Mathematics I - Introduction

2024/2025

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6. It is the universal language

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- Preparation for other courses — Statistics, Microeconomics, ...

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At the end of the course students should be able to

- compute limits and derivatives and investigate functions
- understand definitions (give positive and negative examples) and theorems (explain their meaning, neccessity of the assumptions, apply them in particular situations)
- perform mathematical proofs, give mathematically exact arguments, write mathematical formulae, use quantifiers

Mathematics I

- Introduction
- Limit of a sequence
- Mappings
- Functions of one real variable

Textbooks

- **Hájková et al: Mathematics 1**
- Trench: Introduction to real analysis
- Ghorpade, Limaye: A course in calculus and real analysis
- Zorich: Mathematical analysis I
- Rudin: Principles of mathematical analysis
- Fikhtengoltz: The fundamentals of Mathematical Analysis.

Sets

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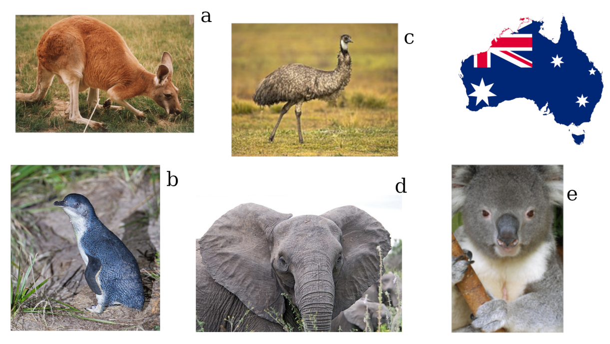
Exercise (True or false)

A - set of all animals living in Australia.

A $a \in A$ B $b \in A$ C $c \in A$ D $d \in A$ E $e \in A$



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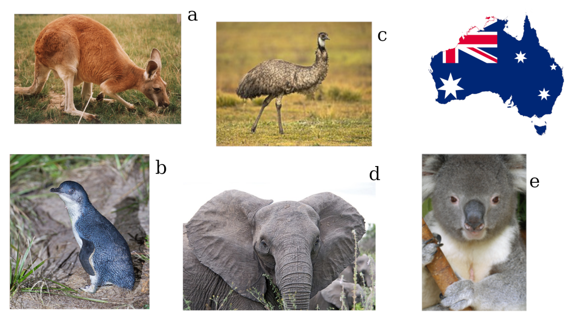

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- $A_1 \times \cdots \times A_m = \{(a_1, \ldots, a_m) : \; a_1 \in A_1, \ldots, a_m \in A_m\}$
  ... the Cartesian product

Sets - questions

Exercise

Let $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{1,3,5,7,9\}$ and $B = \{1,2,3,4,5\}$. Find

1. $A \cup B$
2. $A \cap B$
3. A^c
4. $(B^c)^c$
5. $A \setminus B$
6. $B \setminus A$

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Exercise (True or false)

Let A be a set.

- A $\emptyset \in A$
- B $\emptyset \subset A$
- C $0 = \emptyset$
- D $\{x\} \in \{x, y, z\}$
- E $x \in \{x, y, z\}$

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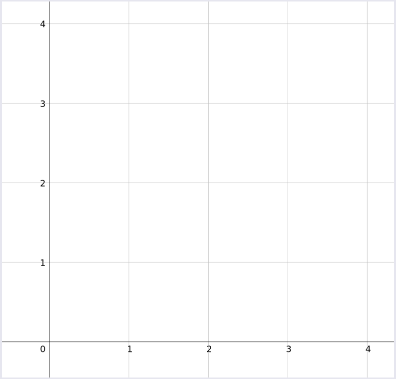
Exercise

Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$. Find $A \times B$, $B \times B$ and sketch them.



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Infinitely many sets: $A_1 \cup A_2 \cup A_3 \cup \ldots$ is equivalent to $\bigcup_{i=1}^{\infty} A_i$, and also to $\bigcup_{i \in \mathbb{N}} A_i$.

Exercise

Let $A_1 = \{0, 1\}, A_2 = \{0, 2\}, A_3 = \{0, 3\}$. Find

1. $\displaystyle\bigcup_{i=1}^{3} A_i$
2. $\displaystyle\bigcap_{i \in \{1,2,3\}} A_i$

de Morgan's laws

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Let $S, A_\alpha, \alpha \in I \neq \emptyset$ be some sets. Then

$$S \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (S \setminus A_\alpha)$$

and

$$S \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (S \setminus A_\alpha).$$

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Exercise

Find statements.

- A Let it be!
- B We all live in a yellow submarine.
- C Is there anybody out there?
- D We don't need any education.

Statements

- $\neg$, also $\overline{\cdots}$, non . . . negation
- $\&$ (also $\wedge$) . . . conjunction, logical “and”
- $||$ (also $\vee$) . . . disjunction (alternative), logical “or”
- $\Rightarrow$ . . . implication
- $\Leftrightarrow$ . . . equivalence; “if and only if”

Exercise

1. Alice does not like chocolate ice cream.
2. Alice likes chocolate and lemon ice cream.
3. Alice likes chocolate or lemon ice cream.
4. If it will be raining tomorrow, we will play board games.
5. We will play board games tomorrow if and only if it will be raining.

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$$V(x), x \in M$$

$$V(x_1, \ldots, x_n), x_1 \in M_1, \ldots, x_n \in M_n$$

Example

$V(x)$: x is even
$M = \{1, 2, 3, 4, 5\}$
$V(3)$ false, $V(4)$ true.

$V(x_1, x_2)$: $x_1 \cdot x_2 = 1$
$M = \{2, \frac{1}{2}, 3, 4\}$
$V(2, \frac{1}{2})$ true, $V(2, 3)$ false.

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Example

$\forall x \in \mathbb{R} : |x| \geq 0$
$\exists x \in \mathbb{Q} : x + 3 \leq 12$
$\exists! x \in \mathbb{R}^+ : x^2 = 42$

If $A(x), x \in M$ and $B(x), x \in M$ are predicates, then

$$\forall x \in M, B(x) \colon A(x) \quad \text{means} \quad \forall x \in M \colon (B(x) \Rightarrow A(x)),$$

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$$\exists x \in M, B(x) : A(x) \quad \text{means} \quad \exists x \in M : (A(x) \;\&\; B(x)).$$

Example

$$\forall x \in \mathbb{R}, x \geq -1 : 1 + 2x \leq (1+x)^2$$
$$\exists x \in \mathbb{R}, x \geq 0 : x \geq x^2$$

Negations of the statements with quantifiers:

$$\neg(\forall x \in M\colon A(x)) \quad \text{is the same as} \quad \exists x \in M\colon \neg A(x),$$

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Example

Find negation

$$\forall x \in \mathbb{R}, x \geq -1 : 1 + 2x \leq (1+x)^2$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x \geq 0, y \geq 0 : \frac{x+y}{2} \geq \sqrt{xy}$$

$$\exists x \in \mathbb{R}, x \geq 0 : x \geq x^2$$

Methods of proofs

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- direct proof
- indirect proof (proof by contrapositive)
- proof by contradiction
- mathematical induction

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- direct proof ($A \Rightarrow B$ follows from $A \Rightarrow C_1 \Rightarrow C_2 \Rightarrow B$)
- indirect proof (proof by contrapositive) ($A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$)
- proof by contradiction ($A \Rightarrow B$ is equivalent to $\neg(A \wedge \neg B)$)
- mathematical induction (base and step of induction)

Methods of proof

Exercise (direct proof) (Cauchy inequality)

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

Exercise (proof by contrapositive)

For a integer n, if n^2 is even, then n is also even.

Exercise (proof by contradiction)

The number $\sqrt{2}$ is irrational.

Exercise (proof by induction)

$$\sum_{i=1}^{n}(2i-1) = n^2$$

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$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\},$$

where $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if and only if $p_1 \cdot q_2 = p_2 \cdot q_1$.

Real numbers

Real numbers

By the set of real numbers $\mathbb{R}$ we will understand a set on which there are operations of addition and multiplication (denoted by $+$ and $\cdot$), and a relation of ordering (denoted by $\leq$), such that it has the following three groups of properties.

I. The properties of addition and multiplication and their relationships.

II. The relationships of the ordering and the operations of addition and multiplication.

III. The infimum axiom.

Properties of addition and multiplication and their relationships:

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Properties of "+":

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),

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- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),
- There is a non-zero element in $\mathbb{R}$ (called identity, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\colon x \cdot y = 1$ (such y is only one, denoted by x^{-1} or $\frac{1}{x}$),

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Relation between "+" and "·":

- $\forall x, y, z \in \mathbb{R}\colon (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

The relationships of the ordering and the operations of addition and multiplication:

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- $\forall x, y \in \mathbb{R}\colon x \leq y \vee y \leq x$ (linear order),
- $\forall x, y, z \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq z) \Rightarrow x \leq z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \leq y \ \ \& \ \ y \leq x) \Rightarrow x = y$ (antisymmetry),
- $\forall x, y, z \in \mathbb{R}\colon x \leq y \Rightarrow x + z \leq y + z$,
- $\forall x, y \in \mathbb{R}\colon (0 \leq x \ \ \& \ \ 0 \leq y) \Rightarrow 0 \leq x \cdot y$.

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Exercise

Which sets are bounded from below? Bounded from above? Bounded?

- A $\mathbb{N}$
- B $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$
- C $\mathbb{R} \setminus \mathbb{Q} \cap (-3, 2]$
- D $\{x \in \mathbb{R} : x < \pi\}$
- E $(-\infty, -1) \cup \{0\} \cup [1, \infty)$

The infimum axiom:

Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

(i) $\forall x \in M : x \geq g$,

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The number g is denoted by $\inf M$ and is called the infimum of the set M.

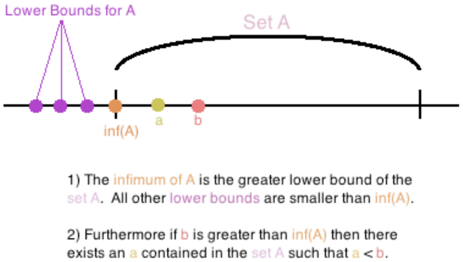

Figure:
https://mathspandorabox.wordpress.com/2016/03/11/the-difference-between-supremum-and-infimum-equivalent-and-equal-set/

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- The infimum of the set M is its greatest lower bound.
- The real numbers exist and are uniquely determined by the properties I–III.

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0,$

(ii) $\forall x \in \mathbb{R}\colon \ -x = (-1) \cdot x,$

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0),$

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n,$

(v) $\forall x, y \in \mathbb{R}\colon (x > 0 \wedge y > 0) \Rightarrow xy > 0,$

(vi) $\forall x \in \mathbb{R}, x \geq 0\ \forall y \in \mathbb{R}, y \geq 0\ \forall n \in \mathbb{N}\colon x < y \Leftrightarrow x^n < y^n.$

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$,
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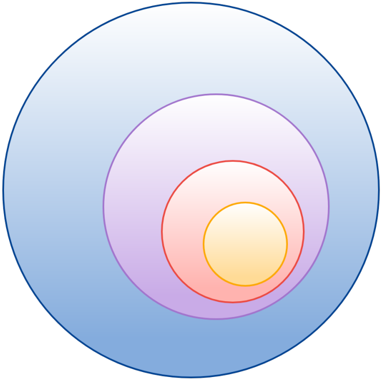
Unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R} : a < x\}, \quad (-\infty, a) = \{x \in \mathbb{R} : x < a\},$$

analogically $(-\infty, a]$, $[a, +\infty)$ and $(-\infty, +\infty)$.

Label the Venn diagram with $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$.



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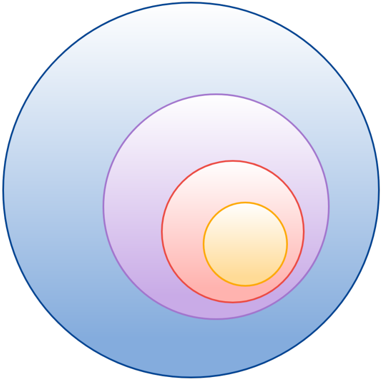

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. If we transfer the addition and multiplication from $\mathbb{R}$ to the above sets, we obtain the usual operations on these sets.
A real number that is not rational is called irrational. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Consequences of the infimum axiom

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M \colon x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G \ \exists x \in M \colon x > G'$,

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The supremum of the set M is denoted by $\sup M$.
The following holds: $\sup M = -\inf(-M)$.

Definition

Let $M \subset \mathbb{R}$. We say that a is a maximum of the set M (denoted by $\max M$) if a is an upper bound of M and $a \in M$.
Analogously we define a minimum of M, denoted by $\min M$.

Exercise

Find infimum, minimum, maximum and supremum:

1. $\{1, 2, 3, 4\}$
2. $[-2, 3]$
3. $(-2, 3)$
4. $(-2, 3]$
5. $[-2, -1) \cup (0, 25]$
6. $(-7, -0) \cup (1, 2)$
7. $[0, \infty)$
8. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
9. $\mathbb{N}$

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Theorem 3 (existence of an integer part)

For every $r \in \mathbb{R}$ there exists an integer part of r, i.e. a number $k \in \mathbb{Z}$ satisfying $k \leq r < k + 1$. The integer part of r is determined uniquely and it is denoted by $[r]$.

Theorem 4 (nth root)

For every $x \in [0, +\infty)$ *and every* $n \in \mathbb{N}$ *there exists a unique* $y \in [0, +\infty)$ *satisfying* $y^n = x$.

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Theorem 5 (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$)

Let $a, b \in \mathbb{R}$, $a < b$. Then there exist $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.

Theorem 6 (existence of an integer part)

For every $x \in \mathbb{R}$ there exists an integer part of r, i.e. a number $n \in \mathbb{Z}$ satisfying $n \leq x < n + 1$. The integer part of x is determined uniquely and it is denoted by $\lfloor x \rfloor$.

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We have $n + 1 > S$, hence $n + 1 \notin M$, i.e. $n + 1 > x$.
At the same time, $n \in M$, hence $n \leq x$. Q.E.D.

Remark

Note that in the proof we did not have to prove that $S \in \mathbb{Z}$, or that $n = S$.
Exercise: prove it.