $\varphi_1, \dots, \varphi_r$ *be functions of* s *variables such that* $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \varphi_j(\boldsymbol{x}) = b_j$, $j = 1, \dots, r$. *Denote* $\boldsymbol{b} = (b_1, \dots, b_r)$. *Let* f *be a function of* r *variables which is continuous at the point* $\boldsymbol{b}$. *If we define a compound function* F *of* s *variables by*

$$F(\boldsymbol{x}) = f(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})),$$

then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} F(\boldsymbol{x}) = f(\boldsymbol{b})$.

V.2. Continuous functions of several variables

V.2. Continuous functions of several variables

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbb{R}$. We say that f is continuous at $\boldsymbol{x}$ with respect to M, if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall \boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M\colon f(\boldsymbol{y}) \in U(f(\boldsymbol{x}), \varepsilon).$$

V.2. Continuous functions of several variables

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbb{R}$. We say that f is continuous at $\boldsymbol{x}$ with respect to M, if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall \boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M\colon f(\boldsymbol{y}) \in U(f(\boldsymbol{x}), \varepsilon).$$

We say that f is continuous at the point $\boldsymbol{x}$ if it is continuous at $\boldsymbol{x}$ with respect to a neighbourhood of $\boldsymbol{x}$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists \delta \in \mathbb{R}, \delta > 0\ \forall \boldsymbol{y} \in U(\boldsymbol{x}, \delta)\colon f(\boldsymbol{y}) \in U(f(\boldsymbol{x}), \varepsilon).$$

Remark

If $\boldsymbol{a}$ is an accumulation point of D_f, then the function f is continuous at $\boldsymbol{a}$ if and only if $\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$.

Other neighborhoods

In the definition of continuity, the circle $U(x, \delta)$ can be changed with the cube $A(x, \delta)$, where
$A(x, r) = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : \quad |y_j - x_j| < r\}$.
We have $U(x, r) \subset A(x, r) \subset U(x, r\sqrt{n})$.

Theorem 11 (Heine)

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f\colon M \to \mathbb{R}$. Then the following are equivalent.

(i) *The function f is continuous at $\boldsymbol{x}$ with respect to M.*

(ii) $\lim\limits_{j\to\infty} f(\boldsymbol{x}^j) = f(\boldsymbol{x})$ *for each sequence* $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ *such that* $\boldsymbol{x}^j \in M$ *for* $j \in \mathbb{N}$ *and* $\lim\limits_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

Examples

1. $f(x,y) = \begin{cases} \frac{x}{y}, & y \neq 0, \\ 0, & y = 0. \end{cases}$

   $M_1 = \mathbb{R}^2; \quad M_2 = \mathbb{R} \times \{0\}\,; \quad M_3 = \{0\} \times \mathbb{R};$
   $M_4 = \{(x,x) :\ x \in \mathbb{R}\}\,.$
2. $f(x,y) = \frac{xy}{x^2+y^2}.$
3. $f(x,y) = \frac{x^2-y^2}{x^2+y^2}; \qquad f(x,y) = \frac{x^2y}{x^2+y^2}$

Repeated limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq \lim_{x\to x_0} \lim_{y\to y_0} f(x,y) \neq \lim_{y\to y_0} \lim_{x\to x_0} f(x,y).$$

Example

$f(x,y) = \frac{x-y+x^2+y^2}{x+y}.$

Theorem 12

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, $f\colon M \to \mathbb{R}$, $g\colon M \to \mathbb{R}$, and $c \in \mathbb{R}$. If f and g are continuous at the point $\boldsymbol{x}$ with respect to M, then the functions cf, $f+g$ a fg are continuous at $\boldsymbol{x}$ with respect to M. If the function g is nonzero at $\boldsymbol{x}$, then also the function f/g is continuous at $\boldsymbol{x}$ with respect to M.

Theorem 13

Let $r, s \in \mathbb{N}$, $M \subset \mathbb{R}^s$, $L \subset \mathbb{R}^r$, and $\boldsymbol{y} \in M$. Let $\varphi_1, \dots, \varphi_r$ be functions defined on M, which are continuous at $\boldsymbol{y}$ with respect to M and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in L$ for each $\boldsymbol{x} \in M$. Let $f\colon L \to \mathbb{R}$ be continuous at the point $[\varphi_1(\boldsymbol{y}), \dots, \varphi_r(\boldsymbol{y})]$ with respect to L. Then the compound function $F\colon M \to \mathbb{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in M,$$

is continuous at $\boldsymbol{y}$ with respect to M.

Definition

Let $M \subset \mathbb{R}^n$ and $f\colon M \to \mathbb{R}$. We say that f is continuous on M if it is continuous at each point $\boldsymbol{x} \in M$ with respect to M.

Definition

Let $M \subset \mathbb{R}^n$ and $f\colon M \to \mathbb{R}$. We say that f is continuous on M if it is continuous at each point $\boldsymbol{x} \in M$ with respect to M.

Remark

The functions $\pi_j\colon \mathbb{R}^n \to \mathbb{R}$, $\pi_j(\boldsymbol{x}) = x_j$, $1 \leq j \leq n$, are continuous on $\mathbb{R}^n$. They are called coordinate projections.

Exercise 3

Every open set in $\mathbb{R}$ can be represented as at most countable disjoint union of open intervals.

Theorem 14

Let f be a continuous function on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following holds:

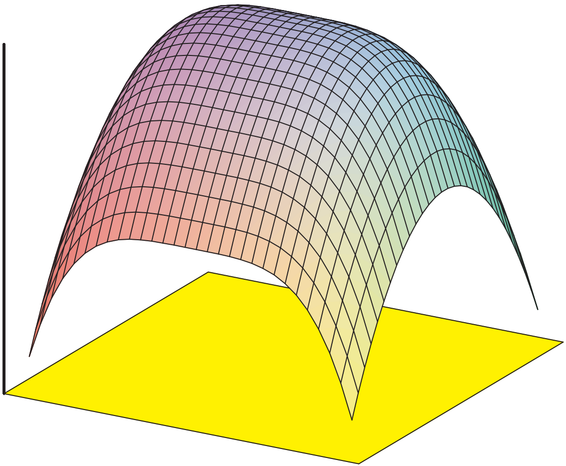
(i) *The set $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) < c\}$ is open in $\mathbb{R}^n$.*

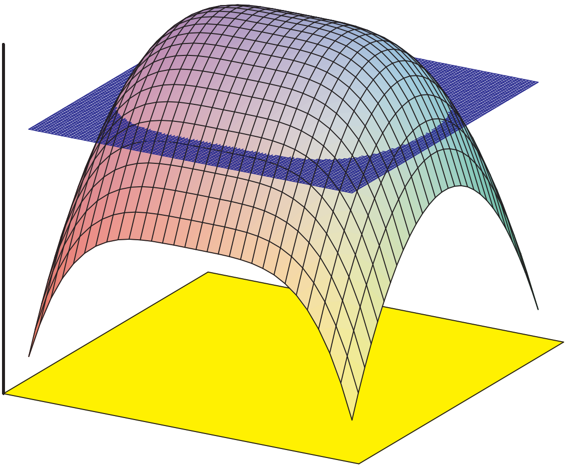
(ii) *The set $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) > c\}$ is open in $\mathbb{R}^n$.*

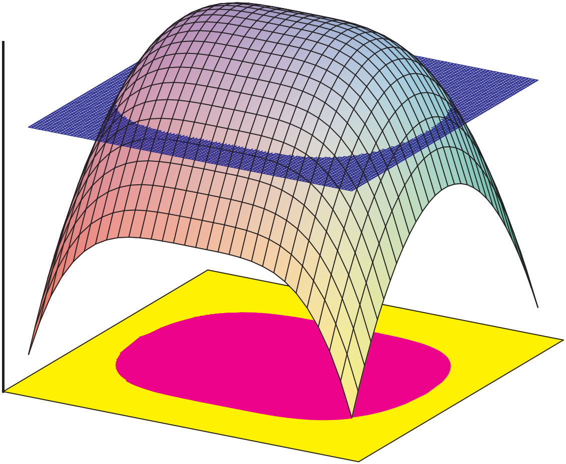
(iii) *The set $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) \leq c\}$ is closed in $\mathbb{R}^n$.*

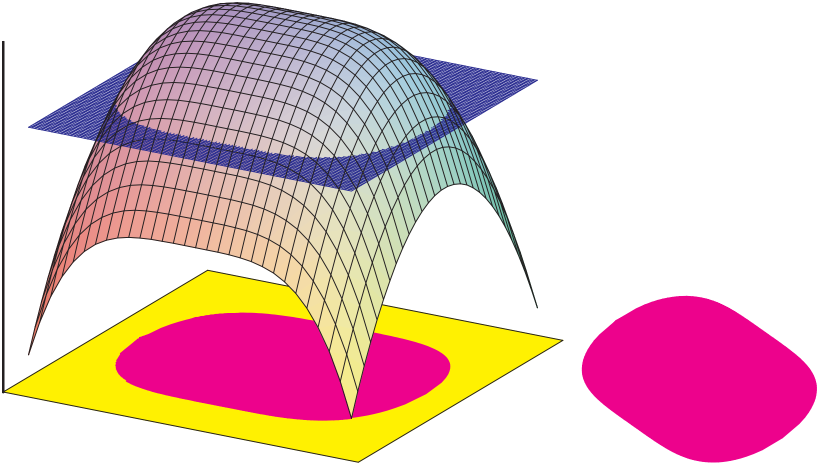
(iv) *The set $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) \geq c\}$ is closed in $\mathbb{R}^n$.*

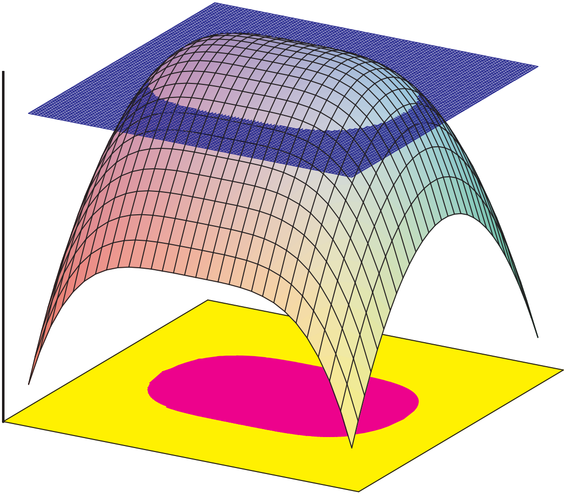
(v) *The set $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) = c\}$ is closed in $\mathbb{R}^n$.*

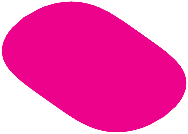


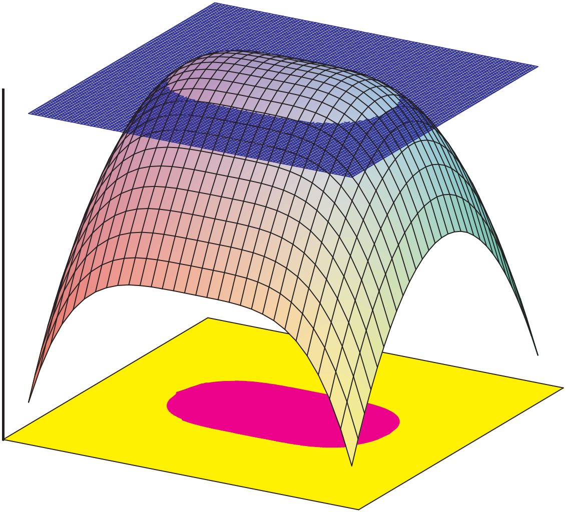


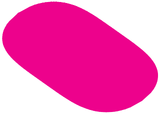












Definition

We say that a set $M \subset \mathbb{R}^n$ is compact if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

Definition

We say that a set $M \subset \mathbb{R}^n$ is compact if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

Theorem 15 (characterisation of compact subsets of $\mathbb{R}^n$)

The set $M \subset \mathbb{R}^n$ is compact if and only if M is bounded and closed.

Definition

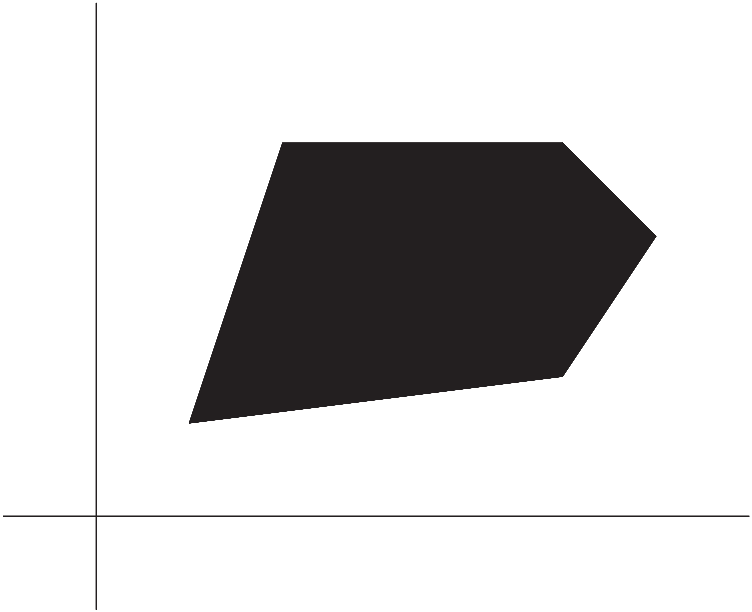
We say that a set $M \subset \mathbb{R}^n$ is compact if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

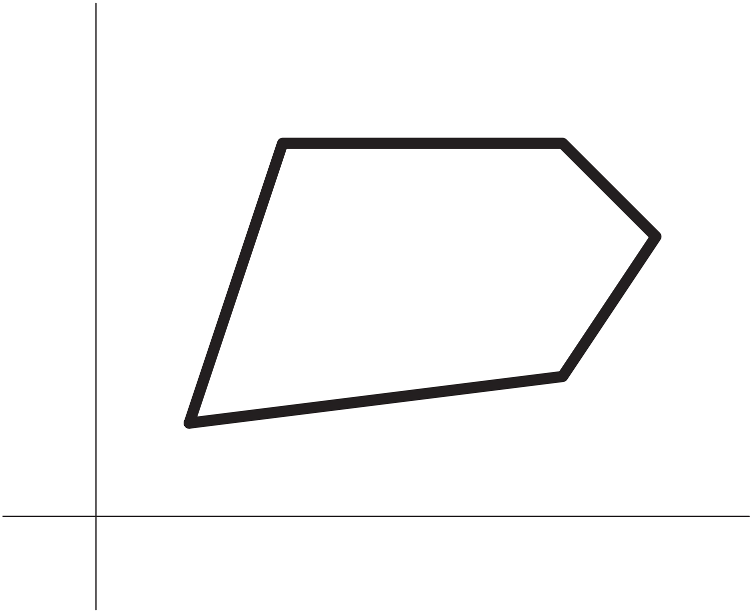
Theorem 15 (characterisation of compact subsets of $\mathbb{R}^n$)

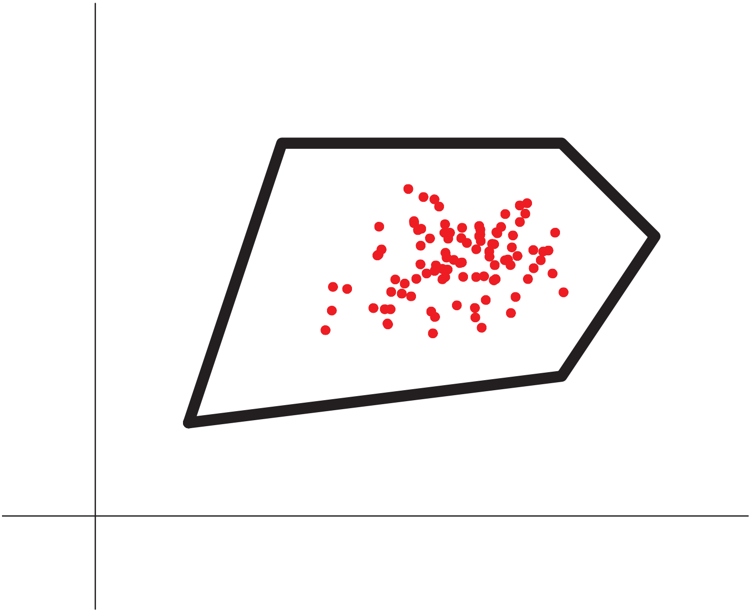
The set $M \subset \mathbb{R}^n$ is compact if and only if M is bounded and closed.

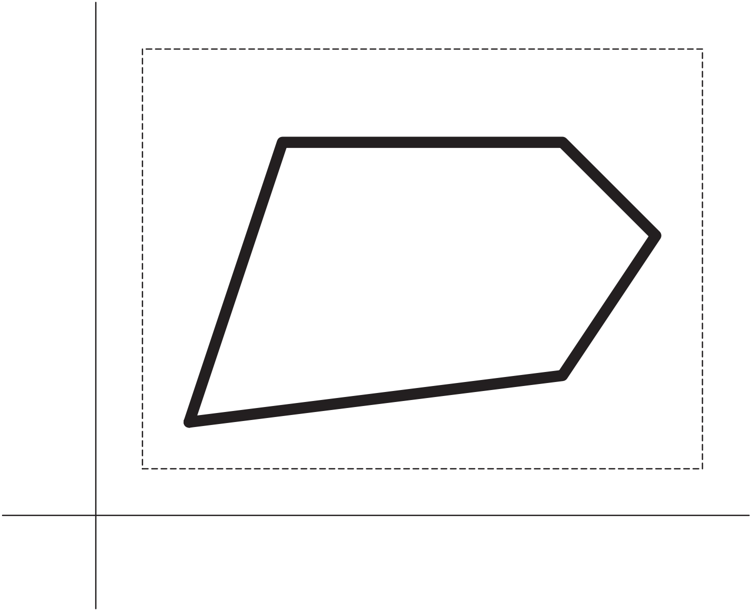
Lemma 16

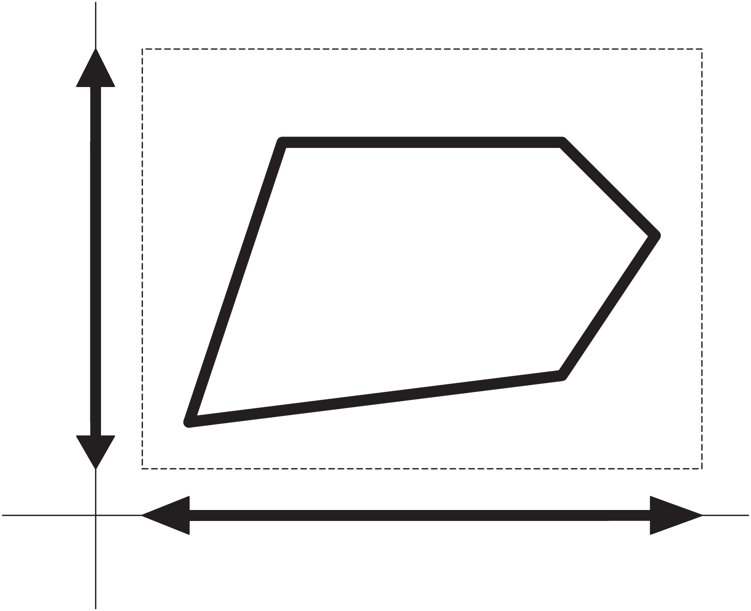
Let $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^n$. Then it has a convergent subsequence.

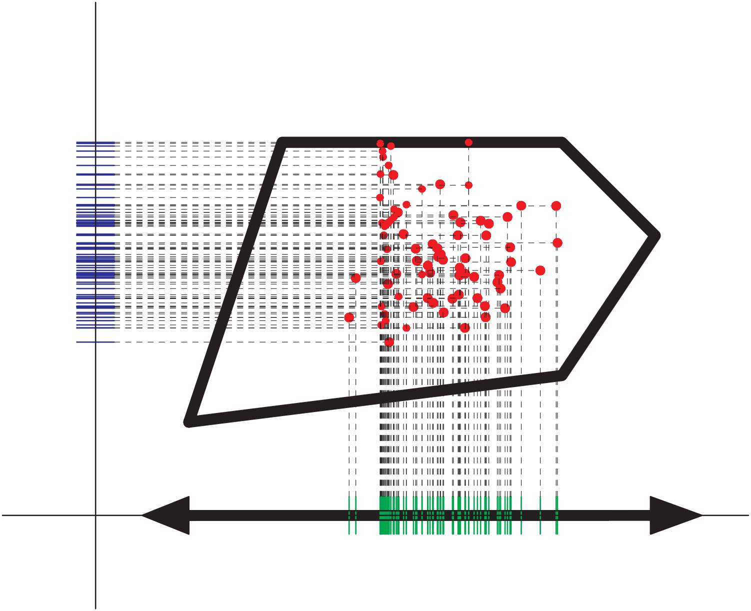


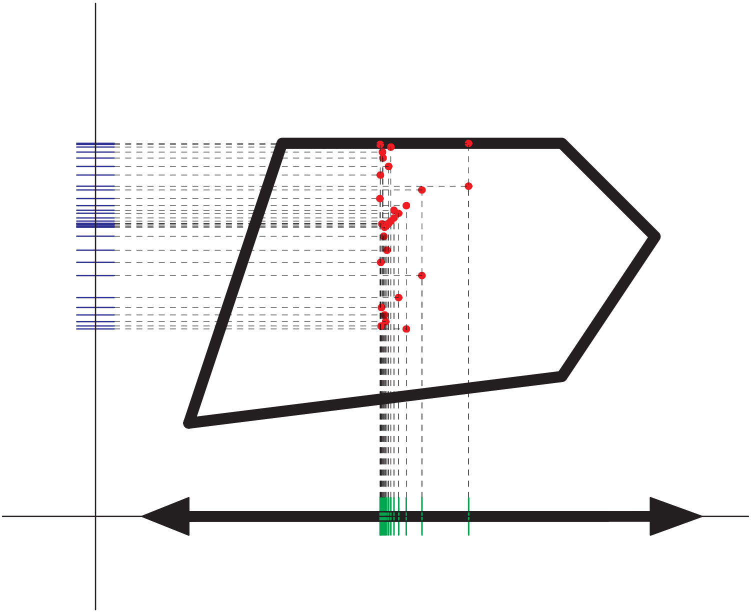


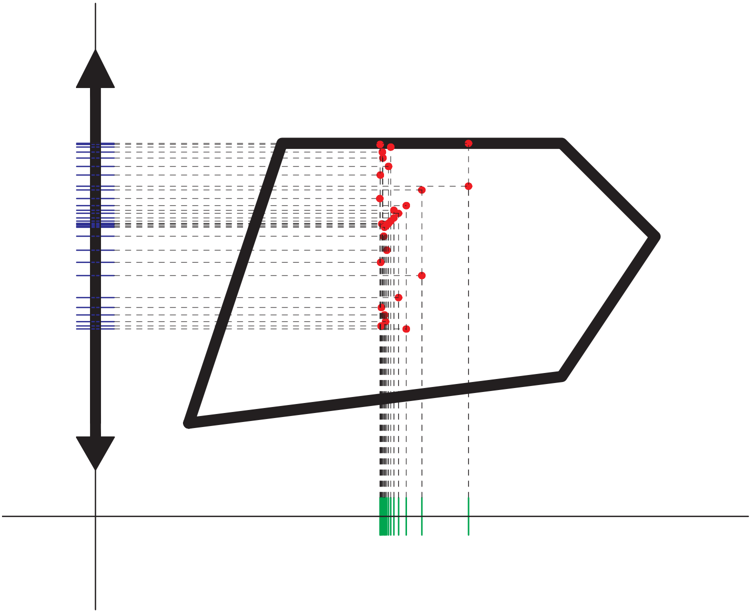


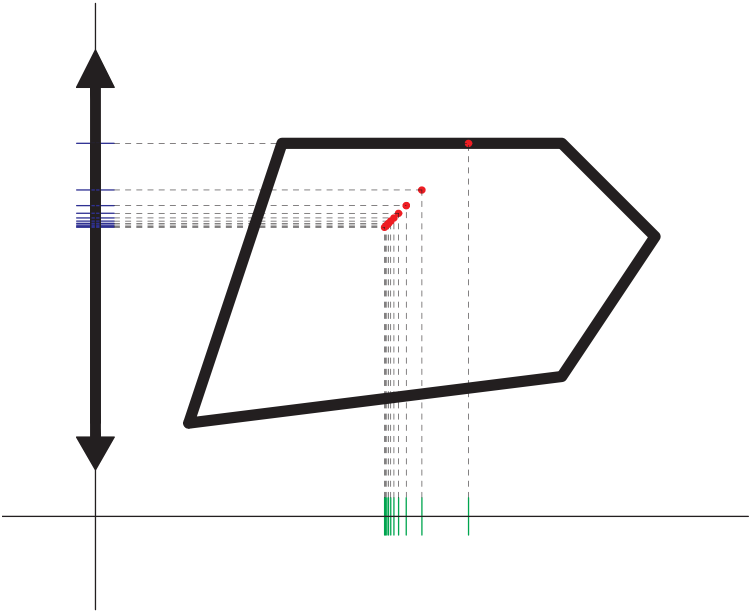












Definition

We say that a system of open sets $\{U_\alpha \subset \mathbb{R}^n\}_{\alpha\in\mathcal{A}}$ is an open covering of a set $M \in \mathbb{R}^n$, if $M \subset \bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha$.

Exercise 4

$M \in \mathbb{R}^n$ is compact iff every open covering of M has a finite subcovering.

Hint 1. $\Leftarrow$ Note that a sequence $\{x_m \in \mathbb{R}^n\}_{m=1}^\infty$ has a convergent subsequence in M iff
$\exists x \in M\ \forall\varepsilon > 0\ \forall N \in \mathbb{N}\ \exists\widetilde{N} > N:\ \rho(x_{\widetilde{N}}, x) < \varepsilon.$
Hence, assuming there is no convergent subsequence of $\{x_m\}_{m=1}^\infty$, we have that
$\forall x \in M\ \exists\varepsilon = \varepsilon(x)\ \exists N \in \mathbb{N}\ \forall\widetilde{N} > N:\ \rho(x, x_{\widetilde{N}}) \geq \varepsilon.$
Now consider $\{U(x, \varepsilon(x))\}_{x\in M}$.

Hint 2. $\Rightarrow$ Note that from every open covering one can find a countable subcovering. To do that, consider the set of all "rational balls" $U(\boldsymbol{x}, r)$, where each of coordinate $\boldsymbol{x}$ and radius r are rational. Now, for each point $x \in M$ there exists U_α such that $x \in U_\alpha$. But at the same time x is contained in some rational ball B. Take that one sufficiently small so that it is also contained in U_α. Once we have a countable set of open sets, $\{U_m\}_{m=1}^{\infty}$, assuming that there is no finite subcovering of it, choose $x_m \in M \setminus \bigcup_{j=1}^{m} U_j$ and then take a convergent subsequence.

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and let f be a function defined at least on M (i.e. $M \subset D_f$). We say that f attains at the point $\boldsymbol{x}$ its

- maximum on M if $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in M$,

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and let f be a function defined at least on M (i.e. $M \subset D_f$). We say that f attains at the point $\boldsymbol{x}$ its

- maximum on M if $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in M$,
- local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M$,

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and let f be a function defined at least on M (i.e. $M \subset D_f$). We say that f attains at the point $\boldsymbol{x}$ its

- maximum on M if $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in M$,
- local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M$,
- strict local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$ for every $\boldsymbol{y} \in \big(U(\boldsymbol{x}, \delta) \setminus \{\boldsymbol{x}\}\big) \cap M$.

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and let f be a function defined at least on M (i.e. $M \subset D_f$). We say that f attains at the point $\boldsymbol{x}$ its

- maximum on M if $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in M$,
- local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in U(\boldsymbol{x}, \delta) \cap M$,
- strict local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$ for every $\boldsymbol{y} \in \big(U(\boldsymbol{x}, \delta) \setminus \{\boldsymbol{x}\}\big) \cap M$.

The notions of a minimum, a local minimum, and a strict local minimum with respect to M are defined in analogous way.

Definition

We say that a function f attains a local maximum at a point $\boldsymbol{x} \in \mathbb{R}^n$ if $\boldsymbol{x}$ is a local maximum with respect to some neighbourhood of $\boldsymbol{x}$.

Definition

We say that a function f attains a local maximum at a point $\boldsymbol{x} \in \mathbb{R}^n$ if $\boldsymbol{x}$ is a local maximum with respect to some neighbourhood of $\boldsymbol{x}$.

Similarly we define local minimum, strict local maximum and strict local minimum.

Theorem 17 (attaining extrema)

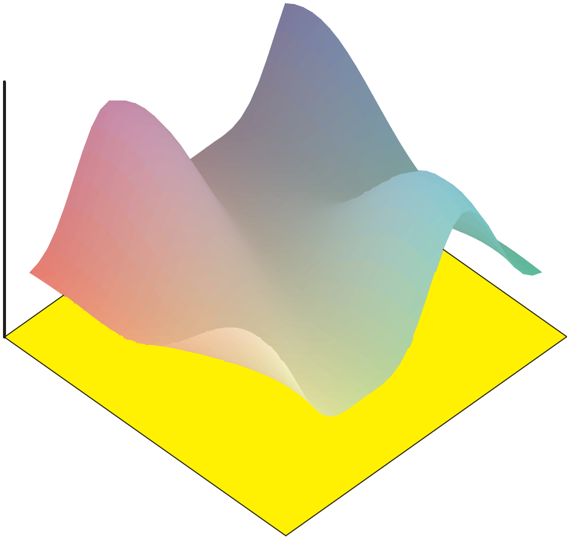
Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f : M \to \mathbb{R}$ a function continuous on M. Then f attains its maximum and minimum on M.

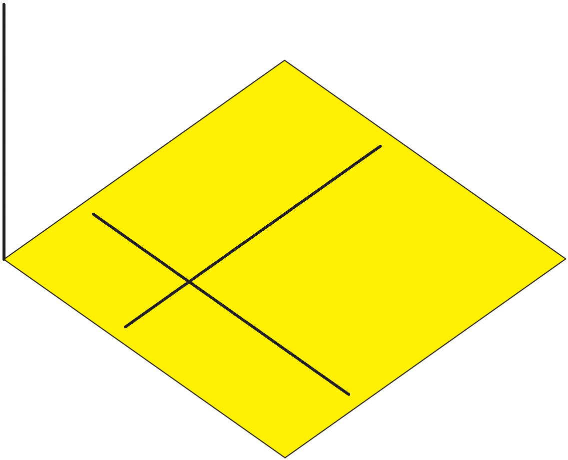
Theorem 17 (attaining extrema)

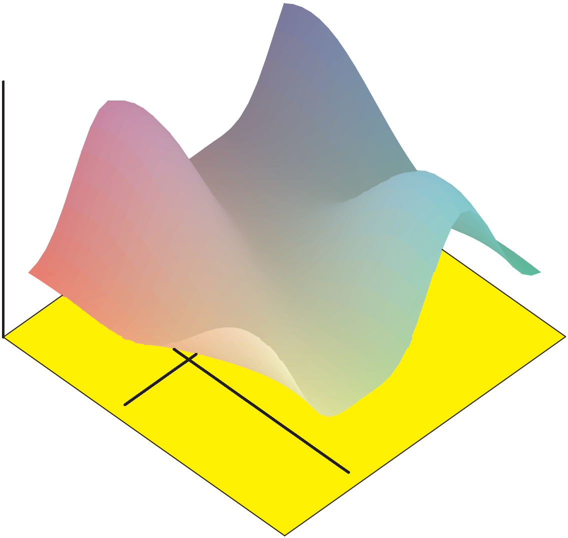
Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f\colon M \to \mathbb{R}$ a function continuous on M. Then f attains its maximum and minimum on M.

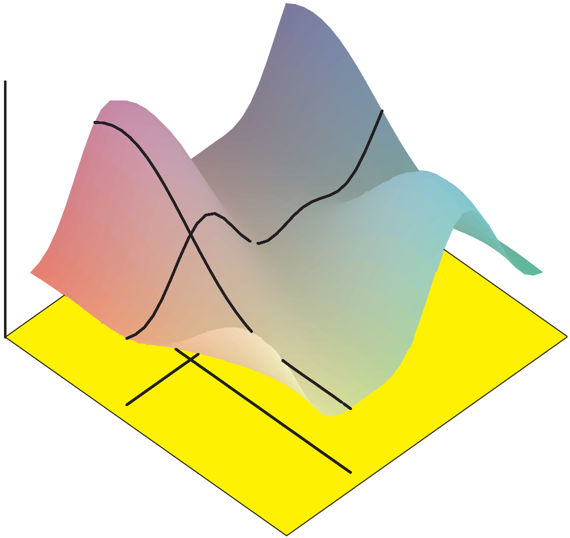
Corollary

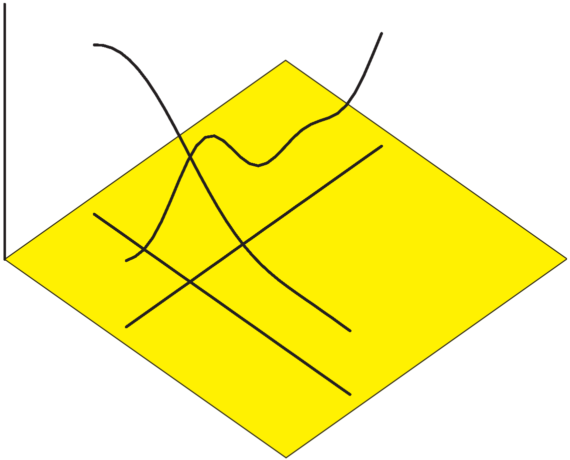
Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f\colon M \to \mathbb{R}$ a continuous function on M. Then f is bounded on M.

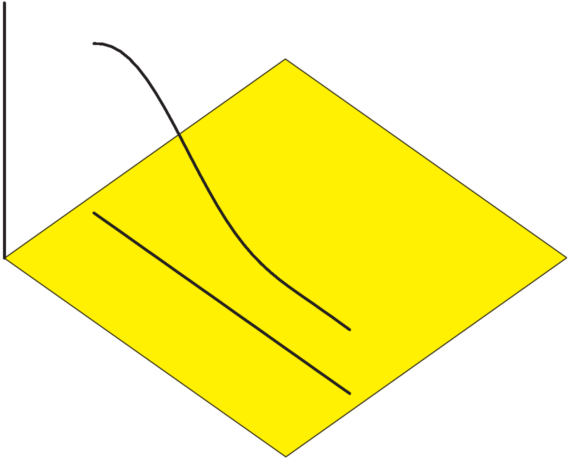


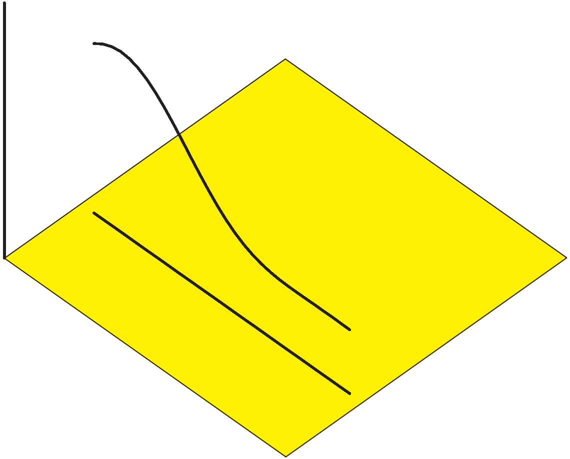


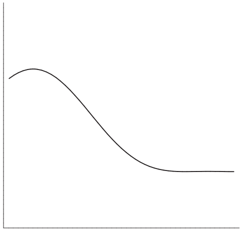


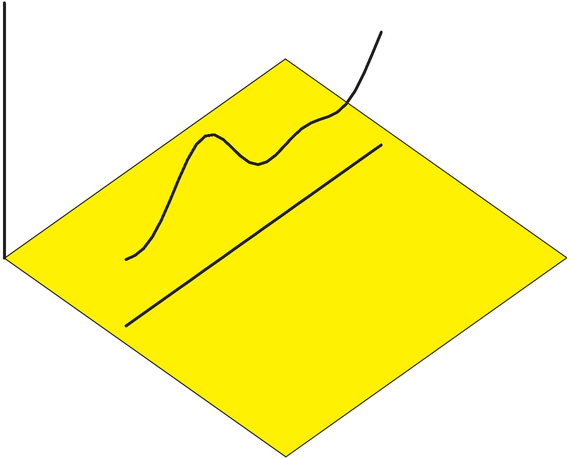


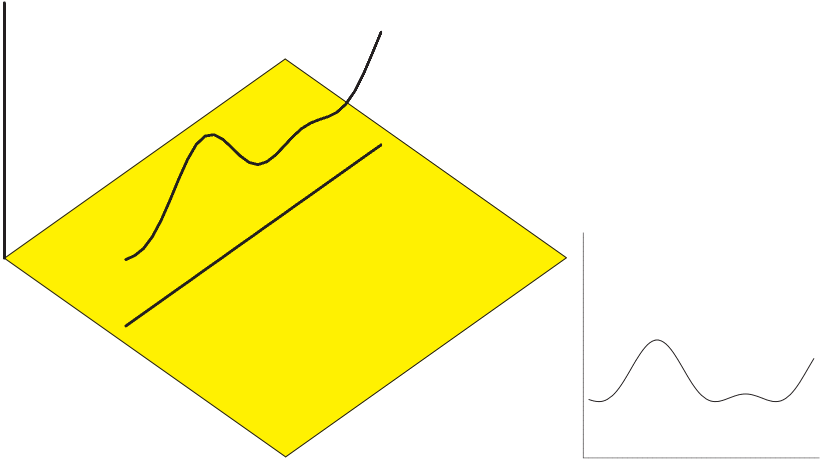












Denote $\boldsymbol{e}^j = (0, \ldots, 0, \underbrace{1}_{j\text{th coordinate}}, 0, \ldots, 0)$.

Denote $\boldsymbol{e}^j = (0, \dots, 0, \underset{j\text{th coordinate}}{1}, 0, \dots, 0)$.

Definition

Let f be a function of n variables, $j \in \{1, \dots, n\}$, $\boldsymbol{a} \in \mathbb{R}^n$. Then the number

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$

is called the partial derivative (of first order) of function f according to jth variable at the point $\boldsymbol{a}$ (if the limit exists).

Denote $\boldsymbol{e}^j = (0, \dots, 0, \underbrace{1}_{j\text{th coordinate}}, 0, \dots, 0)$.

Definition

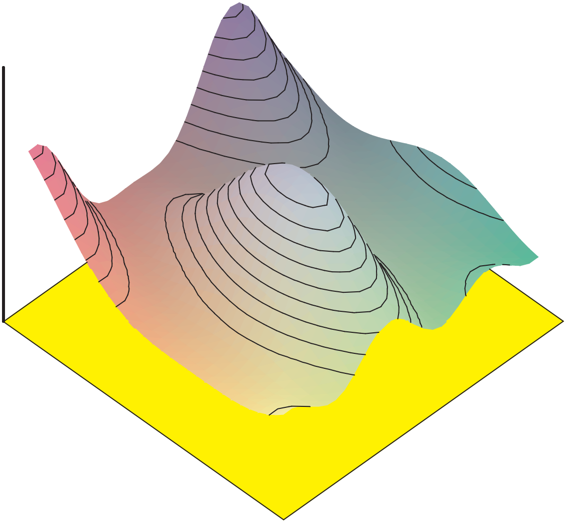
Let f be a function of n variables, $j \in \{1, \dots, n\}$, $\boldsymbol{a} \in \mathbb{R}^n$.
Then the number

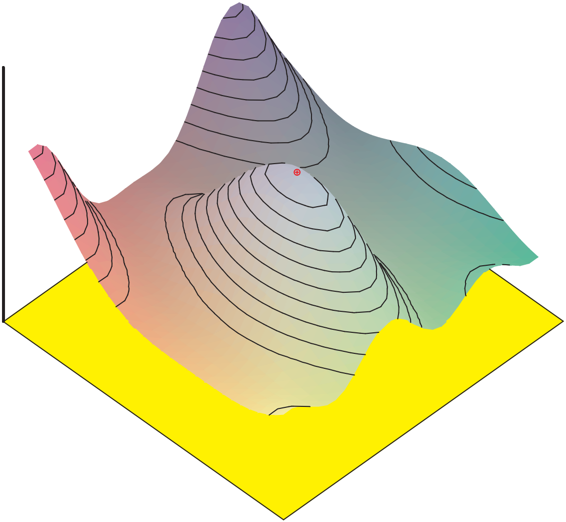
$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$
$$= \lim_{t \to 0} \frac{f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_n)}{t}$$

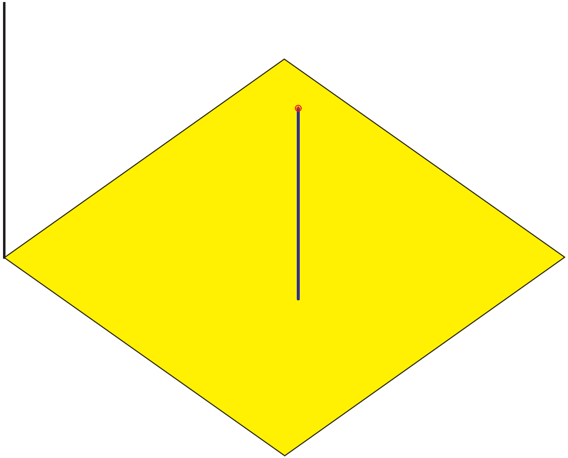
is called the partial derivative (of first order) of function f according to jth variable at the point $\boldsymbol{a}$ (if the limit exists).

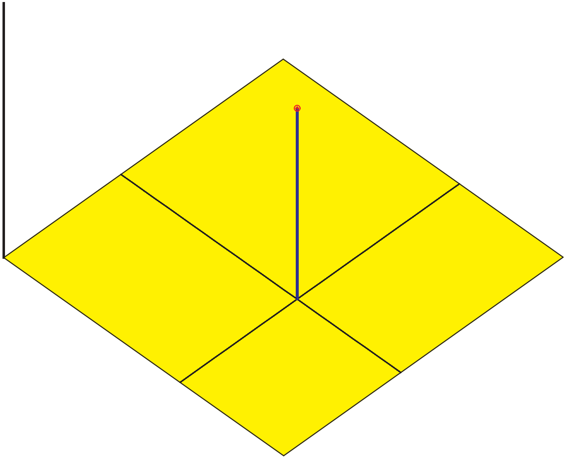
Theorem 18 (necessary condition of the existence of local extremum)

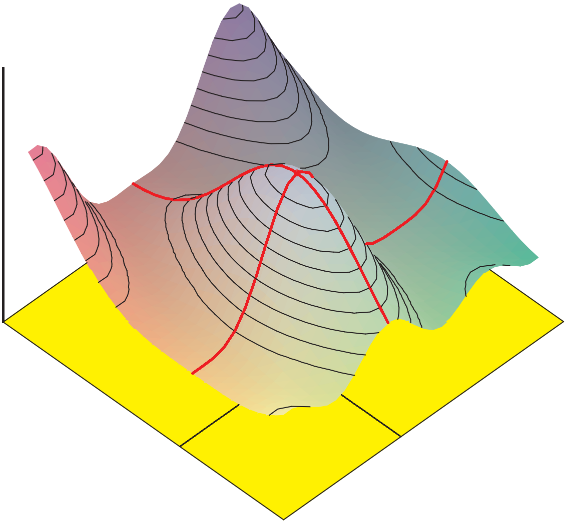
Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and suppose that a function $f\colon G \to \mathbb{R}$ has a local extremum (i.e. a local maximum or a local minimum) at the point $\boldsymbol{a}$. Then for each $j \in \{1, \dots, n\}$ the following holds:
The partial derivative $\dfrac{\partial f}{\partial x_j}(\boldsymbol{a})$ either does not exist or it is equal to zero.

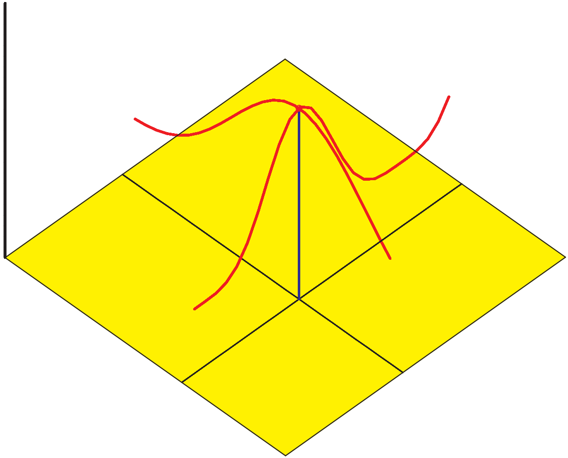


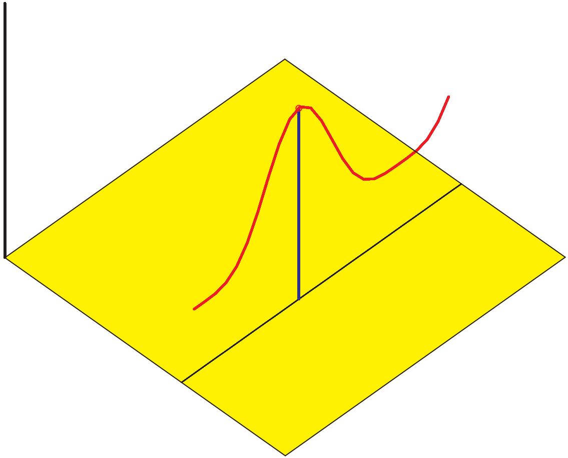


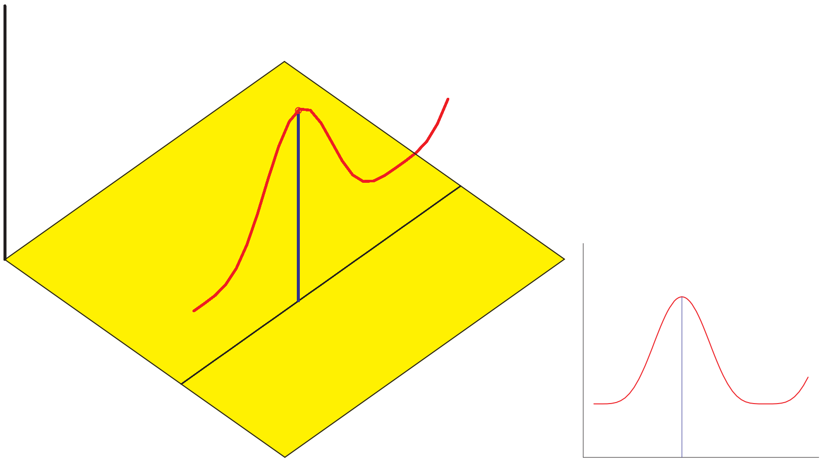












Definition

Let $G \subset \mathbb{R}^n$ be a non-empty open set. If a function $f\colon G \to \mathbb{R}$ has all partial derivatives continuous at each point of the set G (i.e. the function $\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_j}(\boldsymbol{x})$ is continuous on G for each $j \in \{1, \dots, n\}$), then we say that f is of the class $\mathcal{C}^1$ on G. The set of all of these functions is denoted by $C^1(G)$.

Definition

Let $G \subset \mathbb{R}^n$ be a non-empty open set. If a function $f\colon G \to \mathbb{R}$ has all partial derivatives continuous at each point of the set G (i.e. the function $\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_j}(\boldsymbol{x})$ is continuous on G for each $j \in \{1, \dots, n\}$), then we say that f is of the class $\mathcal{C}^1$ on G. The set of all of these functions is denoted by $C^1(G)$.

Remark

If $G \subset \mathbb{R}^n$ is a non-empty open set and and $f, g \in C^1(G)$, then $f + g \in C^1(G)$, $f - g \in C^1(G)$, and $fg \in C^1(G)$. If moreover $g(\boldsymbol{x}) \neq 0$ for each $\boldsymbol{x} \in G$, then $f/g \in C^1(G)$.

Proposition 19 (weak Lagrange theorem)

Let $n \in \mathbb{N}$, $I_1, \dots, I_n \subset \mathbb{R}$ be open intervals, $I = I_1 \times I_2 \times \cdots \times I_n$, $f \in C^1(I)$, and $\boldsymbol{a}, \boldsymbol{b} \in I$. Then there exist points $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^n \in I$ with $\xi^i_j \in [a_j, b_j]$ for each $i, j \in \{1, \dots, n\}$, such that

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{\xi}^i)(b_i - a_i).$$

Consequence

If $f \in \mathcal{C}^1(G)$, then $f \in \mathcal{C}(G)$.

Remark

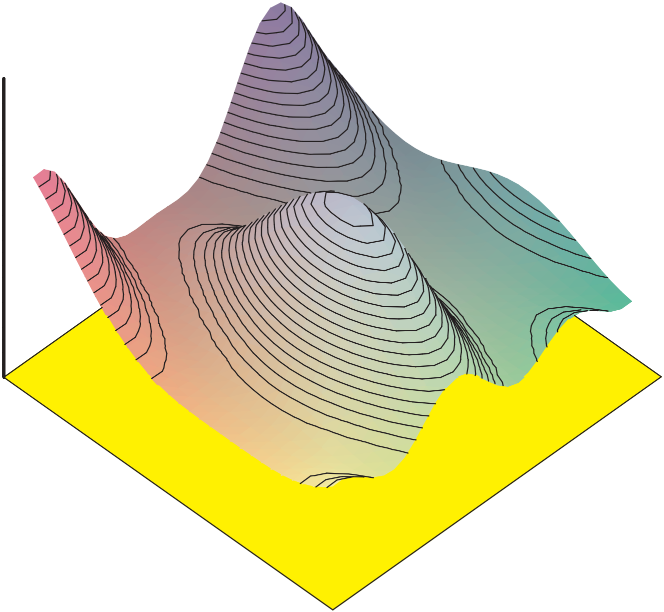
Existence of partial derivatives at *a* **does not** imply continuity at *a*.

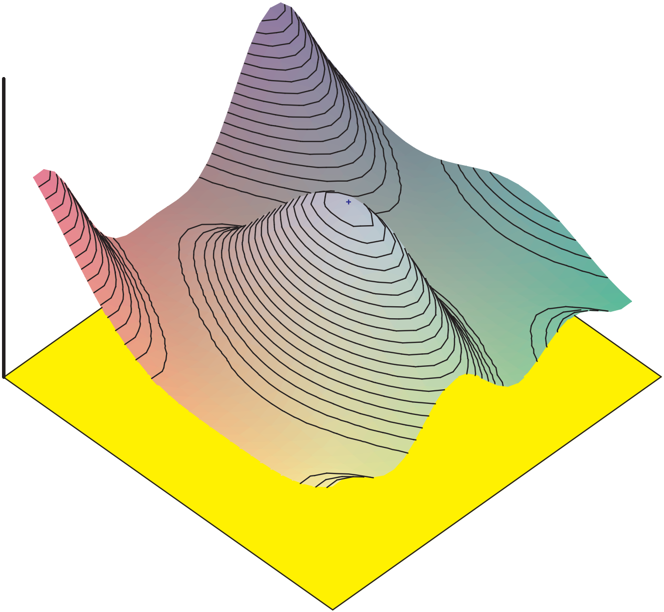
Definition

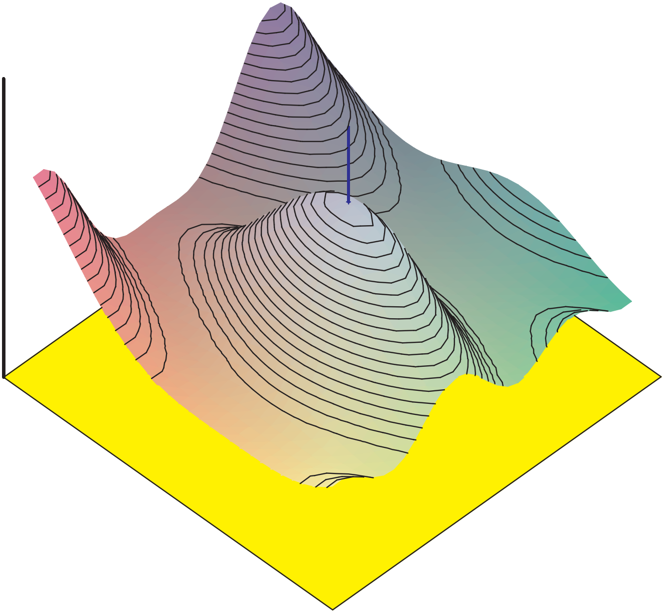
Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in C^1(G)$. Then the graph of the function

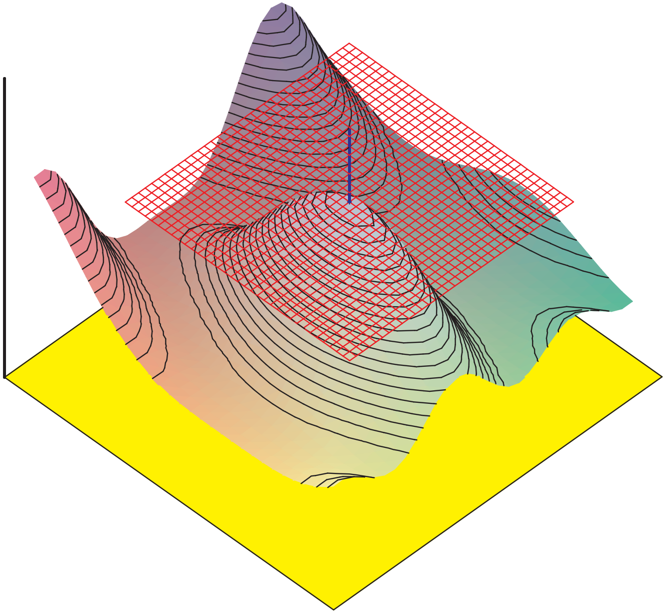
$$T\colon \boldsymbol{x} \mapsto f(\boldsymbol{a}) + \frac{\partial f}{\partial x_1}(\boldsymbol{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\boldsymbol{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\boldsymbol{a})(x_n - a_n), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

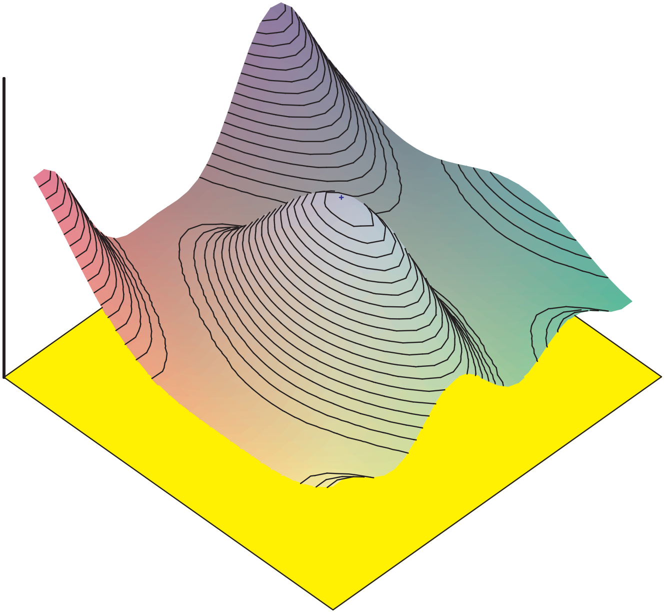
is called the tangent hyperplane to the graph of the function f at the point $(\boldsymbol{a}, f(\boldsymbol{a}))$.

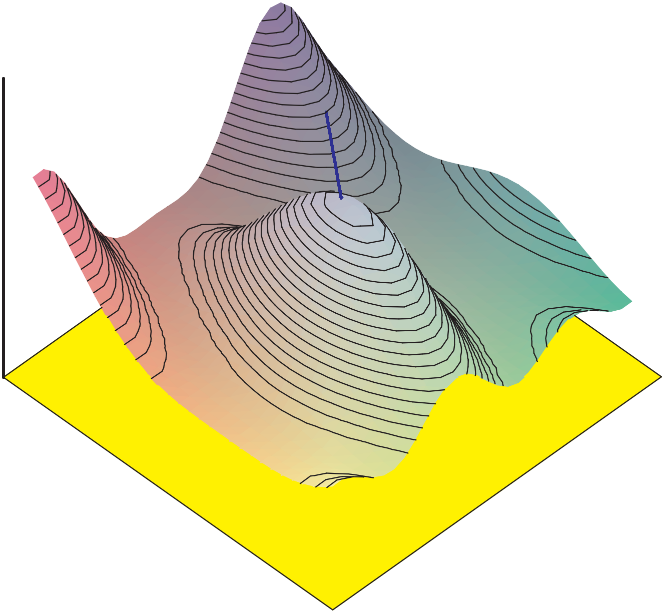


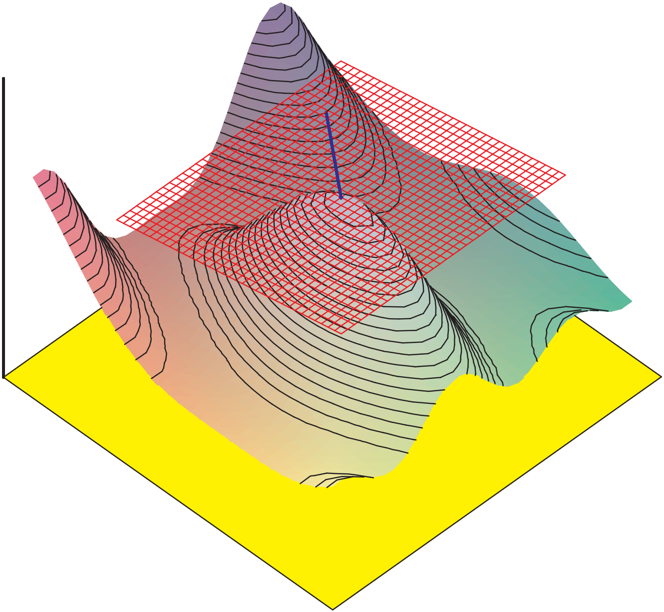












Theorem 20 (tangent hyperplane)

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $(\boldsymbol{a}, f(\boldsymbol{a}))$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 20 (tangent hyperplane)

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $(\boldsymbol{a}, f(\boldsymbol{a}))$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 21

Let $G \subset \mathbb{R}^n$ be an open non-empty set and $f \in C^1(G)$. Then f is continuous on G.

Theorem 20 (tangent hyperplane)

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $(\boldsymbol{a}, f(\boldsymbol{a}))$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 21

Let $G \subset \mathbb{R}^n$ be an open non-empty set and $f \in C^1(G)$. Then f is continuous on G.

Remark

Existence of partial derivatives at a **does not** imply continuity at a.

Theorem 22 (derivative of a composite function; chain rule)

Let $r, s \in \mathbb{N}$ and let $G \subset \mathbb{R}^s$, $H \subset \mathbb{R}^r$ be open sets. Let $\varphi_1, \dots, \varphi_r \in C^1(G)$, $f \in C^1(H)$ and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in H$ for each $\boldsymbol{x} \in G$. Then the compound function $F\colon G \to \mathbb{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in G,$$

is of the class $\mathcal{C}^1$ on G. Let $\boldsymbol{a} \in G$ and $\boldsymbol{b} = [\varphi_1(\boldsymbol{a}), \dots, \varphi_r(\boldsymbol{a})]$. Then for each $j \in \{1, \dots, s\}$ we have

$$\frac{\partial F}{\partial x_j}(\boldsymbol{a}) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(\boldsymbol{b}) \frac{\partial \varphi_i}{\partial x_j}(\boldsymbol{a}).$$

Definition

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in C^1(G)$. The gradient of f at the point $\boldsymbol{a}$ is the vector

$$\nabla f(\boldsymbol{a}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{a})\right].$$

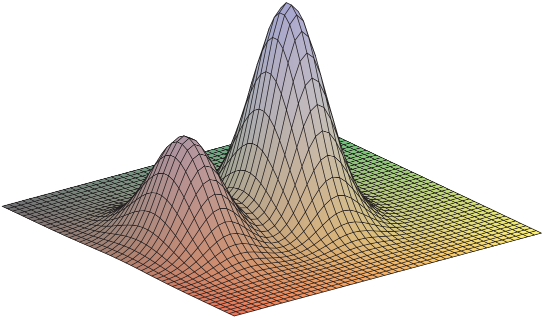
Definition

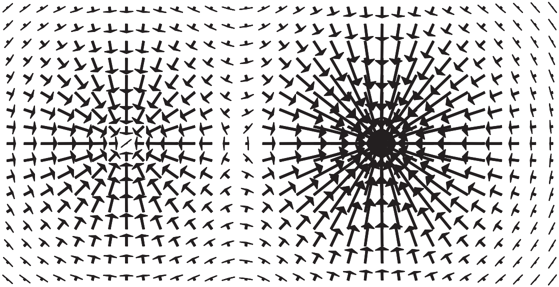
Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in C^1(G)$. The gradient of f at the point $\boldsymbol{a}$ is the vector

$$\nabla f(\boldsymbol{a}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{a})\right].$$

Remark

The gradient of f at a points in the direction of steepest growth of f at a. At every point, the gradient is perpendicular to the contour of f.





Definition

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$. Then the point $\boldsymbol{a}$ is called a stationary (or critical) point of the function f.

Definition

Let $G \subset \mathbb{R}^n$ be an open set, $f\colon G \to \mathbb{R}$, $i, j \in \{1, \dots, n\}$, and suppose that $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists finite for each $\boldsymbol{x} \in G$. Then the partial derivative of the second order of the function f according to ith and jth variable at a point $\boldsymbol{a} \in G$ is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j}(\boldsymbol{a})$$

If $i = j$ then we use the notation $\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a})$.

Definition

Let $G \subset \mathbb{R}^n$ be an open set, $f\colon G \to \mathbb{R}$, $i, j \in \{1, \dots, n\}$, and suppose that $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists finite for each $\boldsymbol{x} \in G$. Then the partial derivative of the second order of the function f according to ith and jth variable at a point $\boldsymbol{a} \in G$ is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial \left(\frac{\partial f}{\partial x_i} \right)}{\partial x_j}(\boldsymbol{a})$$

If $i = j$ then we use the notation $\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a})$.

Similarly we define higher order partial derivatives.

Remark

In general it is not true that $\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a})$.

Remark

In general it is not true that $\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a})$.

Theorem 23 (interchanging of partial derivatives)

Let $i, j \in \{1, \ldots, n\}$ and suppose that a function f has both partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ on a neighbourhood of a point $\boldsymbol{a} \in \mathbb{R}^n$ and that these functions are continuous at $\boldsymbol{a}$. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}).$$

Definition

Let $G \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}$. We say that a function f is of the class $\mathcal{C}^k$ on G, if all partial derivatives of f of all orders up to k are continuous on G. The set of all of these functions is denoted by $C^k(G)$.

Definition

Let $G \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}$. We say that a function f is of the class $\mathcal{C}^k$ on G, if all partial derivatives of f of all orders up to k are continuous on G. The set of all of these functions is denoted by $C^k(G)$.

We say that a function f is of the class $\mathcal{C}^\infty$ on G, if all partial derivatives of all orders of f are continuous on G. The set of all of these functions is denoted by $C^\infty(G)$.

V.4. Implicit function theorem and Lagrange multiplier theorem

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

(i) $F \in C^1(G)$,

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

(i) $F \in C^1(G)$,

(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F \colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

(i) $F \in C^1(G)$,

(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

(iii) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

(i) $F \in C^1(G)$,

(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

(iii) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighbourhood $V \subset \mathbb{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists a unique $y \in V$ satisfying $F(\boldsymbol{x}, y) = 0$.

Theorem 24 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

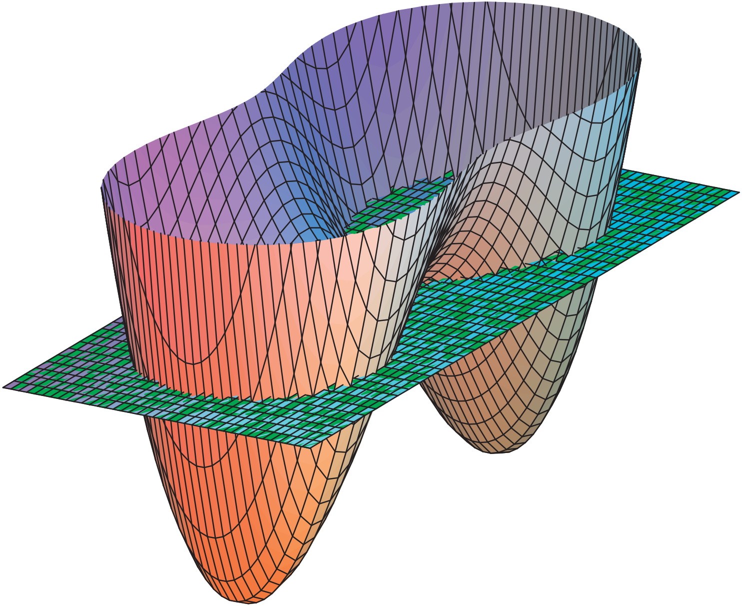
(i) $F \in C^1(G)$,

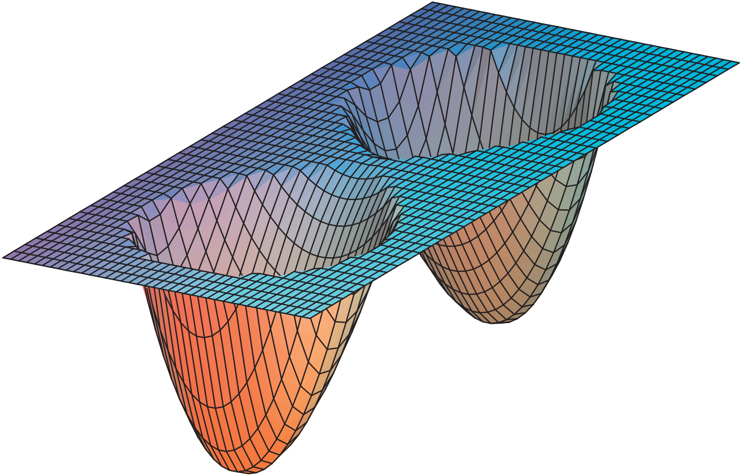
(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

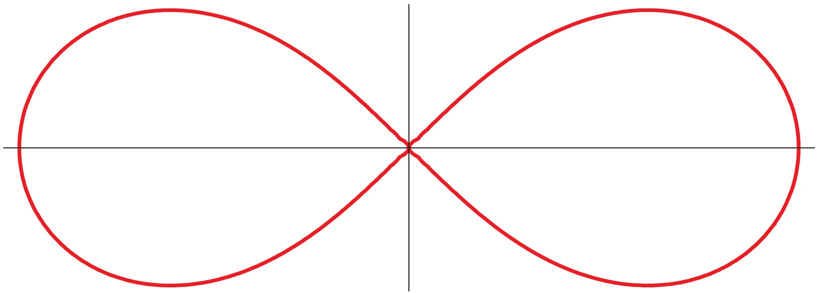
(iii) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

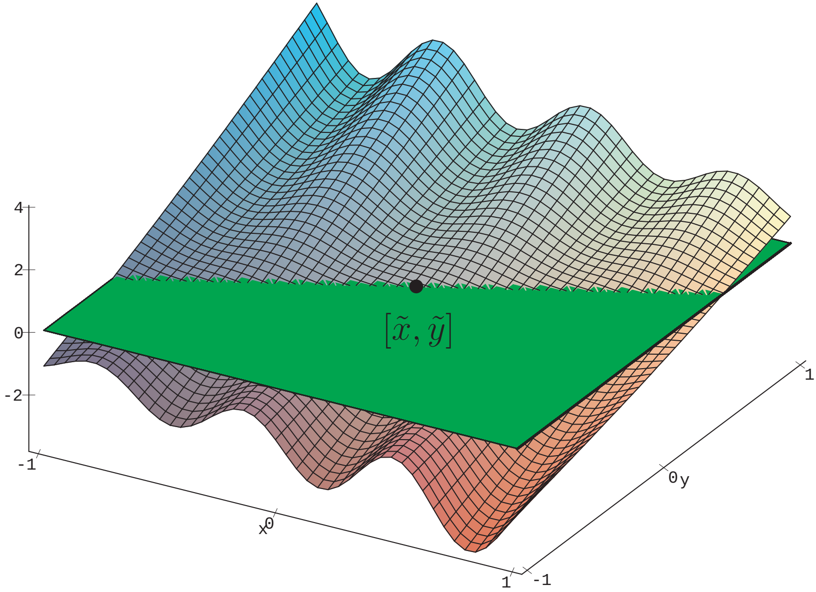
Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighbourhood $V \subset \mathbb{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists a unique $y \in V$ satisfying $F(\boldsymbol{x}, y) = 0$. If we denote this y by $\varphi(\boldsymbol{x})$, then the resulting function φ is in $C^1(U)$ and

$$\frac{\partial \varphi}{\partial x_j}(\boldsymbol{x}) = -\frac{\frac{\partial F}{\partial x_j}(\boldsymbol{x}, \varphi(\boldsymbol{x}))}{\frac{\partial F}{\partial y}(\boldsymbol{x}, \varphi(\boldsymbol{x}))} \quad \text{for } \boldsymbol{x} \in U,\ j \in \{1, \dots, n\}.$$

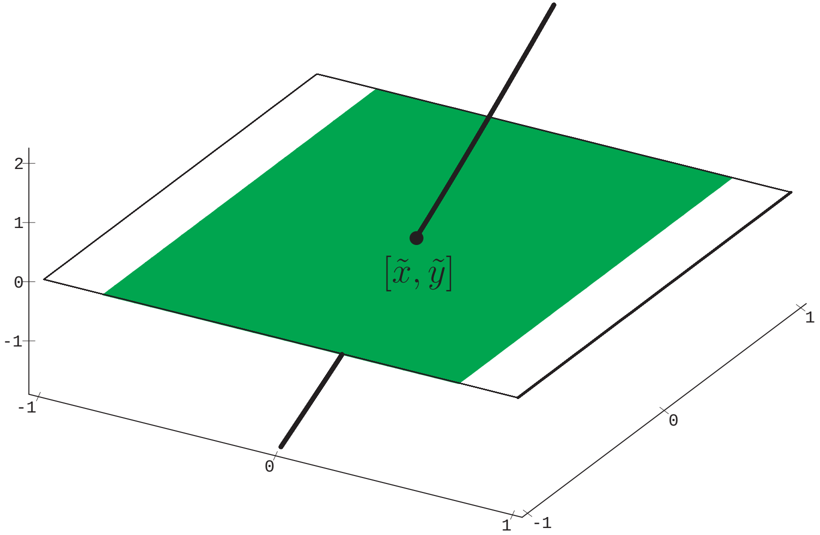




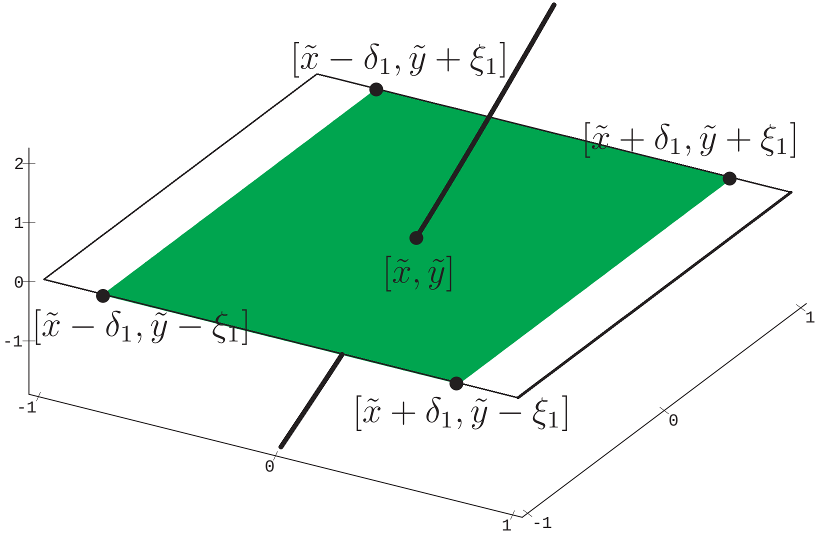




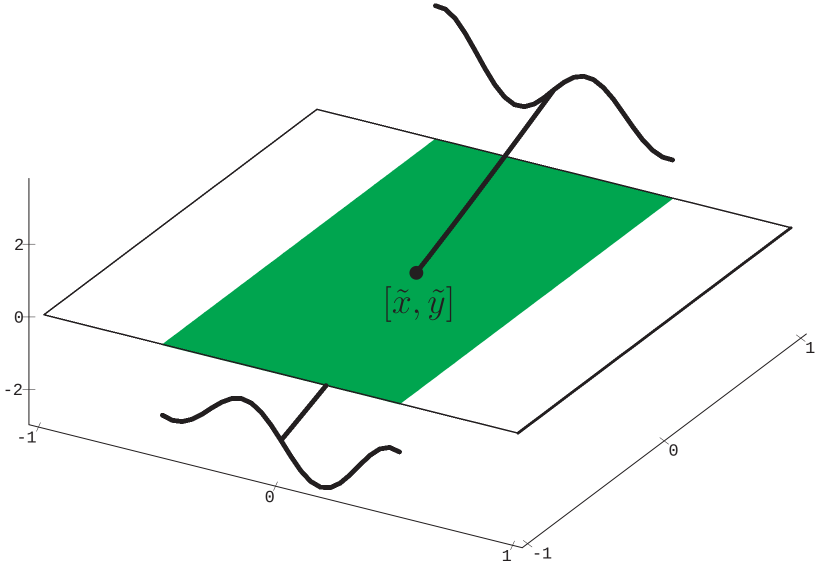
4
2
0
-2
-1
x
0
1
-1
0
y
1
$[\tilde{x}, \tilde{y}]$



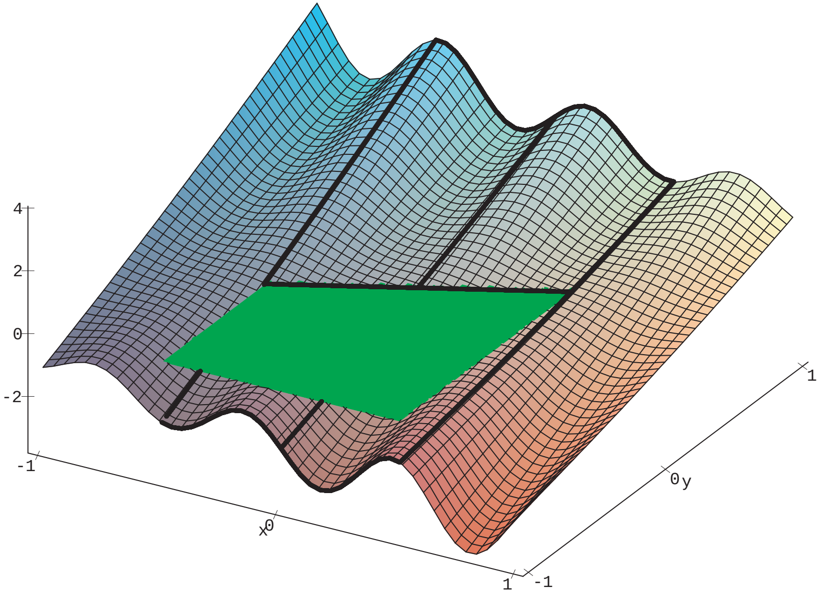
2
1
0
-1
-1
0
1
-1
0
1
$[\tilde{x}, \tilde{y}]$



$[\tilde{x}-\delta_1, \tilde{y}+\xi_1]$
$[\tilde{x}+\delta_1, \tilde{y}+\xi_1]$
$[\tilde{x}, \tilde{y}]$
$[\tilde{x}-\delta_1, \tilde{y}-\xi_1]$
$[\tilde{x}+\delta_1, \tilde{y}-\xi_1]$
2
1
0
-1
-1
0
1
-1
0
1



2
0
-2
-1
0
1
-1
0
1
$[\tilde{x}, \tilde{y}]$


4
2
0
-2
-1
x 0
1
-1
0 y
1

Theorem 25 (Lagrange multiplier theorem)

Let $G \subset \mathbb{R}^2$ be an open set, $f, g \in C^1(G)$, $M = \{[x, y] \in G;\ g(x, y) = 0\}$ and let $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

Theorem 25 (Lagrange multiplier theorem)

Let $G \subset \mathbb{R}^2$ be an open set, $f, g \in C^1(G)$, $M = \{[x, y] \in G;\ g(x, y) = 0\}$ and let $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

(I) $\nabla g(\tilde{x}, \tilde{y}) = \boldsymbol{o}$,

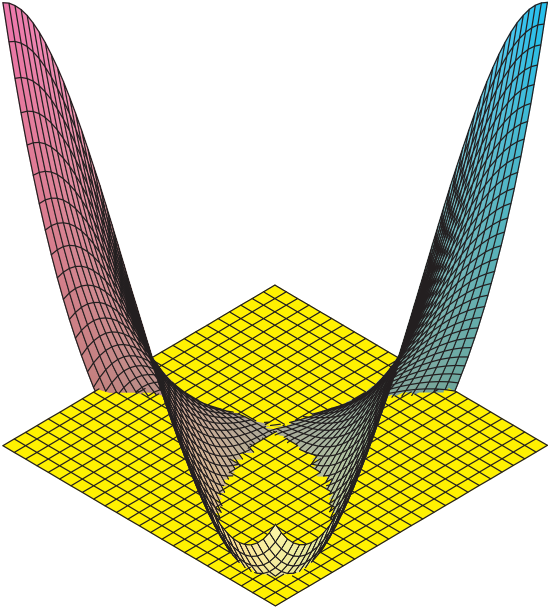
Theorem 25 (Lagrange multiplier theorem)

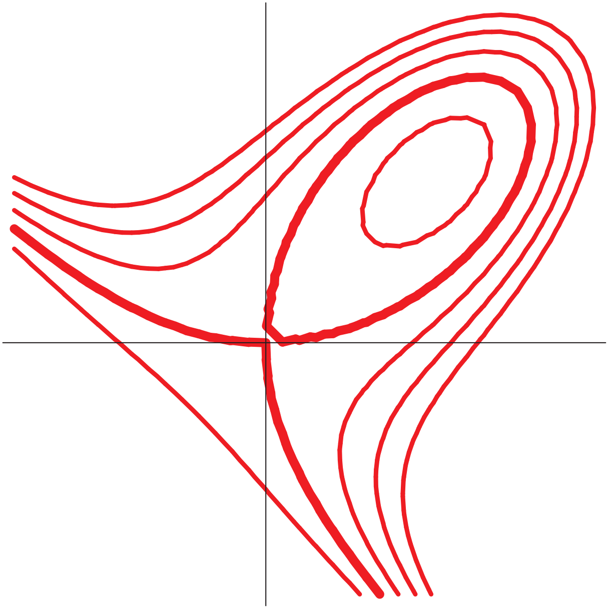
Let $G \subset \mathbb{R}^2$ be an open set, $f, g \in C^1(G)$, $M = \{[x, y] \in G;\ g(x, y) = 0\}$ and let $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

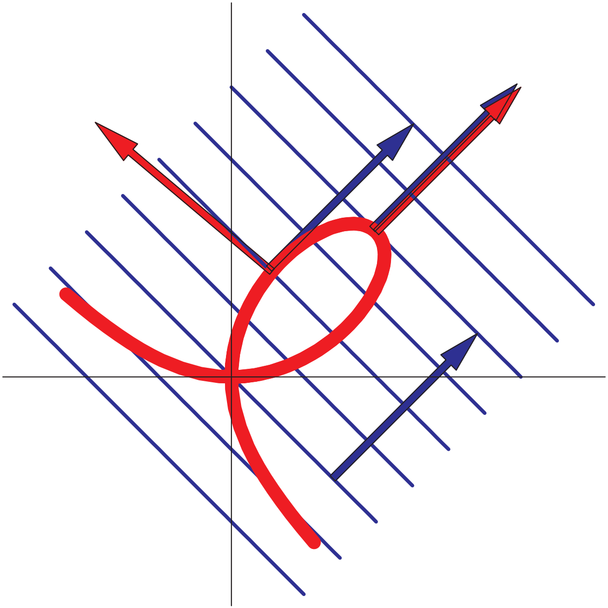
(I) $\nabla g(\tilde{x}, \tilde{y}) = \boldsymbol{o}$,

(II) *there exists $\lambda \in \mathbb{R}$ satisfying*

$$\frac{\partial f}{\partial x}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial x}(\tilde{x}, \tilde{y}) = 0,$$
$$\frac{\partial f}{\partial y}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial y}(\tilde{x}, \tilde{y}) = 0.$$







Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ an open set, $F_j \colon G \to \mathbb{R}$ for $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that

Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ an open set, $F_j \colon G \to \mathbb{R}$ for $j = 1, \ldots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that

(i) $F_j \in C^k(G)$ *for all* $j \in \{1, \ldots, m\}$,

Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ *an open set,* $F_j \colon G \to \mathbb{R}$ *for* $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. *Suppose that*

(i) $F_j \in C^k(G)$ *for all* $j \in \{1, \dots, m\}$,

(ii) $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for all* $j \in \{1, \dots, m\}$,

Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ an open set, $F_j \colon G \to \mathbb{R}$ for $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that

(i) $F_j \in C^k(G)$ *for all* $j \in \{1, \dots, m\}$,

(ii) $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for all* $j \in \{1, \dots, m\}$,

(iii) $$\begin{vmatrix} \frac{\partial F_1}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_1}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_m}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \end{vmatrix} \neq 0.$$

Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ an open set, $F_j\colon G \to \mathbb{R}$ for $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that

(i) $F_j \in C^k(G)$ *for all* $j \in \{1, \dots, m\}$,

(ii) $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for all* $j \in \{1, \dots, m\}$,

(iii) $\begin{vmatrix} \frac{\partial F_1}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_1}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_m}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \end{vmatrix} \neq 0.$

Then there are a neighbourhood $U \subset \mathbb{R}^n$ of $\tilde{\boldsymbol{x}}$ and a neighbourhood $V \subset \mathbb{R}^m$ of $\tilde{\boldsymbol{y}}$ such that for each $\boldsymbol{x} \in U$ there exists a unique $\boldsymbol{y} \in V$ satisfying $F_j(\boldsymbol{x}, \boldsymbol{y}) = 0$ for each $j \in \{1, \dots, m\}$.

Theorem 26 (implicit functions)

Let $m, n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $G \subset \mathbb{R}^{n+m}$ an open set, $F_j \colon G \to \mathbb{R}$ for $j = 1, \dots, m$, $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{y}} \in \mathbb{R}^m$, $[\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}] \in G$. Suppose that

(i) $F_j \in C^k(G)$ *for all* $j \in \{1, \dots, m\}$,

(ii) $F_j(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) = 0$ *for all* $j \in \{1, \dots, m\}$,

(iii) $\begin{vmatrix} \frac{\partial F_1}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_1}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) & \dots & \frac{\partial F_m}{\partial y_m}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \end{vmatrix} \neq 0.$

Then there are a neighbourhood $U \subset \mathbb{R}^n$ of $\tilde{\boldsymbol{x}}$ and a neighbourhood $V \subset \mathbb{R}^m$ of $\tilde{\boldsymbol{y}}$ such that for each $\boldsymbol{x} \in U$ there exists a unique $\boldsymbol{y} \in V$ satisfying $F_j(\boldsymbol{x}, \boldsymbol{y}) = 0$ for each $j \in \{1, \dots, m\}$. If we denote the coordinates of this $\boldsymbol{y}$ by $\varphi_j(\boldsymbol{x})$, then the resulting functions φ_j are in $C^k(U)$.

Remark

The symbol in the condition (iii) of Theorem 26 is called a determinant. The general definition will be given later.

Remark

The symbol in the condition (iii) of Theorem 26 is called a determinant. The general definition will be given later.

For $m = 1$ we have $\begin{vmatrix} a \end{vmatrix} = a$, $a \in \mathbb{R}$. In particular, in this case the condition (iii) in Theorem 26 is the same as the condition (iii) in Theorem 24.

For $m = 2$ we have $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, $a, b, c, d \in \mathbb{R}$.

Theorem 27 (Lagrange multipliers theorem)

Let $m, n \in \mathbb{N}$, $m < n$, $G \subset \mathbb{R}^n$ an open set, $f, g_1, \dots, g_m \in C^1(G)$,

$$M = \{\boldsymbol{z} \in G;\ g_1(\boldsymbol{z}) = 0, g_2(\boldsymbol{z}) = 0, \dots, g_m(\boldsymbol{z}) = 0\}$$

and let $\tilde{\boldsymbol{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

Theorem 27 (Lagrange multipliers theorem)

Let $m, n \in \mathbb{N}$, $m < n$, $G \subset \mathbb{R}^n$ an open set, $f, g_1, \dots, g_m \in C^1(G)$,

$$M = \{\boldsymbol{z} \in G;\ g_1(\boldsymbol{z}) = 0, g_2(\boldsymbol{z}) = 0, \dots, g_m(\boldsymbol{z}) = 0\}$$

and let $\tilde{\boldsymbol{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

(I) *the vectors*

$$\nabla g_1(\tilde{\boldsymbol{z}}), \nabla g_2(\tilde{\boldsymbol{z}}), \dots, \nabla g_m(\tilde{\boldsymbol{z}})$$

are linearly dependent,

Theorem 27 (Lagrange multipliers theorem)

Let $m, n \in \mathbb{N}$, $m < n$, $G \subset \mathbb{R}^n$ an open set, $f, g_1, \dots, g_m \in C^1(G)$,

$$M = \{\boldsymbol{z} \in G;\ g_1(\boldsymbol{z}) = 0, g_2(\boldsymbol{z}) = 0, \dots, g_m(\boldsymbol{z}) = 0\}$$

and let $\tilde{\boldsymbol{z}} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

(I) *the vectors*

$$\nabla g_1(\tilde{\boldsymbol{z}}), \nabla g_2(\tilde{\boldsymbol{z}}), \dots, \nabla g_m(\tilde{\boldsymbol{z}})$$

are linearly dependent,

(II) *there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ satisfying*

$$\nabla f(\tilde{\boldsymbol{z}}) + \lambda_1 \nabla g_1(\tilde{\boldsymbol{z}}) + \lambda_2 \nabla g_2(\tilde{\boldsymbol{z}}) + \cdots + \lambda_m \nabla g_m(\tilde{\boldsymbol{z}}) = \boldsymbol{o}.$$

Remark

- The notion of linearly dependent vectors will be defined later.

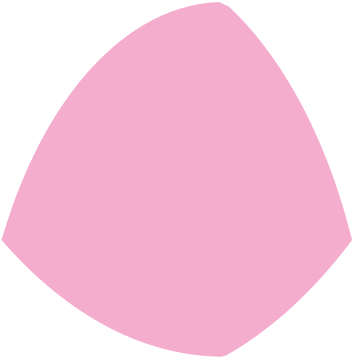
Remark

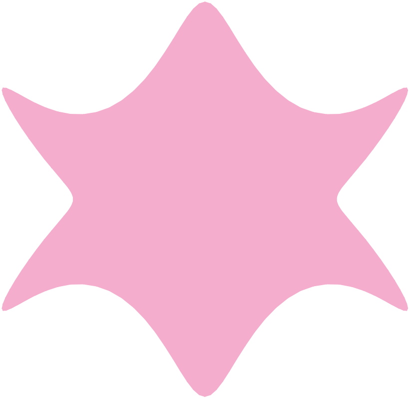
- The notion of linearly dependent vectors will be defined later.
  For $m = 1$: One vector is linearly dependent if it is the zero vector.
  For $m = 2$: Two vectors are linearly dependent if one of them is a multiple of the other one.

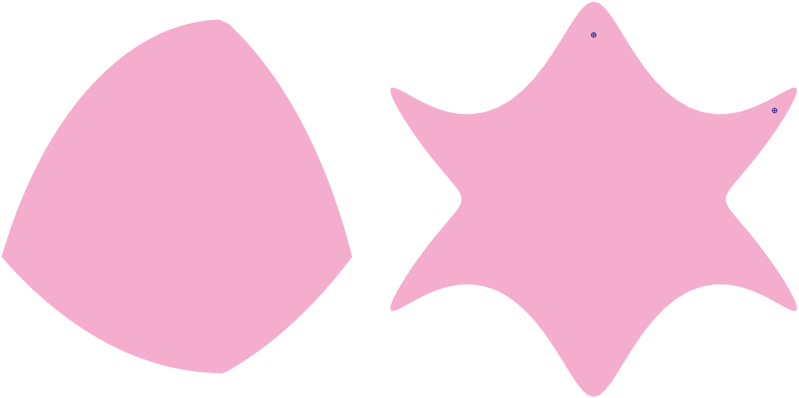
Remark

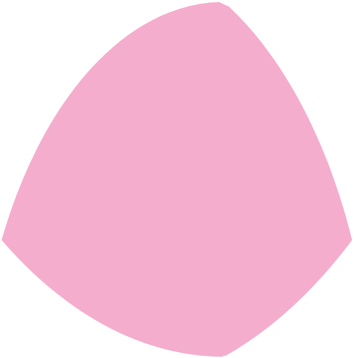
- The notion of linearly dependent vectors will be defined later.
  For $m = 1$: One vector is linearly dependent if it is the zero vector.
  For $m = 2$: Two vectors are linearly dependent if one of them is a multiple of the other one.
- The numbers $\lambda_1, \ldots, \lambda_m$ are called the Lagrange multipliers.

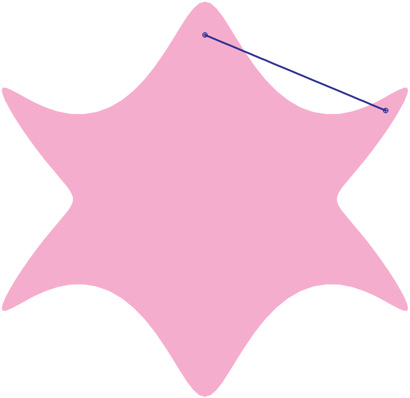
V.5. Concave and quasiconcave functions

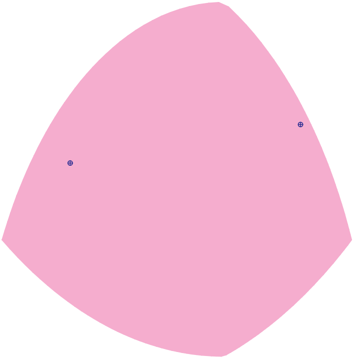


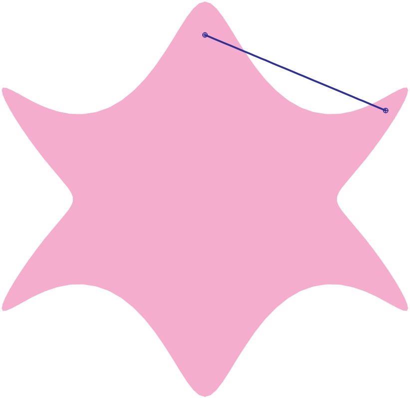


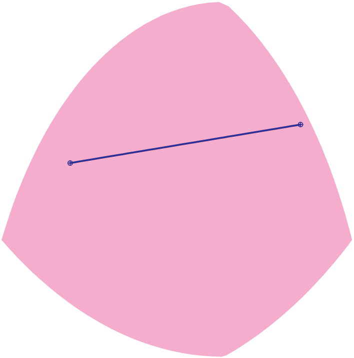


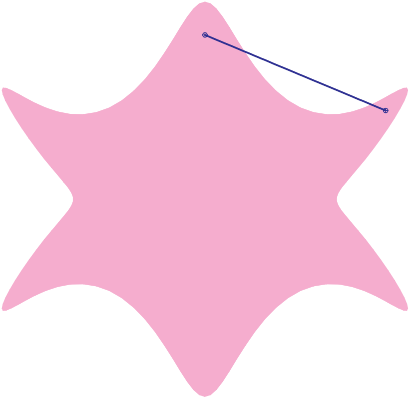


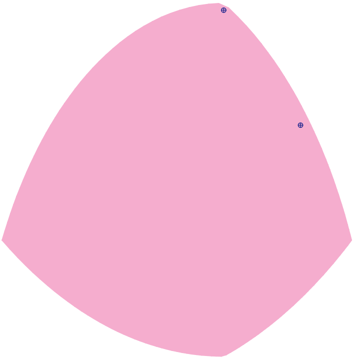


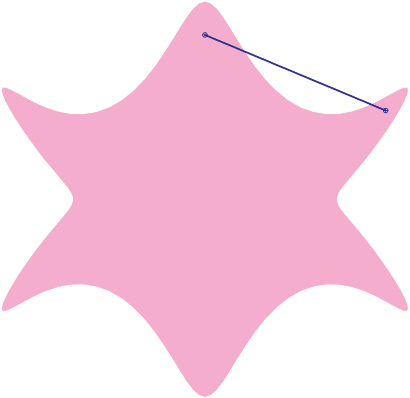


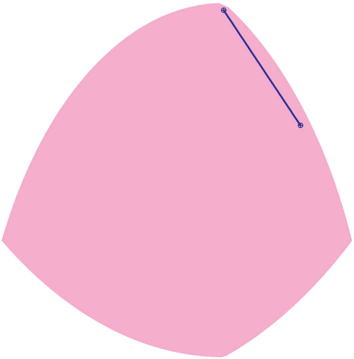


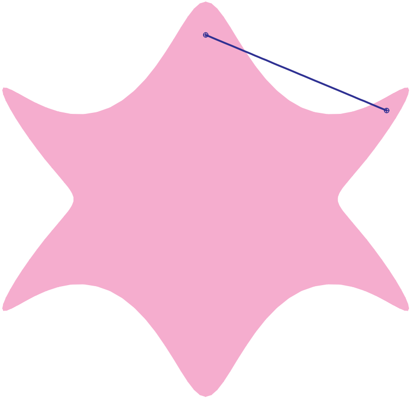


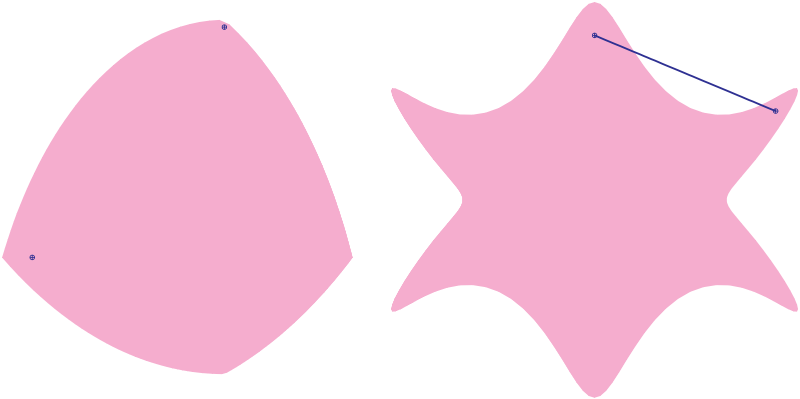


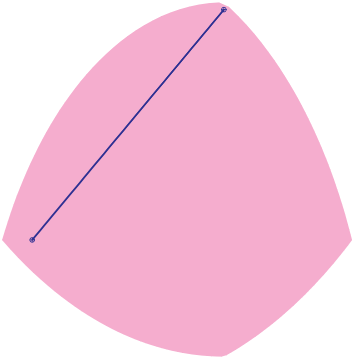


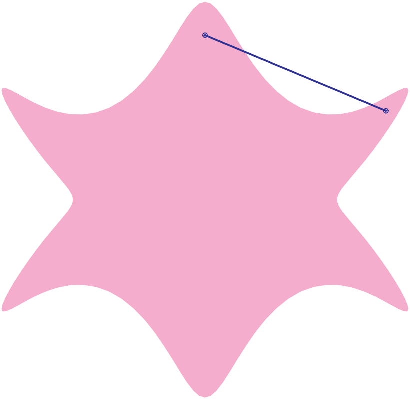






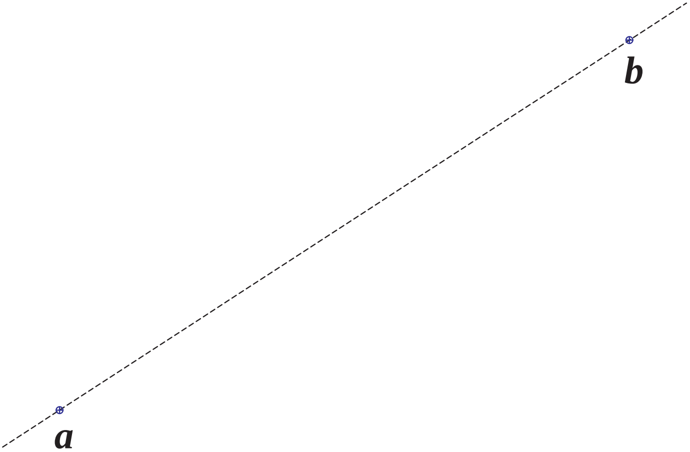


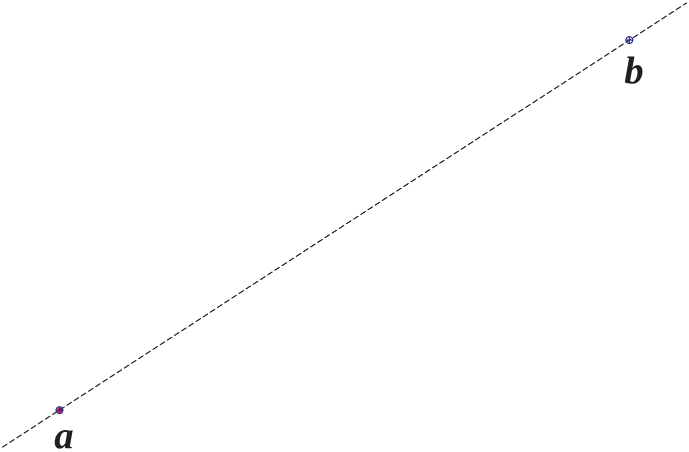




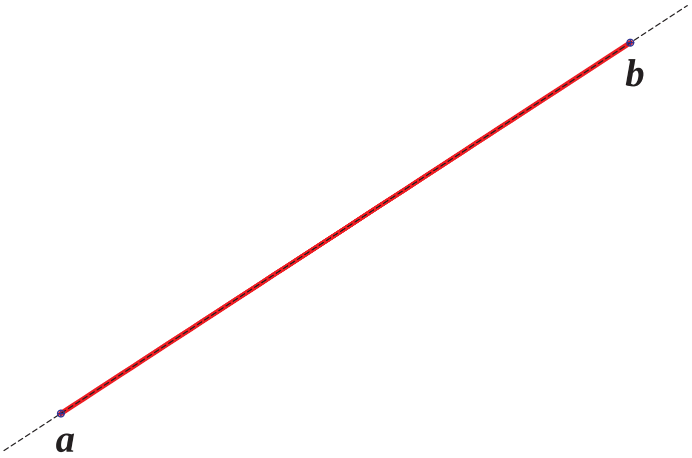
⊕
b

⊕
a

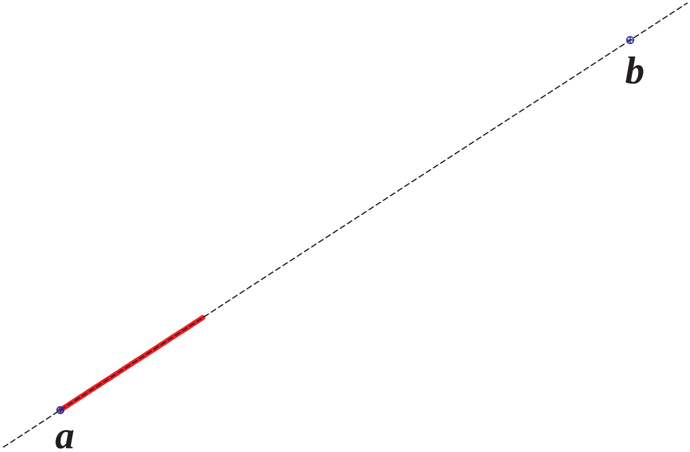

b
a



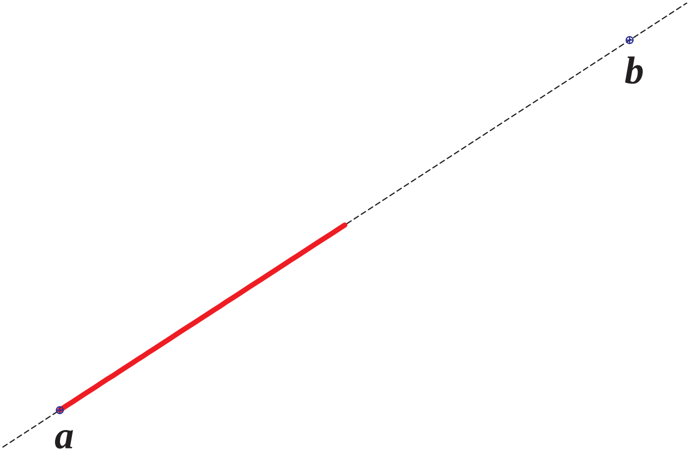
$$\boldsymbol{a} = 1 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \boldsymbol{a} + 0 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



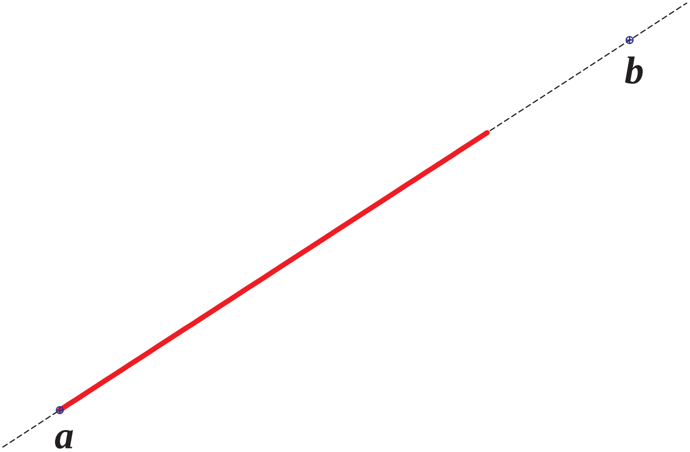
$$\boldsymbol{b} = 0 \cdot \boldsymbol{a} + 1 \cdot \boldsymbol{b} = \boldsymbol{a} + 1 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



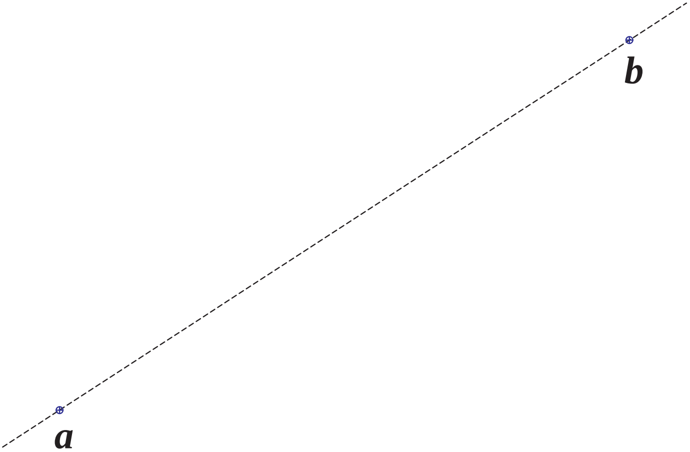
$$\frac{3}{4} \cdot \boldsymbol{a} + \frac{1}{4} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{1}{4} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$\frac{1}{2} \cdot \boldsymbol{a} + \frac{1}{2} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{1}{2} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$\frac{1}{4} \cdot \boldsymbol{a} + \frac{3}{4} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{3}{4} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$t \cdot \boldsymbol{a} + (1-t) \cdot \boldsymbol{b} = \boldsymbol{a} + (1-t) \cdot (\boldsymbol{b} - \boldsymbol{a})$$

Definition

Let $M \subset \mathbb{R}^n$. We say that M is convex if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in M \; \forall t \in [0, 1] \colon \; t\boldsymbol{x} + (1-t)\boldsymbol{y} \in M.$$

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. We say that f is

- concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1] \colon \ f(t\boldsymbol{a}+(1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a})+(1-t)f(\boldsymbol{b}),$$

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. We say that f is

- concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1] \colon \; f(t\boldsymbol{a}+(1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a})+(1-t)f(\boldsymbol{b}),$$

- strictly concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b} \, \forall t \in (0,1) \colon \\ f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}).$$

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. We say that f is

- concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1] \colon \; f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}),$$

- strictly concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b} \, \forall t \in (0,1) \colon \quad f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}).$$

Remark

By changing the inequalities to the opposite we obtain a definition of a *convex* and a *strictly convex* function.

Remark

A function f is convex (strictly convex) if and only if the function $-f$ is concave (strictly concave).
All the theorems in this section are formulated for concave and strictly concave functions. They have obvious analogies that hold for convex and strictly convex functions.

Remark

- If a function f is strictly concave on M, then it is concave on M.

Remark

- If a function f is strictly concave on M, then it is concave on M.
- Let f be a concave function on M. Then f is strictly concave on M if and only if the graph of f "does not contain a segment", i.e.

$$\neg\Big(\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b},\ \forall t \in [0,1]: \\ f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b})\Big)$$

Theorem 28

Let f be a function concave on an open convex set $G \subset \mathbb{R}^n$. Then f is continuous on G.

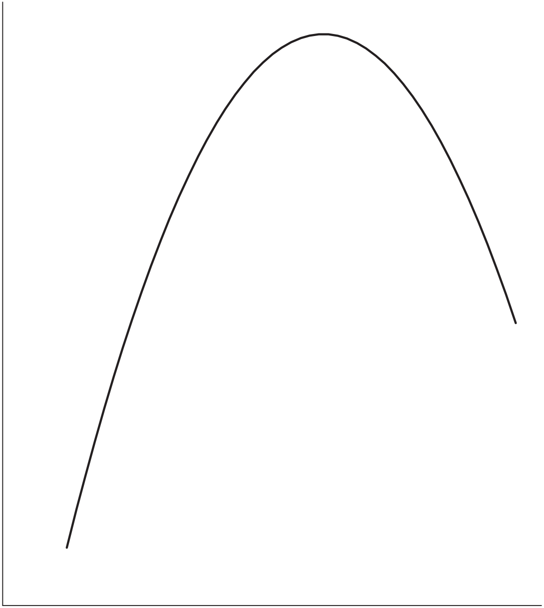
Theorem 28

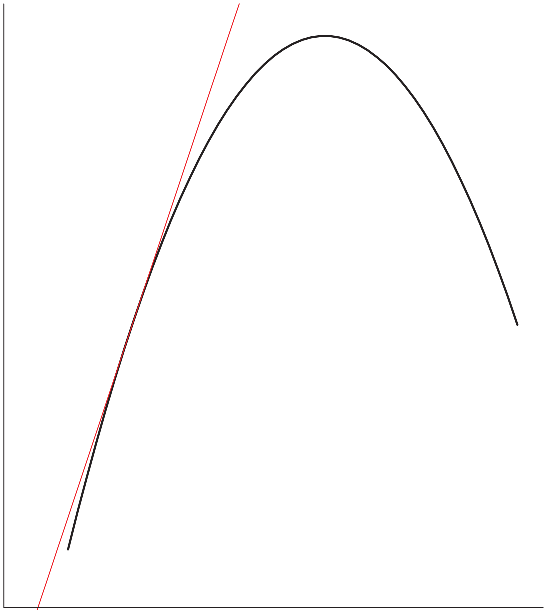
Let f be a function concave on an open convex set $G \subset \mathbb{R}^n$. Then f is continuous on G.

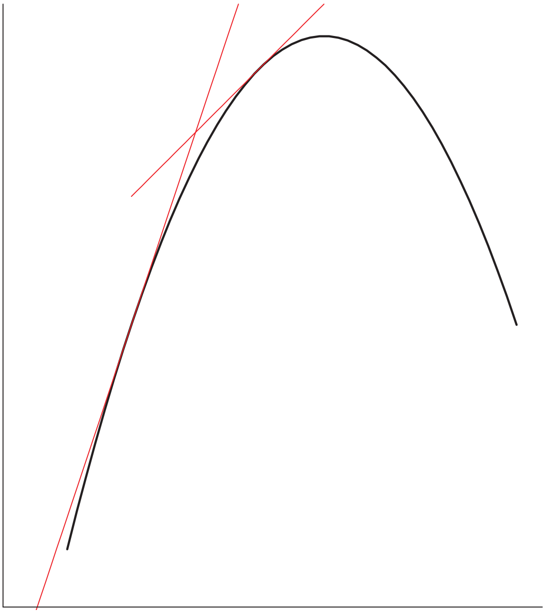
Theorem 29 (characterisation of strictly concave functions of the class $\mathcal{C}^1$)

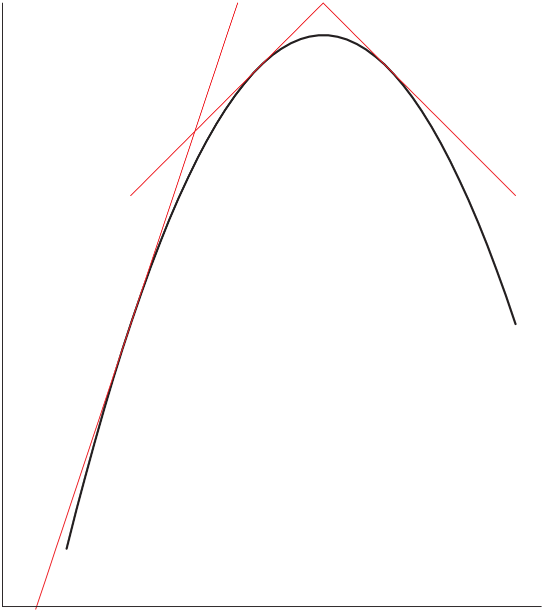
Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is strictly concave on G if and only if

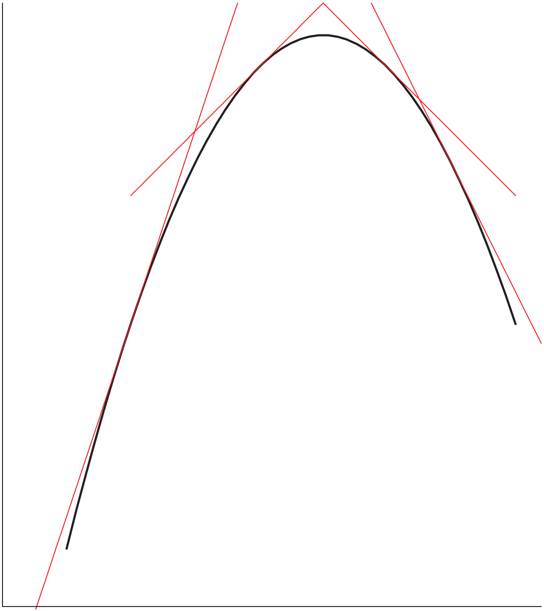
$$\forall \boldsymbol{x}, \boldsymbol{y} \in G, \boldsymbol{x} \neq \boldsymbol{y} \colon f(\boldsymbol{y}) < f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$











Theorem 30 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G\colon\ f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Theorem 30 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G\colon f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Corollary 31

Let $G \subset \mathbb{R}^n$ be a convex open set, $f \in C^1(G)$, and let $\boldsymbol{a} \in G$ be a critical point of f (i.e. $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$). If f is concave on G, then $\boldsymbol{a}$ is a maximum point of f on G.

Theorem 30 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G\colon\ f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Corollary 31

Let $G \subset \mathbb{R}^n$ be a convex open set, $f \in C^1(G)$, and let $\boldsymbol{a} \in G$ be a critical point of f (i.e. $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$). If f is concave on G, then $\boldsymbol{a}$ is a maximum point of f on G. If f is strictly concave on G, then $\boldsymbol{a}$ is a strict maximum point of f on G.

Theorem 32 (level sets of concave functions)

Let f be a function concave on a convex set $M \subset \mathbb{R}^n$. Then for each $\alpha \in \mathbb{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and let f be a function defined on M. We say that f is

- quasiconcave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1] \colon \; f(t\boldsymbol{a}+(1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and let f be a function defined on M. We say that f is

- quasiconcave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1] \colon \, f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

- strictly quasiconcave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \; \forall t \in (0,1) \colon \\ f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\}.$$

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and let f be a function defined on M. We say that f is

- quasiconcave on M if

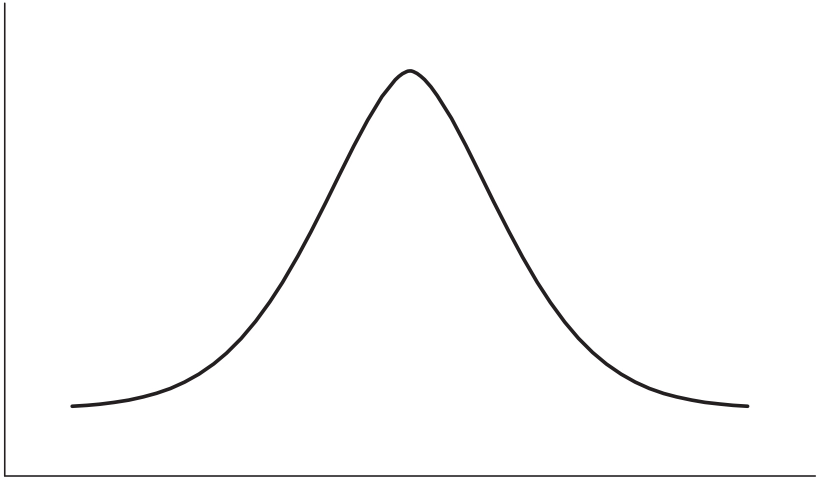
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \, \forall t \in [0,1]\colon \, f(t\boldsymbol{a}+(1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

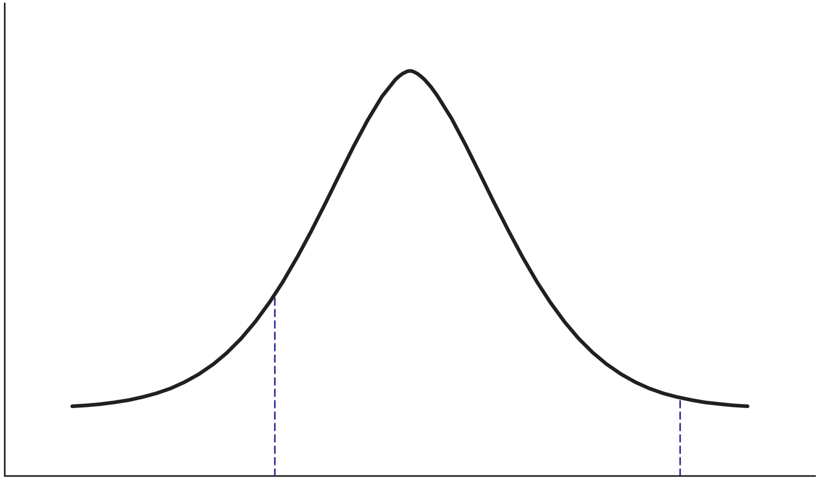
- strictly quasiconcave on M if

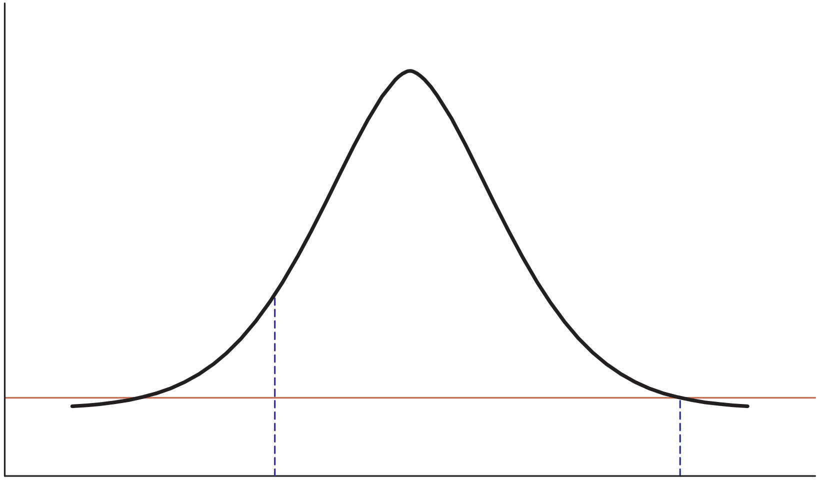
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \ \forall t \in (0,1)\colon \\ f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\}.$$

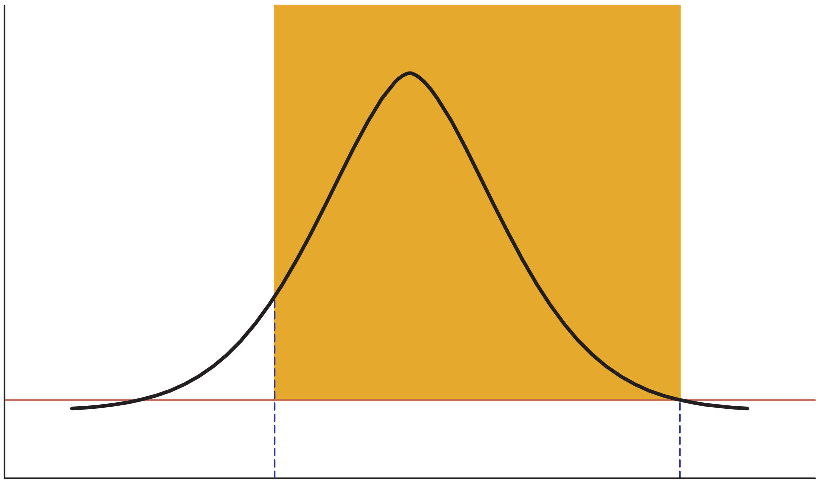
Remark

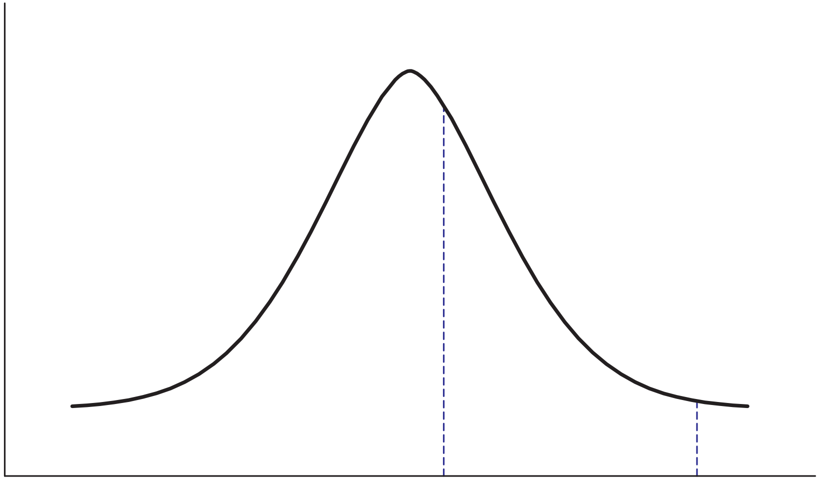
By changing the inequalities to the opposite and changing the minimum to a maximum we obtain a definition of a *quasiconvex* and a *strictly quasiconvex* function.

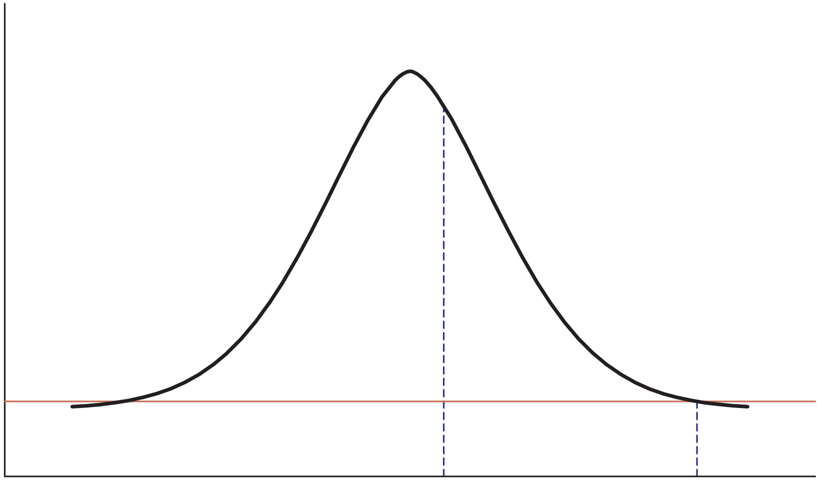


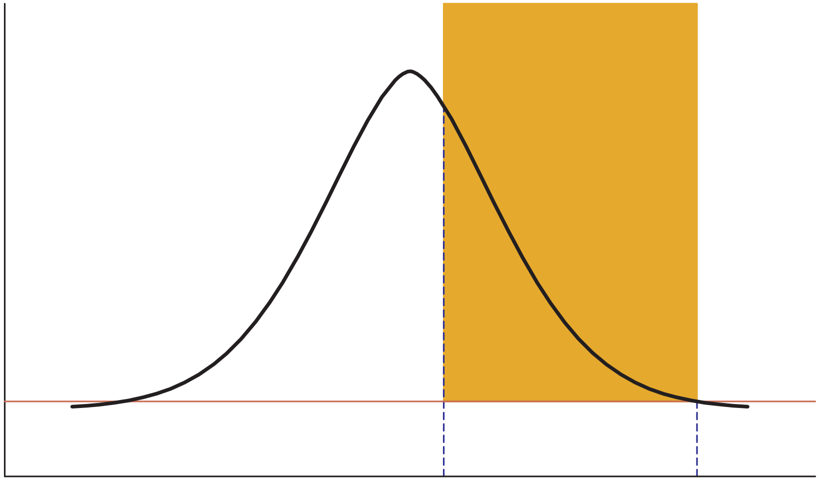


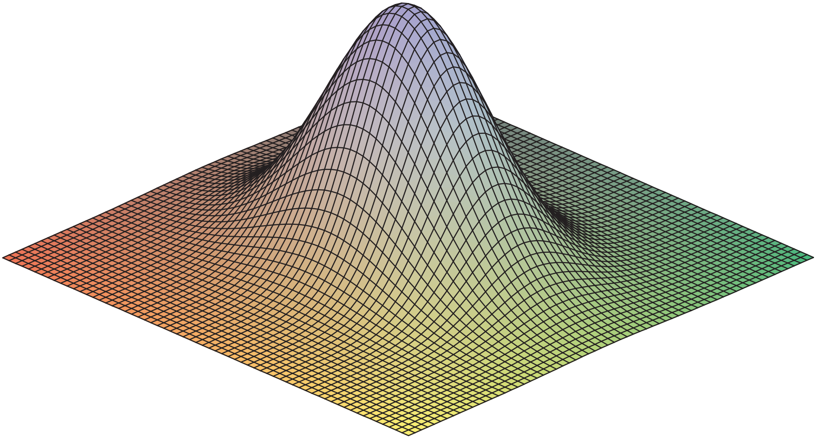












Remark

A function f is quasiconvex (strictly quasiconvex) if and only if the function $-f$ is quasiconcave (strictly quasiconcave).

All the theorems in this section are formulated for quasiconcave and strictly quasiconcave functions. They have obvious analogies that hold for quasiconvex and strictly quasiconvex functions.

Remark

- If a function f is strictly quasiconcave on M, then it is quasiconcave on M.

Remark

- If a function f is strictly quasiconcave on M, then it is quasiconcave on M.
- Let f be a quasiconcave function on M. Then f is strictly quasiconcave on M if and only if the graph of f "does not contain a horizontal segment", i.e.

$$\neg\big(\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \forall t \in [0,1]\colon f(t\boldsymbol{a}+(1-t)\boldsymbol{b}) = f(\boldsymbol{a})\big).$$

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If f is strictly concave on M, then f is strictly quasiconcave on M.

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If f is strictly concave on M, then f is strictly quasiconcave on M.

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If f is strictly concave on M, then f is strictly quasiconcave on M.

Theorem 33 (characterization of quasiconcave functions using level sets)

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. Then f is quasiconcave on M if and only if for each $\alpha \in \mathbb{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

Theorem 34 (a uniqueness of an extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbb{R}^n$. Then there exists at most one point of maximum of f.

Theorem 34 (a uniqueness of an extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbb{R}^n$. Then there exists at most one point of maximum of f.

Corollary

Let $M \subset \mathbb{R}^n$ be a convex, closed, bounded and nonempty set and f a continuous and strictly quasiconcave function on M. Then f attains its maximum at exactly one point.

Theorem 35 (sufficient condition for concave and convex functions in $\mathbb{R}^2$)

Let $G \subset \mathbb{R}^2$ be convex and $f \in C^2(G)$.
If $\frac{\partial^2 f}{\partial x^2} \leq 0$, $\frac{\partial^2 f}{\partial y^2} \leq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ hold on G, then f is concave on G.

Theorem 35 (sufficient condition for concave and convex functions in $\mathbb{R}^2$)

Let $G \subset \mathbb{R}^2$ be convex and $f \in C^2(G)$.

If $\frac{\partial^2 f}{\partial x^2} \leq 0$, $\frac{\partial^2 f}{\partial y^2} \leq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ hold on G, then f is concave on G.

If $\frac{\partial^2 f}{\partial x^2} \geq 0$, $\frac{\partial^2 f}{\partial y^2} \geq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ hold on G, then f is convex on G.

VI. Matrix calculus

VI.1. Basic operations with matrices

VI.1. Basic operations with matrices

Definition

A table of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of type $m \times n$ (shortly, an m-by-n matrix). We also write $(a_{ij})_{\substack{i=1..m \\ j=1..n}}$ for short.

VI.1. Basic operations with matrices

Definition

A table of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of type $m \times n$ (shortly, an m-by-n matrix). We also write $(a_{ij})_{\substack{i=1..m \\ j=1..n}}$ for short.

An n-by-n matrix is called a square matrix of order n.

VI.1. Basic operations with matrices

Definition

A table of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of type $m \times n$ (shortly, an m-by-n matrix). We also write $(a_{ij})_{\substack{i=1..m \\ j=1..n}}$ for short.

An n-by-n matrix is called a square matrix of order n.

The set of all m-by-n matrices is denoted by $M(m \times n)$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the *i*th row of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the *i*th row of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, where $j \in \{1, 2, \dots, n\}$, is called the jth column of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, where $j \in \{1, 2, \dots, n\}$, is called the jth column of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, where $j \in \{1, 2, \dots, n\}$, is called the jth column of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, where $j \in \{1, 2, \dots, n\}$, is called the jth column of the matrix $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, where $i \in \{1, 2, \dots, m\}$, is called the ith row of the matrix $\boldsymbol{A}$.

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, where $j \in \{1, 2, \dots, n\}$, is called the jth column of the matrix $\boldsymbol{A}$.

Definition

We say that two matrices are equal, if they are of the same type and the corresponding elements are equal, i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$ and $\boldsymbol{B} = (b_{uv})_{\substack{u=1..r \\ v=1..s}}$, then $\boldsymbol{A} = \boldsymbol{B}$ if and only if $m = r$, $n = s$ and
$a_{ij} = b_{ij} \; \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, n\}$.

Definition

Let $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$, $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\boldsymbol{B} = (b_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\lambda \in \mathbb{R}$.

The sum of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the matrix defined by

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

Definition

Let $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$, $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\boldsymbol{B} = (b_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\lambda \in \mathbb{R}$.
The sum of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the matrix defined by

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

The product of the real number λ and the matrix $\boldsymbol{A}$ (or the λ-multiple of the matrix $\boldsymbol{A}$) is the matrix defined by

$$\lambda \boldsymbol{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}.$$

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n) \colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon\ \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon\ \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n) \colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n) \colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n) \colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon\ \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon\ \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon\ \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C}_{\boldsymbol{A}} \in M(m \times n)\colon\ \boldsymbol{A} + \boldsymbol{C}_{\boldsymbol{A}} = \boldsymbol{O}$, *(existence of an opposite element)*

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C}_{\boldsymbol{A}} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{C}_{\boldsymbol{A}} = \boldsymbol{O}$, *(existence of an opposite element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$,

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C_A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{C_A} = \boldsymbol{O}$, *(existence of an opposite element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$,
- $\forall \boldsymbol{A} \in M(m \times n)\colon 1 \cdot \boldsymbol{A} = \boldsymbol{A}$,

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C}_{\boldsymbol{A}} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{C}_{\boldsymbol{A}} = \boldsymbol{O}$, *(existence of an opposite element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$,
- $\forall \boldsymbol{A} \in M(m \times n)\colon 1 \cdot \boldsymbol{A} = \boldsymbol{A}$,
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda + \mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$,

Proposition 36 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C}_{\boldsymbol{A}} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{C}_{\boldsymbol{A}} = \boldsymbol{O}$, *(existence of an opposite element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$,
- $\forall \boldsymbol{A} \in M(m \times n)\colon 1 \cdot \boldsymbol{A} = \boldsymbol{A}$,
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda + \mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$,
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \lambda \in \mathbb{R}\colon \lambda(\boldsymbol{A} + \boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$.

Remark

- The matrix $\boldsymbol{O}$ from the previous proposition is called a zero matrix and all its elements are all zeros.

Remark

- The matrix $\boldsymbol{O}$ from the previous proposition is called a zero matrix and all its elements are all zeros.
- The matrix $\boldsymbol{C_A}$ from the previous proposition is called a matrix opposite to $\boldsymbol{A}$. It is determined uniquely, it is denoted by $-\boldsymbol{A}$, and it satisfies $-\boldsymbol{A} = (-a_{ij})_{\substack{i=1..m \\ j=1..n}}$ and $-\boldsymbol{A} = -1 \cdot \boldsymbol{A}$.

Definition

Let $\boldsymbol{A} \in M(m \times n)$, $\boldsymbol{A} = (a_{is})_{\substack{i=1..m \\ s=1..n}}$, $\boldsymbol{B} \in M(n \times k)$, $\boldsymbol{B} = (b_{sj})_{\substack{s=1..n \\ j=1..k}}$. Then the product of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as a matrix $\boldsymbol{AB} \in M(m \times k)$, $\boldsymbol{AB} = (c_{ij})_{\substack{i=1..m \\ j=1..k}}$, where

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}
$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}
$$

Theorem 37 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$. Then:

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l)$:
$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$, *(associativity of multiplication)*

Theorem 37 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$. *Then:*

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l):$
$\boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C}$, *(associativity of multiplication)*

(ii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B}, \boldsymbol{C} \in M(n \times k):$
$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}\boldsymbol{C}$, *(distributivity from the left)*

Theorem 37 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$. Then:

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l):$
$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$, *(associativity of multiplication)*

(ii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B}, \boldsymbol{C} \in M(n \times k):$
$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$, *(distributivity from the left)*

(iii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \boldsymbol{C} \in M(n \times k):$
$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$, *(distributivity from the right)*

Theorem 37 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$*. Then:*

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l):$
$\boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C},$ *(associativity of multiplication)*

(ii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B}, \boldsymbol{C} \in M(n \times k):$
$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}\boldsymbol{C},$ *(distributivity from the left)*

(iii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \boldsymbol{C} \in M(n \times k):$
$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{C},$ *(distributivity from the right)*

(iv) $\exists! \boldsymbol{I} \in M(n \times n)\ \forall \boldsymbol{A} \in M(n \times n): \boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}.$
(existence and uniqueness of an identity matrix $\boldsymbol{I}$*)*

Theorem 37 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$*. Then:*

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l)$:
$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$, *(associativity of multiplication)*

(ii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B}, \boldsymbol{C} \in M(n \times k)$:
$\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$, *(distributivity from the left)*

(iii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \boldsymbol{C} \in M(n \times k)$:
$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$, *(distributivity from the right)*

(iv) $\exists! \boldsymbol{I} \in M(n \times n)\ \forall \boldsymbol{A} \in M(n \times n)$: $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$.
(existence and uniqueness of an identity matrix $\boldsymbol{I}$*)*

Remark

Warning! The matrix multiplication is not commutative.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, then $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, where $b_{uv} = a_{vu}$ for each $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, then $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, where $b_{uv} = a_{vu}$ for each $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, then $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, where $b_{uv} = a_{vu}$ for each $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, then $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, where $b_{uv} = a_{vu}$ for each $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$\boldsymbol{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T=\begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A}=(a_{ij})_{\substack{i=1..m\\ j=1..n}}$, then $\boldsymbol{A}^T=(b_{uv})_{\substack{u=1..n\\ v=1..m}}$, where $b_{uv}=a_{vu}$ for each $u\in\{1,\dots,n\}$, $v\in\{1,2,\dots,m\}$.

Theorem 38 (properties of the transpose of a matrix)

The following is true:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon\ \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

Theorem 38 (properties of the transpose of a matrix)

The following is true:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

(ii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon (\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$,

Theorem 38 (properties of the transpose of a matrix)

The following is true:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

(ii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon (\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$,

(iii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\colon (\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.

Theorem 38 (properties of the transpose of a matrix)

The following is true:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

(ii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon (\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$,

(iii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\colon (\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.

Theorem 38 (properties of the transpose of a matrix)

The following is true:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

(ii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon (\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$,

(iii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\colon (\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.

Definition

We say that the matrix $\boldsymbol{A} \in M(n \times n)$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$.

VI.2. Invertible matrices

VI.2. Invertible matrices

Definition

Let $\boldsymbol{A} \in M(n \times n)$. We say that $\boldsymbol{A}$ is an invertible matrix if there exist $\boldsymbol{B} \in M(n \times n)$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}.$$

VI.2. Invertible matrices

Definition

Let $\boldsymbol{A} \in M(n \times n)$. We say that $\boldsymbol{A}$ is an invertible matrix if there exist $\boldsymbol{B} \in M(n \times n)$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}.$$

Definition

We say that the matrix $\boldsymbol{B} \in M(n \times n)$ is an inverse of a matrix $\boldsymbol{A} \in M(n \times n)$ if $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

VI.2. Invertible matrices

Definition

Let $\boldsymbol{A} \in M(n \times n)$. We say that $\boldsymbol{A}$ is an invertible matrix if there exist $\boldsymbol{B} \in M(n \times n)$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}.$$

Definition

We say that the matrix $\boldsymbol{B} \in M(n \times n)$ is an inverse of a matrix $\boldsymbol{A} \in M(n \times n)$ if $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

Remark

A matrix $\boldsymbol{A} \in M(n \times n)$ is invertible if and only if it has an inverse.

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Theorem 39 (operations with invertible matrices)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ be invertible matrices. Then

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Theorem 39 (operations with invertible matrices)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ be invertible matrices. Then

(i) $\boldsymbol{A}^{-1}$ *is invertible and* $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$,

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Theorem 39 (operations with invertible matrices)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ be invertible matrices. Then

(i) $\boldsymbol{A}^{-1}$ *is invertible and* $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$,

(ii) $\boldsymbol{A}^{T}$ *is invertible and* $\left(\boldsymbol{A}^{T}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{T}$,

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Theorem 39 (operations with invertible matrices)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ be invertible matrices. Then

(i) $\boldsymbol{A}^{-1}$ *is invertible and* $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$,

(ii) $\boldsymbol{A}^{T}$ *is invertible and* $\left(\boldsymbol{A}^{T}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{T}$,

(iii) $\boldsymbol{AB}$ *is invertible and* $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Definition

Let $k, n \in \mathbb{N}$ and $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$. We say that a vector $\boldsymbol{u} \in \mathbb{R}^n$ is a linear combination of the vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ with coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ if

$$\boldsymbol{u} = \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k.$$

Definition

Let $k, n \in \mathbb{N}$ and $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$. We say that a vector $\boldsymbol{u} \in \mathbb{R}^n$ is a linear combination of the vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ with coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ if

$$\boldsymbol{u} = \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k.$$

By a trivial linear combination of vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ we mean the linear combination $0 \cdot \boldsymbol{v}^1 + \cdots + 0 \cdot \boldsymbol{v}^k$.

Definition

Let $k, n \in \mathbb{N}$ and $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k \in \mathbb{R}^n$. We say that a vector $\boldsymbol{u} \in \mathbb{R}^n$ is a linear combination of the vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ with coefficients $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ if

$$\boldsymbol{u} = \lambda_1 \boldsymbol{v}^1 + \dots + \lambda_k \boldsymbol{v}^k.$$

By a trivial linear combination of vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ we mean the linear combination $0 \cdot \boldsymbol{v}^1 + \dots + 0 \cdot \boldsymbol{v}^k$. Linear combination which is not trivial is called non-trivial.

Definition

We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ are linearly dependent if there exists their non-trivial linear combination which is equal to the zero vector.

Definition

We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ are linearly dependent if there exists their non-trivial linear combination which is equal to the zero vector. We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ are linearly independent if they are not linearly dependent, i.e. if whenever $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ satisfy $\lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k = \boldsymbol{o}$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Definition

We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ are linearly dependent if there exists their non-trivial linear combination which is equal to the zero vector. We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ are linearly independent if they are not linearly dependent, i.e. if whenever $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ satisfy $\lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k = \boldsymbol{o}$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Remark

Vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ are linearly dependent if and only if one of them can be expressed as a linear combination of the others.

Definition

Let $\boldsymbol{A} \in M(m \times n)$. The rank of the matrix $\boldsymbol{A}$ is the maximal number of linearly independent row vectors of $\boldsymbol{A}$, i.e. the rank is equal to $k \in \mathbb{N}$ if

(i) there is k linearly independent row vectors of $\boldsymbol{A}$ and

(ii) each l-tuple of row vectors of $\boldsymbol{A}$, where $l > k$, is linearly dependent.

Definition

Let $\boldsymbol{A} \in M(m \times n)$. The rank of the matrix $\boldsymbol{A}$ is the maximal number of linearly independent row vectors of $\boldsymbol{A}$, i.e. the rank is equal to $k \in \mathbb{N}$ if

(i) there is k linearly independent row vectors of $\boldsymbol{A}$ and

(ii) each l-tuple of row vectors of $\boldsymbol{A}$, where $l > k$, is linearly dependent.

The rank of the zero matrix is zero. Rank of $\boldsymbol{A}$ is denoted by $\operatorname{rank}(\boldsymbol{A})$.

Definition

We say that a matrix $\boldsymbol{A} \in M(m \times n)$ is in a row echelon form if for each $i \in \{2, \ldots, m\}$ the ith row of $\boldsymbol{A}$ is either a zero vector or it has more zeros at the beginning than the $(i-1)$th row.

Definition

We say that a matrix $\boldsymbol{A} \in M(m \times n)$ is in a row echelon form if for each $i \in \{2, \ldots, m\}$ the ith row of $\boldsymbol{A}$ is either a zero vector or it has more zeros at the beginning than the $(i-1)$th row.

Remark

The rank of a row echelon matrix is equal to the number of its non-zero rows.

Definition

The elementary row operations on the matrix $\boldsymbol{A}$ are:

(i) interchange of two rows,

Definition

The elementary row operations on the matrix $\boldsymbol{A}$ are:

(i) interchange of two rows,

(ii) multiplication of a row by a non-zero real number,

Definition

The elementary row operations on the matrix $\boldsymbol{A}$ are:

(i) interchange of two rows,
(ii) multiplication of a row by a non-zero real number,
(iii) addition of a multiple of a row to another row.

Definition

The elementary row operations on the matrix $\boldsymbol{A}$ are:

(i) interchange of two rows,

(ii) multiplication of a row by a non-zero real number,

(iii) addition of a multiple of a row to another row.

Definition

A matrix transformation is a finite sequence of elementary row operations. If a matrix $\boldsymbol{B} \in M(m \times n)$ results from the matrix $\boldsymbol{A} \in M(m \times n)$ by applying a transformation T on the matrix $\boldsymbol{A}$, then this fact is denoted by $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{B}$.

Theorem 40 (properties of matrix transformations)

(i) *Let $\boldsymbol{A} \in M(m \times n)$. Then there exists a transformation transforming $\boldsymbol{A}$ to a row echelon matrix.*

Theorem 40 (properties of matrix transformations)

(i) *Let $\boldsymbol{A} \in M(m \times n)$. Then there exists a transformation transforming $\boldsymbol{A}$ to a row echelon matrix.*

(ii) *Let T_1 be a transformation applicable to m-by-n matrices. Then there exists a transformation T_2 applicable to m-by-n matrices such that for any two matrices $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$ we have $\boldsymbol{A} \overset{T_1}{\rightsquigarrow} \boldsymbol{B}$ if and only if $\boldsymbol{B} \overset{T_2}{\rightsquigarrow} \boldsymbol{A}$.*

Theorem 40 (properties of matrix transformations)

(i) *Let $\boldsymbol{A} \in M(m \times n)$. Then there exists a transformation transforming $\boldsymbol{A}$ to a row echelon matrix.*

(ii) *Let T_1 be a transformation applicable to m-by-n matrices. Then there exists a transformation T_2 applicable to m-by-n matrices such that for any two matrices $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$ we have $\boldsymbol{A} \overset{T_1}{\rightsquigarrow} \boldsymbol{B}$ if and only if $\boldsymbol{B} \overset{T_2}{\rightsquigarrow} \boldsymbol{A}$.*

(iii) *Let $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$ and there exist a transformation T such that $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{B}$. Then $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{B})$.*

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ {\color{green}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{red}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{green}\bullet} & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \color{red}{\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \color{green}{\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & \color{green}{\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & \color{green}{\bullet} & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & {\color{red}\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & {\color{green}\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & {\color{green}\bullet} & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & {\color{red}\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & \bullet \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \color{green}{\bullet} \\ 0 & 0 & 0 & 0 & 0 & \color{green}{\bullet} \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & \textcolor{red}{\bullet} \\ 0 & 0 & 0 & 0 & 0 & \textcolor{green}{\bullet} \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & {\color{red}\bullet} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Transformation to a row echelon form

$$\begin{pmatrix} {\color{red}\bullet} & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & {\color{red}\bullet} & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & {\color{red}\bullet} & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & {\color{red}\bullet} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark

Similarly as the elementary row operations one can define also elementary column operations. It can be shown that the elementary column operations do not change the rank of the matrix.

Remark

Similarly as the elementary row operations one can define also elementary column operations. It can be shown that the elementary column operations do not change the rank of the matrix.

Remark

It can be shown that $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}^T)$ for any $\boldsymbol{A} \in M(m \times n)$.

Theorem 41 (representation of a transformation)

Let T be a transformation on $m \times n$ matrices. Then there exists an invertible matrix $\boldsymbol{C}_T \in M(m \times m)$ satisfying: whenever we apply the transformation T to a matrix $\boldsymbol{A} \in M(m \times n)$, we obtain the matrix $\boldsymbol{C}_T\boldsymbol{A}$.

Theorem 41 (representation of a transformation)

Let T be a transformation on $m \times n$ matrices. Then there exists an invertible matrix $\boldsymbol{C}_T \in M(m \times m)$ satisfying: whenever we apply the transformation T to a matrix $\boldsymbol{A} \in M(m \times n)$, we obtain the matrix $\boldsymbol{C}_T\boldsymbol{A}$.

Remark

Also the converse is true: For every invertible matrix $\boldsymbol{C}$ the mapping $\boldsymbol{A} \mapsto \boldsymbol{CA}$ is a transformation.

Lemma 42

Let $\boldsymbol{A} \in M(n \times n)$ *and* $\operatorname{rank}(\boldsymbol{A}) = n$*. Then there exists a transformation transforming* $\boldsymbol{A}$ *to* $\boldsymbol{I}$*.*

Lemma 42

Let $\boldsymbol{A} \in M(n \times n)$ and $\operatorname{rank}(\boldsymbol{A}) = n$. Then there exists a transformation transforming $\boldsymbol{A}$ to $\boldsymbol{I}$.

Theorem 43

Let $\boldsymbol{A} \in M(n \times n)$. Then $\boldsymbol{A}$ is invertible if and only if $\operatorname{rank}(\boldsymbol{A}) = n$.

VI.3. Determinants

VI.3. Determinants

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

VI.3. Determinants

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i,1} & \dots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \dots & a_{i,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

VI.3. Determinants

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i,1} & \dots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \dots & a_{i,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

VI.3. Determinants

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & & a_{i-1,j+1} & \dots & a_{i-1,n} \\ & & & & & & \\ a_{i+1,1} & \dots & a_{i+1,j-1} & & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

VI.3. Determinants

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\boldsymbol{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\boldsymbol{A}$ by omitting the ith row and the jth column.

$$\boldsymbol{A}_{ij} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. The determinant of the matrix $\boldsymbol{A}$ is defined by

$$\det \boldsymbol{A} = \begin{cases} a_{11} & \text{if } n = 1, \\ \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \boldsymbol{A}_{i1} & \text{if } n > 1. \end{cases}$$

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. The determinant of the matrix $\boldsymbol{A}$ is defined by

$$\det \boldsymbol{A} = \begin{cases} a_{11} & \text{if } n = 1, \\ \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \boldsymbol{A}_{i1} & \text{if } n > 1. \end{cases}$$

For $\det \boldsymbol{A}$ we will also use the symbol

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Theorem 44 (cofactor expansion)

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$, $k \in \{1, \ldots, n\}$. *Then*

$$\det \boldsymbol{A} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det \boldsymbol{A}_{ik} \quad \textit{(expansion along kth column),}$$

$$\det \boldsymbol{A} = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det \boldsymbol{A}_{kj} \quad \textit{(expansion along kth row).}$$

Lemma 45

Let $j, n \in \mathbb{N}$, $j \leq n$, and the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(n \times n)$ coincide at each row except for the jth row. Let the jth row of $\boldsymbol{A}$ be equal to the sum of the jth rows of $\boldsymbol{B}$ and $\boldsymbol{C}$. Then $\det \boldsymbol{A} = \det \boldsymbol{B} + \det \boldsymbol{C}$.

Lemma 45

Let $j, n \in \mathbb{N}$, $j \leq n$, and the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(n \times n)$ coincide at each row except for the jth row. Let the jth row of $\boldsymbol{A}$ be equal to the sum of the jth rows of $\boldsymbol{B}$ and $\boldsymbol{C}$. Then $\det \boldsymbol{A} = \det \boldsymbol{B} + \det \boldsymbol{C}$.

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1+v_1 & \dots & u_n+v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 & \dots & u_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ v_1 & \dots & v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Theorem 46 (determinant and transformations)

Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

(i) *If the matrix* $\boldsymbol{A}'$ *is created from the matrix* $\boldsymbol{A}$ *by multiplying one row in* $\boldsymbol{A}$ *by a real number* μ, *then* $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$.

Theorem 46 (determinant and transformations)

Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

(i) *If the matrix* $\boldsymbol{A}'$ *is created from the matrix* $\boldsymbol{A}$ *by multiplying one row in* $\boldsymbol{A}$ *by a real number* μ, *then* $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$.

(ii) *If the matrix* $\boldsymbol{A}'$ *is created from* $\boldsymbol{A}$ *by interchanging two rows in* $\boldsymbol{A}$ *(i.e. by applying the elementary row operation of the first type), then* $\det \boldsymbol{A}' = -\det \boldsymbol{A}$.

Theorem 46 (determinant and transformations)

Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

(i) *If the matrix $\boldsymbol{A}'$ is created from the matrix $\boldsymbol{A}$ by multiplying one row in $\boldsymbol{A}$ by a real number μ, then $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$.*

(ii) *If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by interchanging two rows in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the first type), then $\det \boldsymbol{A}' = -\det \boldsymbol{A}$.*

(iii) *If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by adding a μ-multiple of a row in $\boldsymbol{A}$ to another row in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the third type), then $\det \boldsymbol{A}' = \det \boldsymbol{A}$.*

Theorem 46 (determinant and transformations)

Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

(i) *If the matrix $\boldsymbol{A}'$ is created from the matrix $\boldsymbol{A}$ by multiplying one row in $\boldsymbol{A}$ by a real number μ, then* $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$*.*

(ii) *If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by interchanging two rows in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the first type), then* $\det \boldsymbol{A}' = -\det \boldsymbol{A}$*.*

(iii) *If the matrix $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by adding a μ-multiple of a row in $\boldsymbol{A}$ to another row in $\boldsymbol{A}$ (i.e. by applying the elementary row operation of the third type), then* $\det \boldsymbol{A}' = \det \boldsymbol{A}$*.*

(iv) *If $\boldsymbol{A}'$ is created from $\boldsymbol{A}$ by applying a transformation, then* $\det \boldsymbol{A} \neq 0$ *if and only if* $\det \boldsymbol{A}' \neq 0$*.*

Remark

The determinant of a matrix with a zero row is equal to zero.

Remark

The determinant of a matrix with a zero row is equal to zero. The determinant of a matrix with two identical rows is also equal to zero.

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. We say that $\boldsymbol{A}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \ldots, n\}$.

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. We say that $\boldsymbol{A}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \ldots, n\}$. We say that $\boldsymbol{A}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$, $i, j \in \{1, \ldots, n\}$.

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. We say that $\boldsymbol{A}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \dots, n\}$. We say that $\boldsymbol{A}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$, $i, j \in \{1, \dots, n\}$.

Theorem 47 (determinant of a triangular matrix)

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$ be an upper or lower triangular matrix. Then

$$\det \boldsymbol{A} = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}.$$

Theorem 48 (determinant and invertibility)

Let $\boldsymbol{A} \in M(n \times n)$*. Then* $\boldsymbol{A}$ *is invertible if and only if* $\det \boldsymbol{A} \neq 0$*.*

Theorem 49 (determinant of a product)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$*. Then* $\det \boldsymbol{AB} = \det \boldsymbol{A} \cdot \det \boldsymbol{B}$*.*

Theorem 49 (determinant of a product)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$*. Then* $\det \boldsymbol{AB} = \det \boldsymbol{A} \cdot \det \boldsymbol{B}$*.*

Theorem 50 (determinant of a transpose)

Let $\boldsymbol{A} \in M(n \times n)$*. Then* $\det \boldsymbol{A}^T = \det \boldsymbol{A}$*.*

VI.4. Systems of linear equations

A system of m equations in n unknowns $x_1, \dots, x_n$:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}
\tag{S}
$$

where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$.

A system of m equations in n unknowns $x_1, \dots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{S}$$

where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$. The matrix form is

$$\boldsymbol{Ax} = \boldsymbol{b},$$

where $\boldsymbol{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \in M(m \times n)$, is called the coefficient matrix, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M(m \times 1)$ is called the vector of the right-hand side and $\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M(n \times 1)$ is the vector of unknowns.

Definition

The matrix

$$(\boldsymbol{A}|\boldsymbol{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

is called the augmented matrix of the system (S).

Proposition 51 (solutions of a transformed system)

Let $\boldsymbol{A} \in M(m \times n)$, $\boldsymbol{b} \in M(m \times 1)$ *and let* T *be a transformation of matrices with* m *rows. Denote* $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{A}'$, $\boldsymbol{b} \overset{T}{\rightsquigarrow} \boldsymbol{b}'$. *Then for any* $\boldsymbol{y} \in M(n \times 1)$ *we have* $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b}$ *if and only if* $\boldsymbol{A}'\boldsymbol{y} = \boldsymbol{b}'$, *i.e. the systems* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *and* $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ *have the same set of solutions.*

Theorem 52 (Rouché-Fontené)

The system (S) *has a solution if and only if its coefficient matrix has the same rank as its augmented matrix.*

Systems of n equations in n variables

Systems of n equations in n variables

Theorem 53 (solvability of an $n \times n$ system)

Let $\boldsymbol{A} \in M(n \times n)$. Then the following statements are equivalent:

(i) *the matrix $\boldsymbol{A}$ is invertible,*

Systems of n equations in n variables

Theorem 53 (solvability of an $n \times n$ system)

Let $\boldsymbol{A} \in M(n \times n)$. Then the following statements are equivalent:

(i) *the matrix $\boldsymbol{A}$ is invertible,*

(ii) *for each $\boldsymbol{b} \in M(n \times 1)$ the system* (S) *has a unique solution,*

Systems of n equations in n variables

Theorem 53 (solvability of an $n \times n$ system)

Let $\boldsymbol{A} \in M(n \times n)$. Then the following statements are equivalent:

(i) *the matrix $\boldsymbol{A}$ is invertible,*

(ii) *for each $\boldsymbol{b} \in M(n \times 1)$ the system* (S) *has a unique solution,*

(iii) *for each $\boldsymbol{b} \in M(n \times 1)$ the system* (S) *has at least one solution,*

Systems of n equations in n variables

Theorem 53 (solvability of an $n \times n$ system)

Let $\boldsymbol{A} \in M(n \times n)$. Then the following statements are equivalent:

(i) *the matrix $\boldsymbol{A}$ is invertible,*

(ii) *for each $\boldsymbol{b} \in M(n \times 1)$ the system* (S) *has a unique solution,*

(iii) *for each $\boldsymbol{b} \in M(n \times 1)$ the system* (S) *has at least one solution,*

(iv) $\det \boldsymbol{A} \neq 0$.

Theorem 54 (Cramer's rule)

Let $\boldsymbol{A} \in M(n \times n)$ *be an invertible matrix,* $\boldsymbol{b} \in M(n \times 1)$*,* $\boldsymbol{x} \in M(n \times 1)$*, and* $\boldsymbol{Ax} = \boldsymbol{b}$*. Then*

$$x_j = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{vmatrix}}{\det \boldsymbol{A}}$$

for $j = 1, \dots, n$*.*

VI.5. Definiteness of matrices

VI.5. Definiteness of matrices

Definition

We say that a symmetric matrix $\boldsymbol{A} \in M(n \times n)$ is

- positive definite (PD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,

VI.5. Definiteness of matrices

Definition

We say that a symmetric matrix $\boldsymbol{A} \in M(n \times n)$ is

- positive definite (PD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- negative definite (ND), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} < 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,

VI.5. Definiteness of matrices

Definition

We say that a symmetric matrix $\boldsymbol{A} \in M(n \times n)$ is

- positive definite (PD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- negative definite (ND), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} < 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- positive semidefinite (PSD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} \geq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$,

VI.5. Definiteness of matrices

Definition

We say that a symmetric matrix $\boldsymbol{A} \in M(n \times n)$ is

- positive definite (PD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- negative definite (ND), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} < 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- positive semidefinite (PSD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} \geq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$,
- negative semidefinite (NSD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} \leq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$,

VI.5. Definiteness of matrices

Definition

We say that a symmetric matrix $\boldsymbol{A} \in M(n \times n)$ is

- positive definite (PD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- negative definite (ND), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} < 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- positive semidefinite (PSD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} \geq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$,
- negative semidefinite (NSD), if $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} \leq 0$ for all $\boldsymbol{u} \in \mathbb{R}^n$,
- indefinite (ID), if there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} > 0$ and $\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} < 0$.

Proposition 55 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ be diagonal (i.e. $a_{ij} = 0$ whenever $i \neq j$). Then

- *$\boldsymbol{A}$ is PD if and only if $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 55 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ *be diagonal (i.e.* $a_{ij} = 0$ *whenever* $i \neq j$*). Then*

- $\boldsymbol{A}$ *is PD if and only if* $a_{ii} > 0$ *for all* $i = 1, 2, \dots, n$*,*
- $\boldsymbol{A}$ *is ND if and only if* $a_{ii} < 0$ *for all* $i = 1, 2, \dots, n$*,*

Proposition 55 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ be diagonal (i.e. $a_{ij} = 0$ whenever $i \neq j$). Then

- *$\boldsymbol{A}$ is PD if and only if $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*
- *$\boldsymbol{A}$ is ND if and only if $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,*
- *$\boldsymbol{A}$ is PSD if and only if $a_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 55 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ be diagonal (i.e. $a_{ij} = 0$ whenever $i \neq j$). Then

- $\boldsymbol{A}$ *is PD if and only if* $a_{ii} > 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is ND if and only if* $a_{ii} < 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is PSD if and only if* $a_{ii} \geq 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is NSD if and only if* $a_{ii} \leq 0$ *for all* $i = 1, 2, \ldots, n$,

Proposition 55 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ *be diagonal (i.e.* $a_{ij} = 0$ *whenever* $i \neq j$*). Then*

- $\boldsymbol{A}$ *is PD if and only if* $a_{ii} > 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is ND if and only if* $a_{ii} < 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is PSD if and only if* $a_{ii} \geq 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is NSD if and only if* $a_{ii} \leq 0$ *for all* $i = 1, 2, \ldots, n$,
- $\boldsymbol{A}$ *is ID if and only if there exist* $i, j \in \{1, 2, \ldots, n\}$ *such that* $a_{ii} > 0$ *and* $a_{jj} < 0$.

Proposition 56 (necessary conditions for definiteness)

Let $\boldsymbol{A} \in M(n \times n)$ be a symmetric matrix. Then

- *If $\boldsymbol{A}$ is PD, then $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 56 (necessary conditions for definiteness)

Let $\boldsymbol{A} \in M(n \times n)$ be a symmetric matrix. Then

- *If $\boldsymbol{A}$ is PD, then $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is ND, then $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 56 (necessary conditions for definiteness)

Let $\boldsymbol{A} \in M(n \times n)$ be a symmetric matrix. Then

- *If $\boldsymbol{A}$ is PD, then $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is ND, then $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is PSD, then $a_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 56 (necessary conditions for definiteness)

Let $\boldsymbol{A} \in M(n \times n)$ be a symmetric matrix. Then

- *If $\boldsymbol{A}$ is PD, then $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is ND, then $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is PSD, then $a_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\boldsymbol{A}$ is NSD, then $a_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$,*

Proposition 56 (necessary conditions for definiteness)

Let $\boldsymbol{A} \in M(n \times n)$ *be a symmetric matrix. Then*

- *If* $\boldsymbol{A}$ *is PD, then* $a_{ii} > 0$ *for all* $i = 1, 2, \ldots, n$*,*
- *If* $\boldsymbol{A}$ *is ND, then* $a_{ii} < 0$ *for all* $i = 1, 2, \ldots, n$*,*
- *If* $\boldsymbol{A}$ *is PSD, then* $a_{ii} \geq 0$ *for all* $i = 1, 2, \ldots, n$*,*
- *If* $\boldsymbol{A}$ *is NSD, then* $a_{ii} \leq 0$ *for all* $i = 1, 2, \ldots, n$*,*
- *If there exist* $i, j \in \{1, 2, \ldots, n\}$ *such that* $a_{ii} > 0$ *and* $a_{jj} < 0$*, then* $\boldsymbol{A}$ *is ID.*

Theorem 57 (Sylvester's criterion)

Let $\boldsymbol{A} = (a_{ij}) \in M(n \times n)$ be a symmetric matrix. Then $\boldsymbol{A}$ is

- *positive definite if and only if*

$$\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \text{for all } k = 1, \dots, n,$$

Theorem 57 (Sylvester's criterion)

Let $\boldsymbol{A} = (a_{ij}) \in M(n \times n)$ *be a symmetric matrix. Then* $\boldsymbol{A}$ *is*

- *positive definite if and only if*

$$\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \text{for all } k = 1, \dots, n,$$

- *negative definite if and only if*

$$(-1)^k \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \text{for all } k = 1, \dots, n,$$

- positive semidefinite if and only if

$$\begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \geq 0$$

for each k-tuple of integers $1 \leq i_1 < \cdots < i_k \leq n$, $k = 1, \dots, n$,

- positive semidefinite if and only if

$$\begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \geq 0$$

  for each k-tuple of integers $1 \leq i_1 < \cdots < i_k \leq n$, $k = 1, \dots, n$,

- negative semidefinite if and only if

$$(-1)^k \begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \geq 0$$

  for each k-tuple of integers $1 \leq i_1 < \cdots < i_k \leq n$, $k = 1, \dots, n$.

Definition

Let $f \in C^2(G)$. Then the matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2 x_2}(x) \end{pmatrix}$$

is called Hessian matrix of f.

Definition

Let $f \in C^2(G)$. Then the matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2 x_2}(x) \end{pmatrix}$$

is called Hessian matrix of f.

Theorem 58

Let $G \subset \mathbb{R}^n$ be convex and $f \in C^2(G)$. If the Hessian matrix of f is positive semidefinite for every $x \in G$, then f is convex on G.

Definition

Let $f \in C^2(G)$. Then the matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2 x_2}(x) \end{pmatrix}$$

is called Hessian matrix of f.

Theorem 58

Let $G \subset \mathbb{R}^n$ be convex and $f \in C^2(G)$. If the Hessian matrix of f is positive semidefinite for every $x \in G$, then f is convex on G. If the Hessian matrix of f is positive definite for every $x \in G$, then f is strictly convex on G.

VII. Antiderivatives and Riemann integral

VII.1. Antiderivatives

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Remark

An antiderivative of f is sometimes called a function primitive to f.

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Remark

An antiderivative of f is sometimes called a function primitive to f.
If F is an antiderivative of f on I, then F is continuous on I.

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Remark

An antiderivative of f is sometimes called a function primitive to f.
If F is an antiderivative of f on I, then F is continuous on I.

Theorem 59 (Uniqueness of an antiderivative)

Let F and G be antiderivatives of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.

Remark

The set of all antiderivatives of f on an open interval I is denoted by

$$\int f(x)\,\mathrm{d}x.$$

Remark

The set of all antiderivatives of f on an open interval I is denoted by

$$\int f(x)\,\mathrm{d}x.$$

The fact that F is an antiderivative of f on I is expressed by

$$\int f(x)\,\mathrm{d}x \overset{c}{=} F(x), \quad x \in I.$$

Table of basic antiderivatives

Table of basic antiderivatives

- $\int x^n \,\mathrm{d}x \stackrel{c}{=} \dfrac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \overset{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \overset{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \stackrel{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \stackrel{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\displaystyle\int \frac{1}{x} \,\mathrm{d}x \stackrel{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \overset{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \overset{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\displaystyle\int \frac{1}{x} \,\mathrm{d}x \overset{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,
- $\displaystyle\int e^x \,\mathrm{d}x \overset{c}{=} e^x$ on $\mathbb{R}$,

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \overset{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \overset{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\displaystyle\int \frac{1}{x} \,\mathrm{d}x \overset{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,
- $\displaystyle\int e^x \,\mathrm{d}x \overset{c}{=} e^x$ on $\mathbb{R}$,
- $\displaystyle\int \sin x \,\mathrm{d}x \overset{c}{=} -\cos x$ on $\mathbb{R}$,

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \stackrel{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \stackrel{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\displaystyle\int \frac{1}{x} \,\mathrm{d}x \stackrel{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,
- $\displaystyle\int e^x \,\mathrm{d}x \stackrel{c}{=} e^x$ on $\mathbb{R}$,
- $\displaystyle\int \sin x \,\mathrm{d}x \stackrel{c}{=} -\cos x$ on $\mathbb{R}$,
- $\displaystyle\int \cos x \,\mathrm{d}x \stackrel{c}{=} \sin x$ on $\mathbb{R}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \stackrel{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \stackrel{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \stackrel{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi+k\pi)$, $k \in \mathbb{Z}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \stackrel{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \stackrel{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \stackrel{c}{=} \operatorname{arctg} x$ on $\mathbb{R}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \overset{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \overset{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} \operatorname{arctg} x$ on $\mathbb{R}$,
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arcsin x$ on $(-1,1)$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \overset{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \overset{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi+k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} \operatorname{arctg} x$ on $\mathbb{R}$,
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arcsin x$ on $(-1,1)$,
- $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arccos x$ on $(-1,1)$.

Theorem 60 (Existence of an antiderivative)

Let f be a continuous function on an open interval I. Then f has an antiderivative on I.

Theorem 61 (Linearity of antiderivatives)

Suppose that f has an antiderivative F on an open interval I, g has an antiderivative G on I, and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.

Theorem 62 (substitution)

(i) *Let F be an antiderivative of f on (a, b). Let $\varphi\colon (\alpha, \beta) \to (a, b)$ have a finite derivative at each point of (α, β). Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha, \beta).$$

Theorem 62 (substitution)

(i) *Let F be an antiderivative of f on (a, b). Let $\varphi\colon (\alpha, \beta) \to (a, b)$ have a finite derivative at each point of (α, β). Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha, \beta).$$

(ii) *Let φ be a function with a finite derivative in each point of (α, β) such that the derivative is either everywhere positive or everywhere negative, and such that $\varphi\big((\alpha, \beta)\big) = (a, b)$. Let f be a function defined on (a, b) and suppose that*

$$\int f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t \overset{c}{=} G(t) \quad \text{on } (\alpha, \beta).$$

Then

$$\int f(x)\,\mathrm{d}x \overset{c}{=} G\big(\varphi^{-1}(x)\big) \quad \text{on } (a, b).$$

Theorem 63 (integration by parts)

Let I be an open interval and let the functions f and g be continuous on I. Let F be an antiderivative of f on I and G an antiderivative of g on I. Then

$$\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x \quad \textit{on } I.$$

Example

Denote $I_n = \int \frac{1}{(1+x^2)^n}\,\mathrm{d}x$, $n \in \mathbb{N}$. Then

$$I_{n+1} = \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n} I_n, x \in \mathbb{R}, \quad n \in \mathbb{N},$$

$$I_1 \overset{c}{=} \operatorname{arctg} x, x \in \mathbb{R}.$$

Definition

A rational function is a ratio of two polynomials, where the polynomial in the denominator is not a zero polynomial.

Definition

A rational function is a ratio of two polynomials, where the polynomial in the denominator is not a zero polynomial.

Theorem ("fundamental theorem of algebra")

Let $n \in \mathbb{N}$, $a_0, \dots, a_n \in \mathbb{C}$, $a_n \neq 0$. Then the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

Lemma 64 (polynomial division)

Let P and Q be polynomials (with complex coefficients) such that Q is not a zero polynomial. Then there are uniquely determined polynomials S and R satisfying:

- $\deg R < \deg Q$,
- $P(x) = S(x)Q(x) + R(x)$ *for all* $x \in \mathbb{C}$.

If P and Q have real coefficients then so have S and R.

Lemma 64 (polynomial division)

Let P and Q be polynomials (with complex coefficients) such that Q is not a zero polynomial. Then there are uniquely determined polynomials S and R satisfying:

- $\deg R < \deg Q$,
- $P(x) = S(x)Q(x) + R(x)$ *for all* $x \in \mathbb{C}$.

If P and Q have real coefficients then so have S and R.

Corollary

If P is a polynomials and $\lambda \in \mathbb{C}$ *its root (i.e.* $P(\lambda) = 0$*), then there is a polynomial S satisfying* $P(x) = (x - \lambda)S(x)$ *for all* $x \in \mathbb{C}$.

Theorem 65 (factorisation into monomials)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial of degree* $n \in \mathbb{N}$. *Then there are numbers* $x_1, \ldots, x_n \in \mathbb{C}$ *such that*

$$P(x) = a_n (x - x_1) \cdots (x - x_n), \quad x \in \mathbb{C}.$$

Theorem 65 (factorisation into monomials)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree $n \in \mathbb{N}$. Then there are numbers $x_1, \ldots, x_n \in \mathbb{C}$ such that

$$P(x) = a_n(x - x_1)\cdots(x - x_n), \quad x \in \mathbb{C}.$$

Definition

Let P be a polynomial that is not zero, $\lambda \in \mathbb{C}$, and $k \in \mathbb{N}$. We say that λ is a root of multiplicity k of the polynomial P if there is a polynomial S satisfying $S(\lambda) \neq 0$ and $P(x) = (x - \lambda)^k S(x)$ for all $x \in \mathbb{C}$.

Theorem 66 (roots of a polynomial with real coefficients)

Let P be a polynomial with real coefficients and $\lambda \in \mathbb{C}$ a root of P of multiplicity $k \in \mathbb{N}$. Then the also the conjugate number $\overline{\lambda}$ is a root of P of multiplicity k.

Theorem 67 (factorisation of a polynomial with real coefficients)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n with real coefficients. Then there exist real numbers $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$ and natural numbers $p_1, \ldots, p_k$, $q_1, \ldots, q_l$ such that

- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,

Theorem 67 (factorisation of a polynomial with real coefficients)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n with real coefficients. Then there exist real numbers $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$ and natural numbers $p_1, \ldots, p_k$, $q_1, \ldots, q_l$ such that

- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,
- *no two polynomials from $x - x_1, x - x_2, \ldots, x - x_k$, $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ have a common root,*

Theorem 67 (factorisation of a polynomial with real coefficients)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n with real coefficients. Then there exist real numbers $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$ and natural numbers $p_1, \ldots, p_k$, $q_1, \ldots, q_l$ such that

- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,
- *no two polynomials from $x - x_1, x - x_2, \ldots, x - x_k$, $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ have a common root,*
- *the polynomials $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ have no real root.*

Theorem 68 (decomposition to partial fractions)

Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

$$Q(x) = a_n(x-x_1)^{p_1} \cdots (x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1} \cdots (x^2+\alpha_l x+\beta_l)^{q_l}$$

be a factorisation of from Theorem 67. Then there exist unique real numbers $A_1^1, \dots, A_{p_1}^1, \dots, A_1^k, \dots, A_{p_k}^k$, $B_1^1, C_1^1, \dots, B_{q_1}^1, C_{q_1}^1, \dots, B_1^l, C_1^l, \dots, B_{q_l}^l, C_{q_l}^l$ such that

$\frac{P(x)}{Q(x)} =$

Theorem 68 (decomposition to partial fractions)

Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

$$Q(x) = a_n(x-x_1)^{p_1}\cdots(x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1}\cdots(x^2+\alpha_l x+\beta_l)^{q_l}$$

be a factorisation of from Theorem 67. Then there exist unique real numbers $A_1^1,\dots,A_{p_1}^1,\dots,A_1^k,\dots,A_{p_k}^k$, $B_1^1, C_1^1,\dots,B_{q_1}^1, C_{q_1}^1,\dots,B_1^l, C_1^l,\dots,B_{q_l}^l, C_{q_l}^l$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-x_1)} + \dots + \frac{A_{p_1}^1}{(x-x_1)^{p_1}} +$$

Theorem 68 (decomposition to partial fractions)

Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

$$Q(x) = a_n(x-x_1)^{p_1} \cdots (x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1} \cdots (x^2+\alpha_l x+\beta_l)^{q_l}$$

be a factorisation of from Theorem 67. Then there exist unique real numbers $A_1^1, \dots, A_{p_1}^1, \dots, A_1^k, \dots, A_{p_k}^k$, $B_1^1, C_1^1, \dots, B_{q_1}^1, C_{q_1}^1, \dots, B_1^l, C_1^l, \dots, B_{q_l}^l, C_{q_l}^l$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-x_1)} + \dots + \frac{A_{p_1}^1}{(x-x_1)^{p_1}} + \dots + \frac{A_1^k}{(x-x_k)} + \dots + \frac{A_{p_k}^k}{(x-x_k)^{p_k}} +$$

$$+$$

Theorem 68 (decomposition to partial fractions)

Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

$$Q(x) = a_n(x-x_1)^{p_1} \cdots (x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1} \cdots (x^2+\alpha_l x+\beta_l)^{q_l}$$

be a factorisation of from Theorem 67. Then there exist unique real numbers $A_1^1, \dots, A_{p_1}^1, \dots, A_1^k, \dots, A_{p_k}^k$, $B_1^1, C_1^1, \dots, B_{q_1}^1, C_{q_1}^1, \dots, B_1^l, C_1^l, \dots, B_{q_l}^l, C_{q_l}^l$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-x_1)} + \cdots + \frac{A_{p_1}^1}{(x-x_1)^{p_1}} + \cdots + \frac{A_1^k}{(x-x_k)} + \cdots + \frac{A_{p_k}^k}{(x-x_k)^{p_k}} +$$

$$+ \frac{B_1^1 x+C_1^1}{(x^2+\alpha_1 x+\beta_1)} + \cdots + \frac{B_{q_1}^1 x+C_{q_1}^1}{(x^2+\alpha_1 x+\beta_1)^{q_1}} + \cdots +$$

$$+$$

Theorem 68 (decomposition to partial fractions)

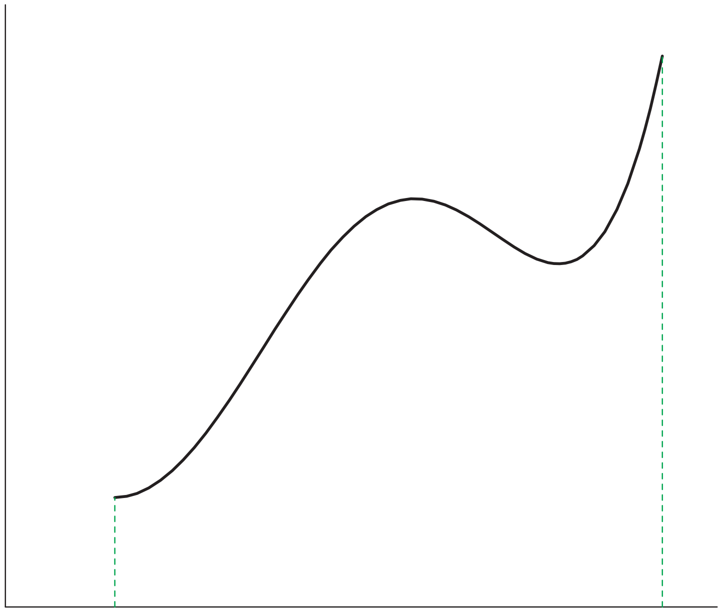
Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

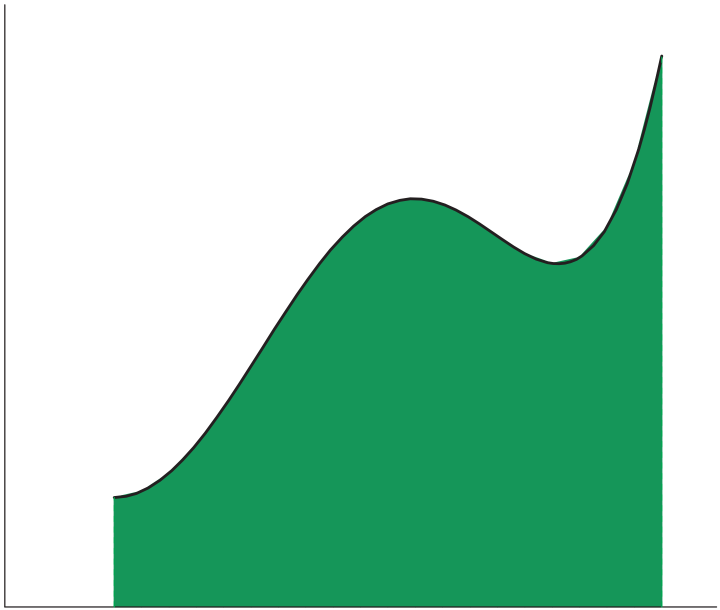
$$Q(x) = a_n(x-x_1)^{p_1}\cdots(x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1}\cdots(x^2+\alpha_l x+\beta_l)^{q_l}$$

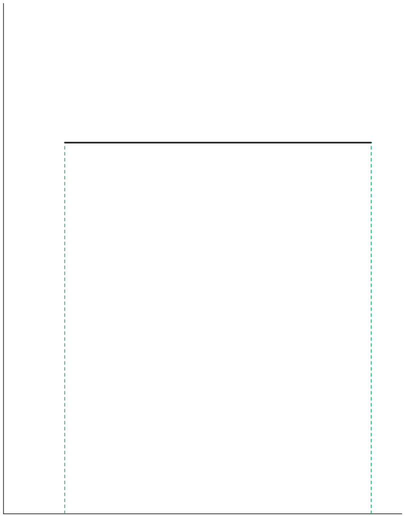
be a factorisation of from Theorem 67. Then there exist unique real numbers $A_1^1, \dots, A_{p_1}^1, \dots, A_1^k, \dots, A_{p_k}^k$, $B_1^1, C_1^1, \dots, B_{q_1}^1, C_{q_1}^1, \dots, B_1^l, C_1^l, \dots, B_{q_l}^l, C_{q_l}^l$ such that

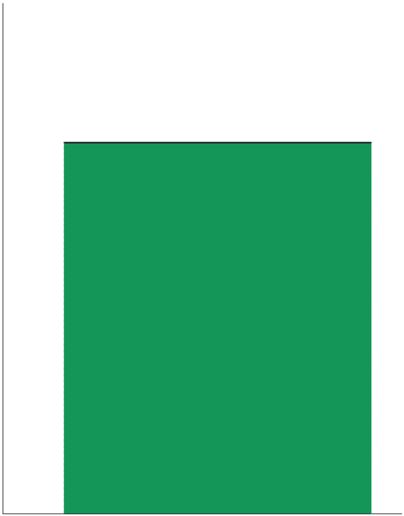
$$\begin{aligned}\frac{P(x)}{Q(x)} = {} & \frac{A_1^1}{(x-x_1)} + \dots + \frac{A_{p_1}^1}{(x-x_1)^{p_1}} + \dots + \frac{A_1^k}{(x-x_k)} + \dots + \frac{A_{p_k}^k}{(x-x_k)^{p_k}} + \\ & + \frac{B_1^1 x + C_1^1}{(x^2+\alpha_1 x+\beta_1)} + \dots + \frac{B_{q_1}^1 x + C_{q_1}^1}{(x^2+\alpha_1 x+\beta_1)^{q_1}} + \dots + \\ & + \frac{B_1^l x + C_1^l}{(x^2+\alpha_l x+\beta_l)} + \dots + \frac{B_{q_l}^l x + C_{q_l}^l}{(x^2+\alpha_l x+\beta_l)^{q_l}}, \; x \in \mathbb{R} \setminus \{x_1, \dots, x_k\}.\end{aligned}$$

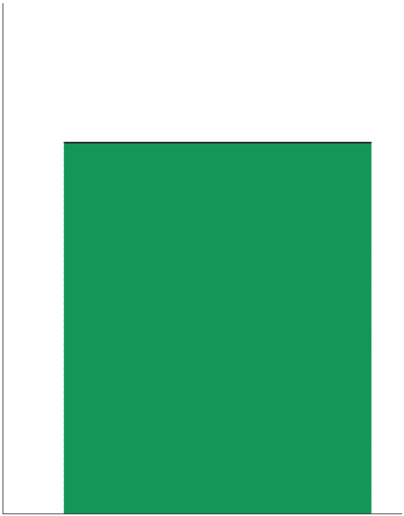
VII.2. Riemann integral

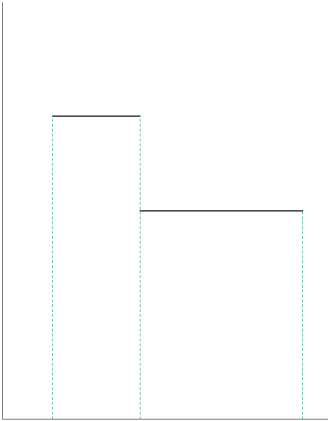


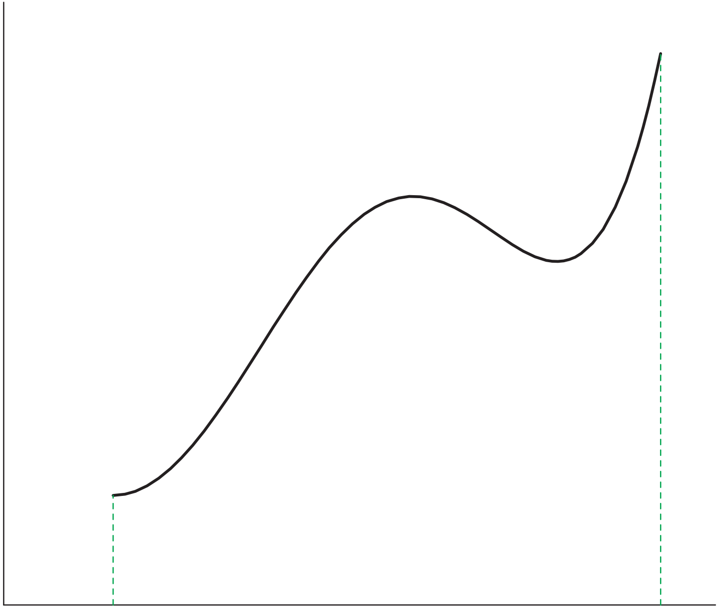


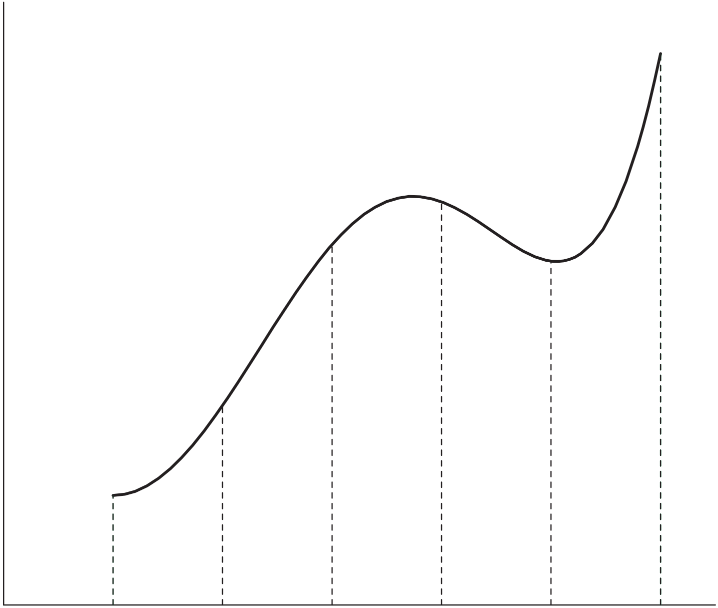


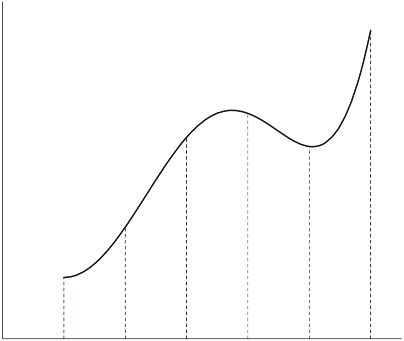


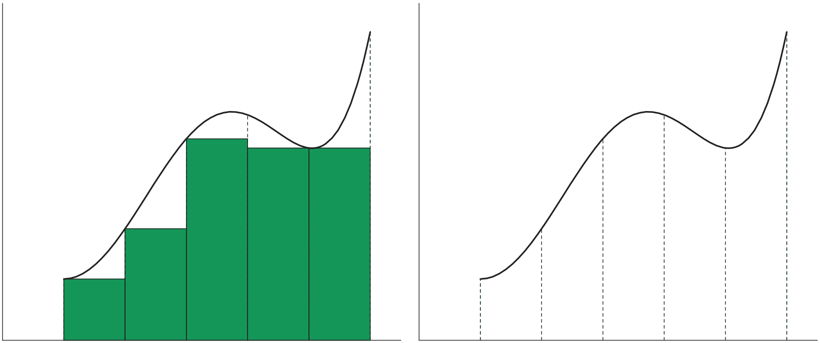


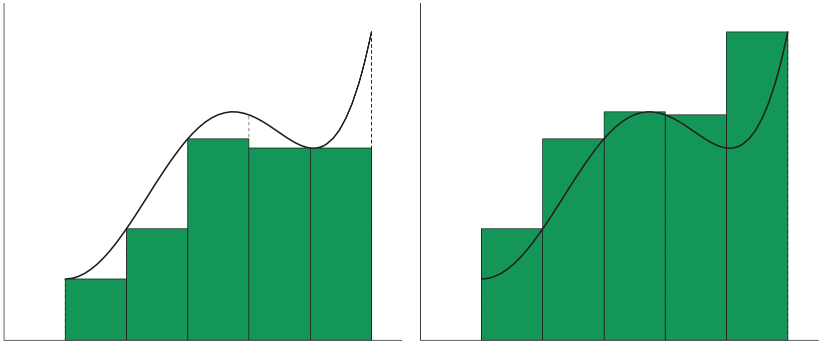


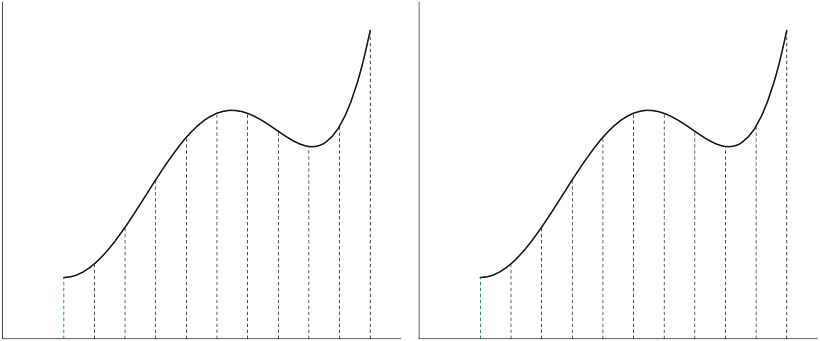


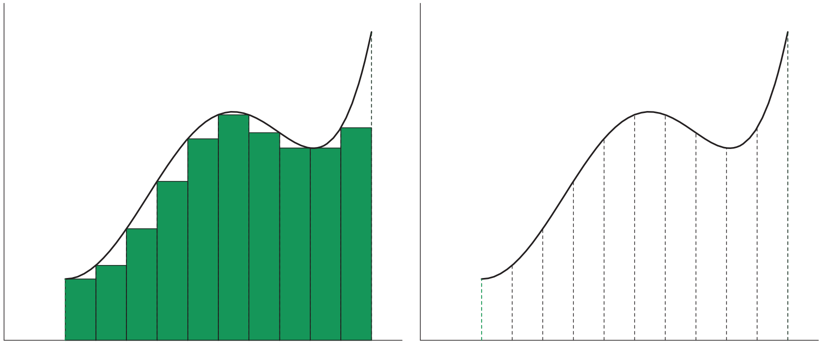


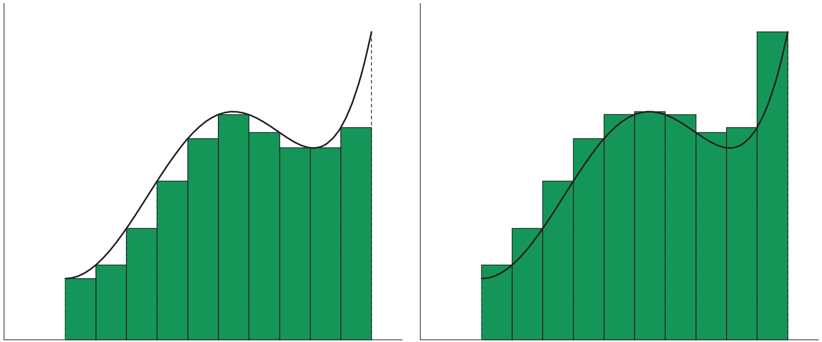












Definition

A finite sequence $\{x_j\}_{j=0}^n$ is called a partition of the interval $[a, b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ are called the partition points.

Definition

A finite sequence $\{x_j\}_{j=0}^{n}$ is called a partition of the interval $[a, b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ are called the partition points.
We say that a partition D' of an interval $[a, b]$ is a refinement of the partition D of $[a, b]$ if each partition point of D is also a partition point of D'.

Definition

Suppose that $a, b \in \mathbb{R}$, $a < b$, the function f is bounded on $[a, b]$, and $D = \{x_j\}_{j=0}^{n}$ is a partition of $[a, b]$. Denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \text{ where } M_j = \sup\{f(x);\ x \in [x_{j-1}, x_j]\},$$

Definition

Suppose that $a, b \in \mathbb{R}$, $a < b$, the function f is bounded on $[a, b]$, and $D = \{x_j\}_{j=0}^{n}$ is a partition of $[a, b]$. Denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}),\ \text{where } M_j = \sup\{f(x);\ x \in [x_{j-1}, x_j]\},$$

$$\underline{S}(f, D) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),\ \text{where } m_j = \inf\{f(x);\ x \in [x_{j-1}, x_j]\},$$

Definition

Suppose that $a, b \in \mathbb{R}$, $a < b$, the function f is bounded on $[a, b]$, and $D = \{x_j\}_{j=0}^{n}$ is a partition of $[a, b]$. Denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \text{ where } M_j = \sup\{f(x);\ x \in [x_{j-1}, x_j]\},$$

$$\underline{S}(f, D) = \sum_{j=1}^{n} m_j(x_j - x_{j-1}), \text{ where } m_j = \inf\{f(x);\ x \in [x_{j-1}, x_j]\},$$

$$\overline{\int_a^b} f = \inf\left\{\overline{S}(f, D);\ D \text{ is a partition of } [a, b]\right\},$$

$$\underline{\int_a^b} f = \sup\left\{\underline{S}(f, D);\ D \text{ is a partition of } [a, b]\right\}.$$

Definition

We say that a function f has the Riemann integral over the interval $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$.

Definition

We say that a function f has the Riemann integral over the interval $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. The value of the integral of f over $[a, b]$ is then equal to the common value of $\overline{\int_a^b} f = \underline{\int_a^b} f$.

Definition

We say that a function f has the Riemann integral over the interval $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. The value of the integral of f over $[a, b]$ is then equal to the common value of $\overline{\int_a^b} f = \underline{\int_a^b} f$. We denote it by $\displaystyle\int_a^b f$.

Definition

We say that a function f has the Riemann integral over the interval $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. The value of the integral of f over $[a, b]$ is then equal to the common value of $\overline{\int_a^b} f = \underline{\int_a^b} f$. We denote it by $\displaystyle\int_a^b f$. If $a > b$, then we define $\displaystyle\int_a^b f = -\int_b^a f$, and in case that $a = b$ we put $\displaystyle\int_a^b f = 0$.

Remark

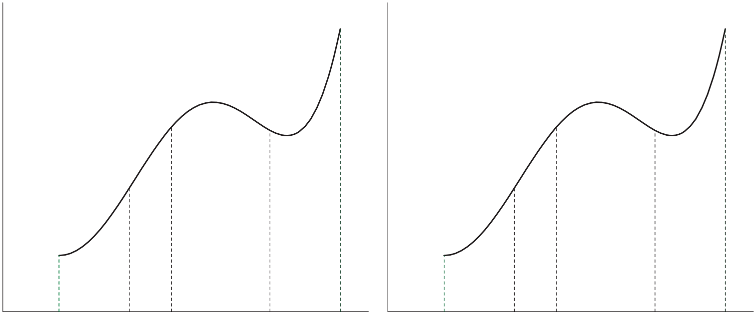
In the case that a function f has the Riemann integral, we write $f \in \mathcal{R}([a, b])$.

Remark

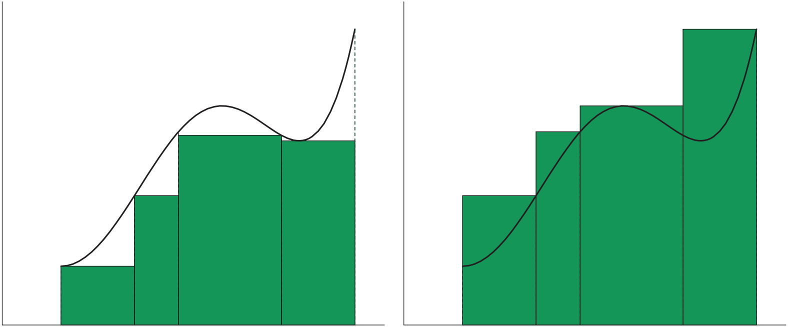
Let D, D' be partitions of $[a, b]$, D' refines D, and let f be a bounded function on $[a, b]$. Then

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

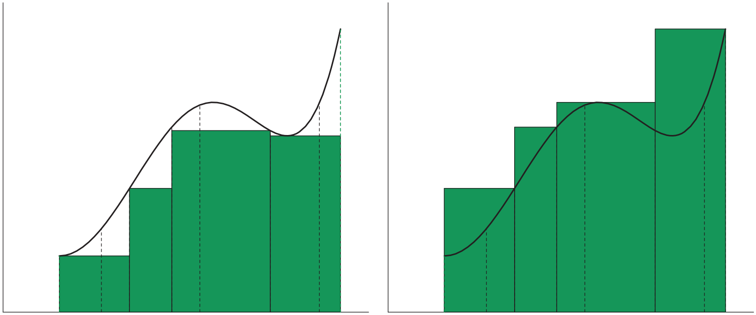
$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$



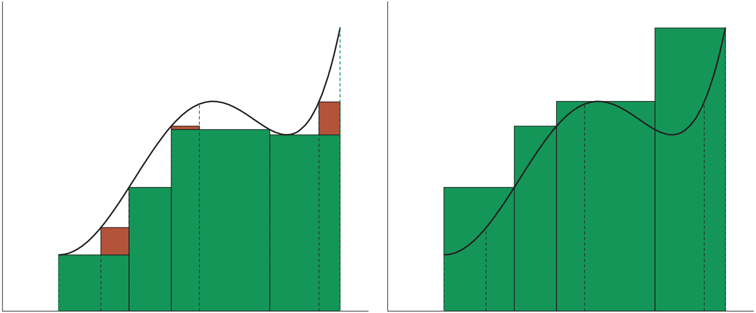
$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$



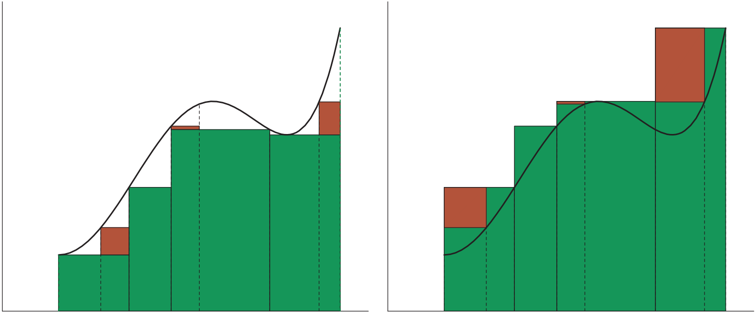
$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$



$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$



$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$



Remark

Let D, D' be partitions of $[a, b]$, D' refines D, and let f be a bounded function on $[a, b]$. Then

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

Remark

Let D, D' be partitions of $[a, b]$, D' refines D, and let f be a bounded function on $[a, b]$. Then

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

Suppose that D_1, D_2 are partitions of $[a, b]$ and a partition D' refines both D_1 and D_2. Then

$$\underline{S}(f, D_1) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D_2).$$

Remark

Let D, D' be partitions of $[a, b]$, D' refines D, and let f be a bounded function on $[a, b]$. Then

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

Suppose that D_1, D_2 are partitions of $[a, b]$ and a partition D' refines both D_1 and D_2. Then

$$\underline{S}(f, D_1) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D_2).$$

It easily follows that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Exercise 5

Show that the following function is not Riemann integrable:

$$f(x) = \begin{cases} \frac{1}{q}, \text{ if } x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\}, q > 0, p, q \in \mathbb{Z}, \\ \qquad\qquad \text{and } \frac{p}{q} \text{ is irreducible fraction,} \\ 0, x = 0 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Theorem 69

Let $a, b \in \mathbb{R},\ a < b$, and $f : (a, b) \to \mathbb{R}$ is Riemann integrable. Then f is a bounded function.

Theorem 70

(criteria of existence of Riemann integral)
Let $a, b \in \mathbb{R},\ a < b$, *and* $f : (a, b) \to \mathbb{R}$ *is a bounded function. Then*

1. 1.1 *if* $\exists \int_a^b f(x)dx = I$, *then* $\forall \varepsilon > 0$ *there exists a partition* D *of the interval* $[a, b]$ *such that*

   $$I - \varepsilon < \underline{S}(f, D) \leq \overline{S}(f, D) < I + \varepsilon$$

   1.2 *if* $\exists I \in \mathbb{R}\ \forall \varepsilon > 0 \exists D$ *partition of* $[a, b]$ *such that*

   $$I - \varepsilon < \underline{S}(f, D) \leq \overline{S}(f, D) < I + \varepsilon,$$

   then $\exists \int_a^b f(x)dx = I$.

2. $f \in \mathcal{R}([a, b])$ *if and onlny if* $\forall \varepsilon > 0$ *there exists a partition* D *of the interval* $[a, b]$ *such that*

   $$\overline{S}(f, D) - \underline{S}(f, D) <$$

Theorem 71

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

Theorem 71

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

(ii) *Suppose that $c \in (a, b)$ and f has the Riemann integral over the intervals $[a, c]$ and $[c, b]$. Then f has the Riemann integral over $[a, b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1}$$

Theorem 71

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

(ii) *Suppose that $c \in (a, b)$ and f has the Riemann integral over the intervals $[a, c]$ and $[c, b]$. Then f has the Riemann integral over $[a, b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1}$$

Remark

The formula (1) holds for all $a, b, c \in \mathbb{R}$ if the integral of f exists over the interval $\big[\min\{a, b, c\}, \max\{a, b, c\}\big]$.

Theorem 72 (linearity of the Riemann integral)

Let f and g be functions with Riemann integral over $[a, b]$ and let $\alpha \in \mathbb{R}$. Then

(i) *the function αf has the Riemann integral over $[a, b]$ and*

$$\int_a^b \alpha f = \alpha \int_a^b f,$$

Theorem 72 (linearity of the Riemann integral)

Let f and g be functions with Riemann integral over $[a, b]$ and let $\alpha \in \mathbb{R}$. Then

(i) *the function αf has the Riemann integral over $[a, b]$ and*
$$\int_a^b \alpha f = \alpha \int_a^b f,$$

(ii) *the function $f + g$ has the Riemann integral over $[a, b]$ and*
$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

Theorem 73

Let $a, b \in \mathbb{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$. Then:

(i) *If $f(x) \leq g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Theorem 73

Let $a, b \in \mathbb{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$. Then:

(i) *If $f(x) \leq g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

(ii) *The function $|f|$ has the Riemann integral over $[a, b]$ and*

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Theorem 74

Let f be a function continuous on an interval $[a, b]$, $a, b \in \mathbb{R}$. Then f has the Riemann integral on $[a, b]$.

Theorem 75

Let f be a function continuous on an interval (a, b) and let $c \in (a, b)$. If we denote $F(x) = \int_c^x f(t)\,\mathrm{d}t$ for $x \in (a, b)$, then $F'(x) = f(x)$ for each $x \in (a, b)$. In other words, F is an antiderivative of f on (a, b).

Theorem 76 (Newton-Leibniz formula)

Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2}$$

Theorem 76 (Newton-Leibniz formula)

Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2}$$

Remark

The Newton-Leibniz formula (2) holds even if $b < a$ (if $F' = f$ on $(b-\varepsilon, a+\varepsilon)$).

Theorem 76 (Newton-Leibniz formula)

Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{2}$$

Remark

The Newton-Leibniz formula (2) holds even if $b < a$ (if $F' = f$ on $(b-\varepsilon, a+\varepsilon)$). Let us denote

$$[F]_a^b = F(b) - F(a).$$

Theorem 77 (integration by parts)

Suppose that the functions f, g, f' a g' are continuous on an interval $[a, b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 77 (integration by parts)

Suppose that the functions f, g, f' a g' are continuous on an interval $[a, b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Theorem 78 (substitution)

Let the function f be continuous on an interval $[a, b]$. Suppose that the function φ has a continuous derivative on $[\alpha, \beta]$ and φ maps $[\alpha, \beta]$ into the interval $[a, b]$. Then

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$