8. $L = \lim_{x \to 0} \dfrac{e^{x^2+x} - \sin x + 3\cos x - 4}{\operatorname{arctg}^3 x}$

1) jmenovatel: $\operatorname{arctg} x = x(1+\omega(x))$

$\operatorname{arctg}^3 x = x^3(1+\omega_2(x))$

2) $e^{x^2+x} = 1 + (x+x^2) + \frac{1}{2}(x+x^2)^2 + \frac{1}{6}(x+x^2)^3 \underbrace{+\, \omega_3(x+x^2)\cdot(x+x^2)^3}_{= \omega_4(x)\cdot x^3}$

$= 1 + x + x^2 + \frac{1}{2}x^2 + x^3 + \frac{1}{2}x^4 + \frac{1}{6}x^3 + \omega_5(x)\cdot x^3$

$= 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \omega_5(x)\,x^3$

3) $-\sin x + 3\cos x - 4 = -\left(x - \frac{x^3}{6} + \omega_6(x)\cdot x^3\right) + 3\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \omega_7(x)\cdot x^4\right) - 4 =$

$= (3-4) + x(-1) + x^2\left(\frac{-3}{2}\right) + \frac{1}{6}x^3 + \omega_8(x)\cdot x^3$

$= -1 - x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + \omega_8(x)\cdot x^3$

4) čitatel:

$\left(1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \omega_5(x)\,x^3\right) + \left(-1 - x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + \omega_8(x)\,x^3\right) = \frac{4}{3}x^3 + \omega_9(x)\,x^3$

$= \frac{4}{3}x^3(1+\omega_{10}(x))$

5) $L = \lim_{x \to 0} \dfrac{\frac{4}{3}\cancel{x^3}(1+\omega_{10}(x))}{\cancel{x^3}(1+\omega_2(x))} = \frac{4}{3}$

15. $L=\lim\limits_{x\to 0}\dfrac{(1+\sin x)^x-e^{x^2}+\frac{x^3}{2}}{x^4}$

a) $-e^{x^2}+\frac{x^3}{2} = -\big(1+x^2+\frac{1}{2}x^4+\underbrace{\omega(x^2)\cdot x^4}_{\omega_2(x)\cdot x^4}\big)+\frac{x^3}{2} = -1-x^2+\frac{x^3}{2}-\frac{1}{2}x^4+\omega_3(x)\cdot x^4$

b) $(1+\sin x)^x = e^{x\log(1+\sin x)}$

vidíme, že $x\log(1+\sin x) = x\cdot\sin x(1+\omega_4(x)) = x^2(1+\omega_5(x))$, pak uděláme rozvoj exp až do y^2, $e^y = 1+y+\frac{y^2}{2}+\omega_6(y)\cdot y^2$

$\exp(x\log(1+\sin x)) =$

$= 1+x\log(1+\sin x)+\frac{1}{2}\big(x\log(1+\sin x)\big)^2+\underbrace{\omega_6(x\log(1+\sin x))\cdot\big(x\log(1+\sin x)\big)^2}_{=\omega_7(x)\cdot x^4}$

c) Pro $x\log(1+\sin x)$ potřebujeme rozvoj až do x^4, pak pro $\log(1+\sin x)$ až do x^3:

$\log(1+\sin x) = \sin x-\frac{1}{2}\sin^2 x+\frac{1}{3}\sin^3 x+\underbrace{\omega_8(\sin x)\cdot\sin^3 x}_{\omega_9(x)\cdot x^3}$

$=\left(x-\frac{x^3}{6}+\omega_{10}(x)\cdot x^3\right)-\frac{1}{2}\left(x-\frac{x^3}{6}+\omega_{10}(x)\cdot x^3\right)^2+\frac{1}{3}\left(x-\frac{x^3}{6}+\omega_{10}(x)\cdot x^3\right)^3+\omega_9(x)\cdot x^3$

$=\left(x-\frac{x^3}{6}+\omega_{10}(x)\cdot x^3\right)-\frac{1}{2}\Big(x^2+\underbrace{\omega_{11}(x)\cdot x^3}_{\text{členy od } x^4 \text{ a výše}}\Big)+\frac{1}{3}\Big(x^3+\underbrace{\omega_{12}(x)\cdot x^3}_{\text{členy } x^5\ldots}\Big)+\omega_9(x)\cdot x^3$

$= x-\frac{x^2}{2}+\frac{x^3}{6}+\omega_{13}(x)\cdot x^3$

d) dosadíme zpět do $\exp(x\log(1+\sin x))$: $\quad x\log(1+\sin x) = x^2-\frac{x^3}{2}+\frac{x^4}{6}+\omega_{13}(x)\cdot x^4$

$\exp(x\log(1+\sin x)) =$

$=1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{6}+\omega_{13}(x)\cdot x^4\right)+\frac{1}{2}\left(x^2-\frac{x^3}{2}+\frac{x^4}{6}+\omega_{13}(x)\cdot x^4\right)^2+\omega_7(x)\cdot x^4$

$=1+\left(x^2-\frac{x^3}{2}+\frac{x^4}{6}+\omega_{13}(x)x^4\right)+\left(\frac{1}{2}x^4+\omega_{14}(x)x^4\right)+\omega_7(x)x^4$

$=1+x^2-\frac{x^3}{2}+\frac{2}{3}x^4+\omega_{15}(x)x^4$

e) čitatel: $\left(1+x^2-\frac{x^3}{2}+\frac{2}{3}x^4+\omega_{15}(x)x^4\right)+\left(-1-x^2+\frac{x^3}{2}-\frac{1}{2}x^4+\omega_3(x)x^4\right)$

$=\frac{1}{6}x^4+\omega_{16}(x)x^4$

f) $L=\lim\limits_{x\to 0}\dfrac{\frac{1}{6}\cancel{x^4}(1+\omega_{17}(x))}{\cancel{x^4}}=\frac{1}{6}$