Mathematics I - Sequences

24/25

II. Limit of a sequence

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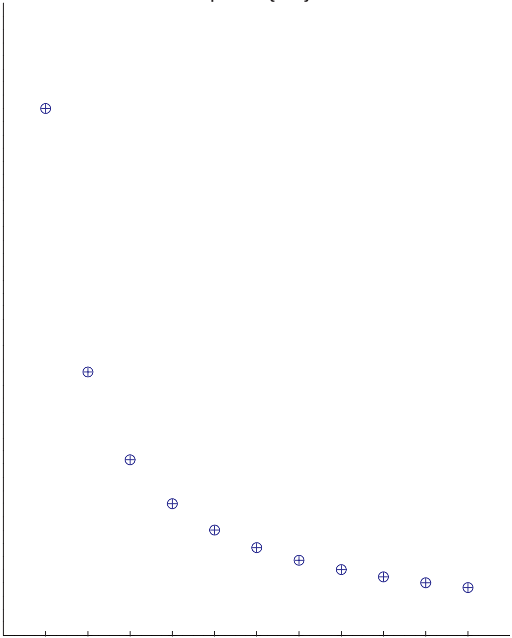
By the set of all members of the sequence $\{a_n\}_{n=1}^{\infty}$ we understand the set

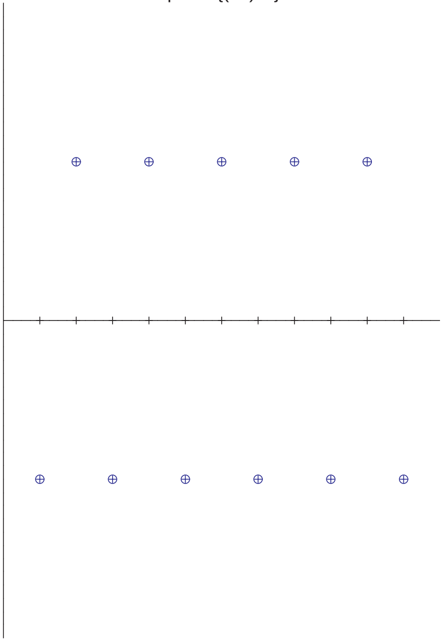
$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N} \colon a_n = x\}.$$

`https://www.geogebra.org/calculator/q7vv3gjp`

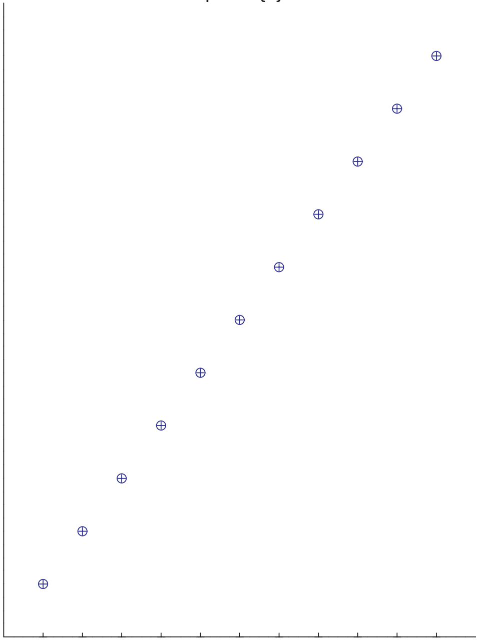
Posloupnost {1/n}

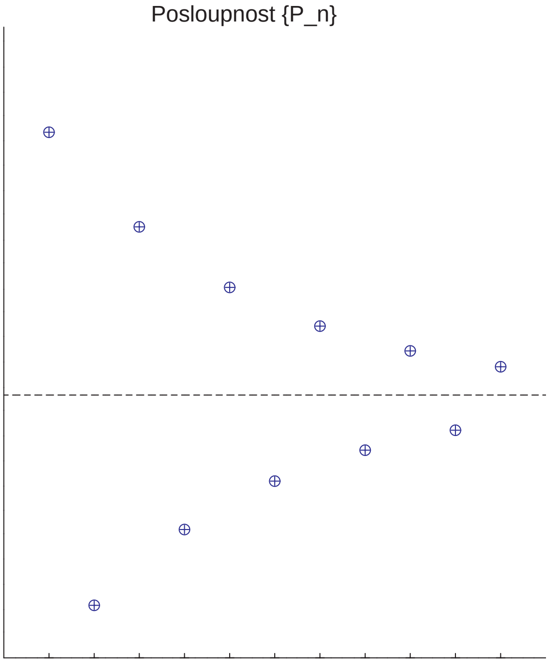


Posloupnost {(–1)^n}



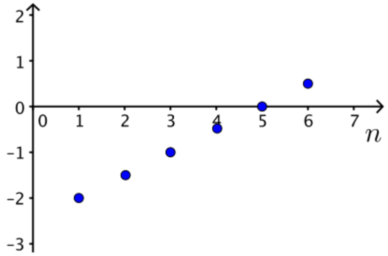
Posloupnost {n}




Posloupnost {P_n}

Exercise

Find the formula for a_n.



A. $a_n = (-\frac{1}{2})^n - \frac{3}{2}$

B. $a_n = \frac{1}{2}n + 5$

C. $a_n = \frac{1}{2}n - 2$

D. $a_n = -\frac{1}{2}n + \frac{5}{2}$

E. $a_n = \frac{1}{2}n - \frac{5}{2}$

Figure:
https://www.cpp.edu/conceptests/question-library/mat116.shtml

Exercise

Find the first 4 terms of a sequences

A $a_n = \dfrac{(-1)^n}{n}$

B $a_n = \dfrac{n+1}{n}$

Exercise

Find the formula for the following sequence

A $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dots$

B $-1, \dfrac{1}{2}, \dfrac{-1}{3}, \dfrac{1}{4}, \dfrac{-1}{5} \dots$

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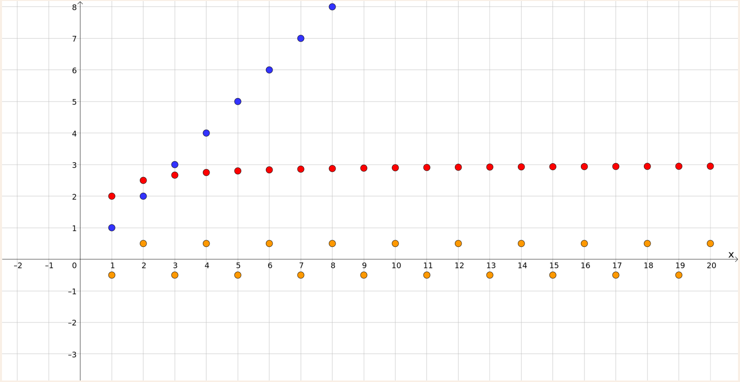
Exercise

Which of these sequences are bounded?



A blue B red C yellow

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Exercise

Find non-decreasing sequences.

A $a_n = \ln n$

B $a_n = e^{-n}$

C $a_n = -4$

D $a_n = \dfrac{(-1)^n}{3^n}$

E $a_n = (-2)^n$

Exercise

Check, if the sequence is monotone:

1. $a_n = \dfrac{n}{4 + n^2}$
2. $a_n = \dfrac{n}{n + 1}$

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- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.

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Exercise

Let $a_n = 1, 2, 3, 4, 5, \ldots, b_n = (-1)^n$. Find

A $a_n + b_n$ B a_n / b_n C $3a_n$

Definition

We say that a sequence $\{a_n\}$ has a limit which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \geq n_0$ we have $|a_n - A| < \varepsilon$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 \colon \; |a_n - A| < \varepsilon.$$

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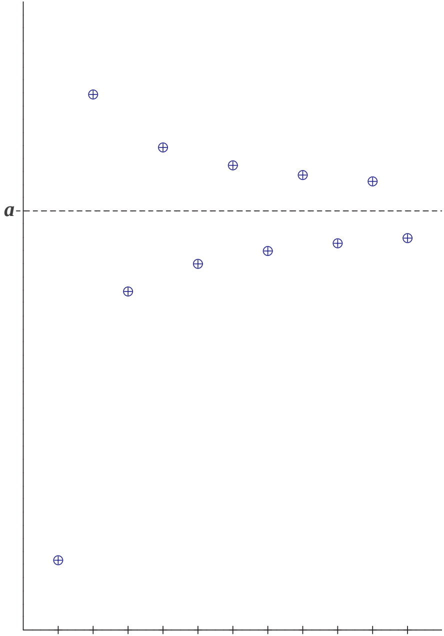
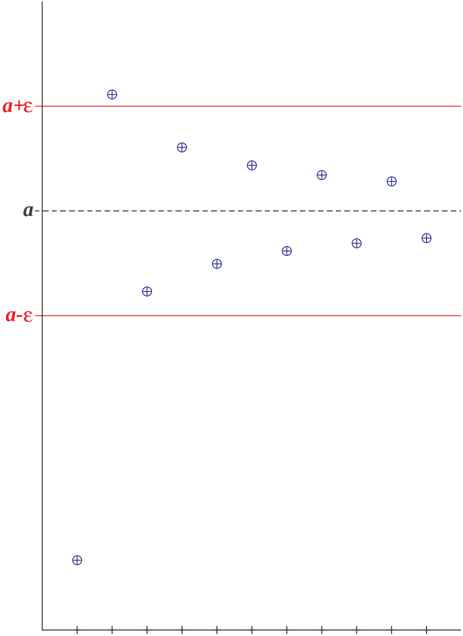
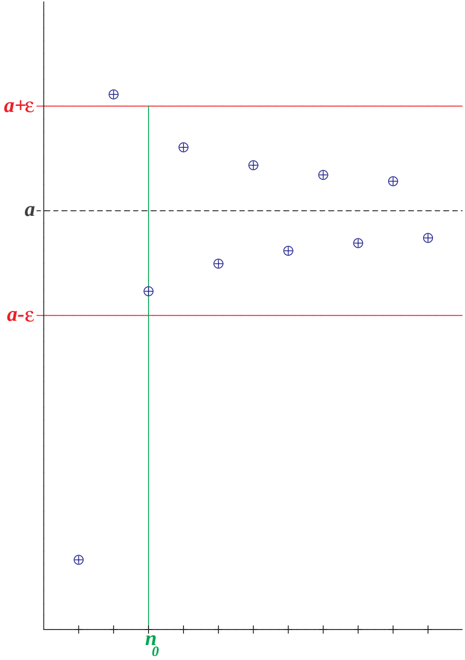
$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 : \ |a_n - A| < \varepsilon.$$

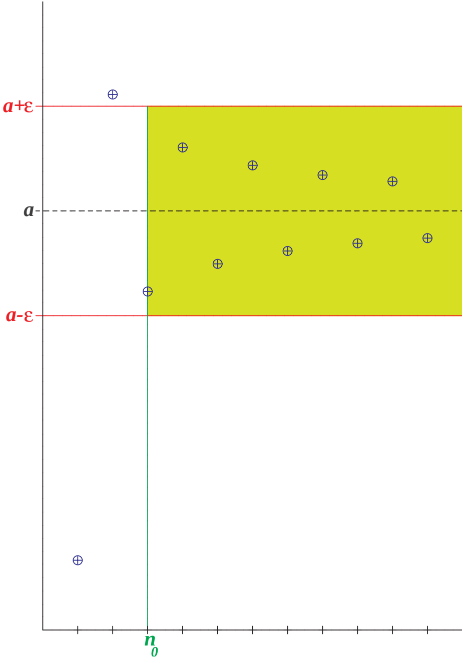
We say that a sequence $\{a_n\}$ is convergent if there exists $A \in \mathbb{R}$ which is a limit of $\{a_n\}$.

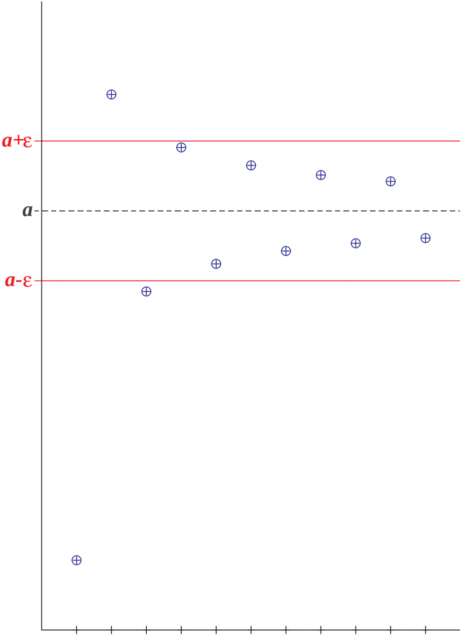
`https://www.geogebra.org/m/GAcTpGCh`

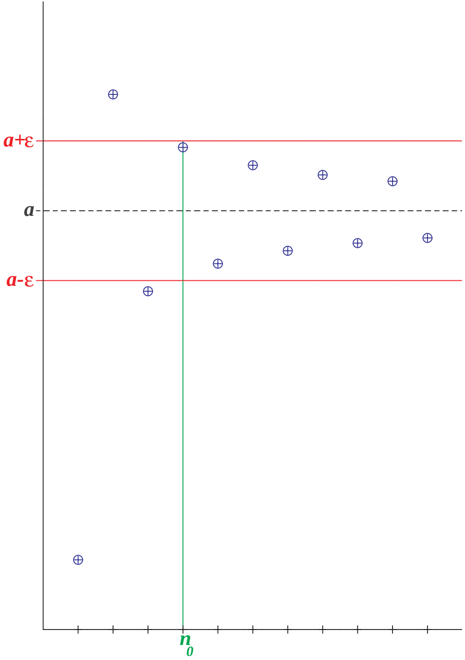

a

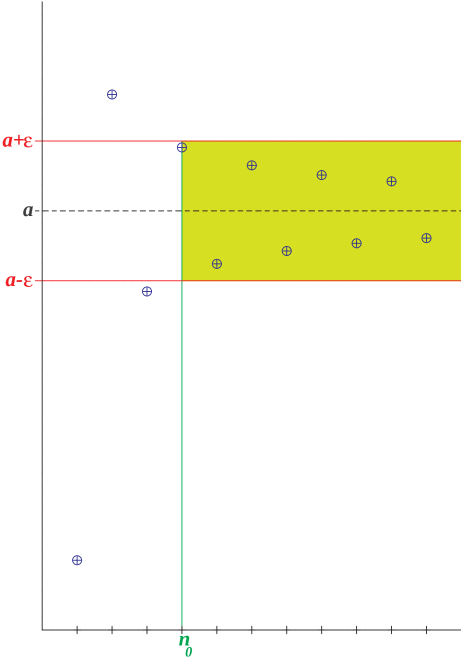

a+ε
a
a-ε

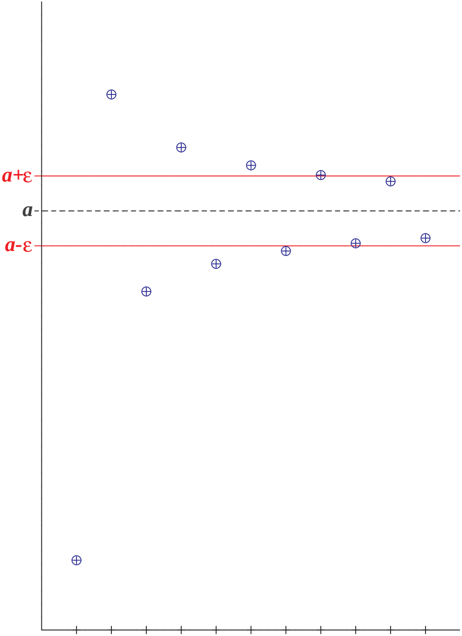

a+ε
a
a-ε
n0

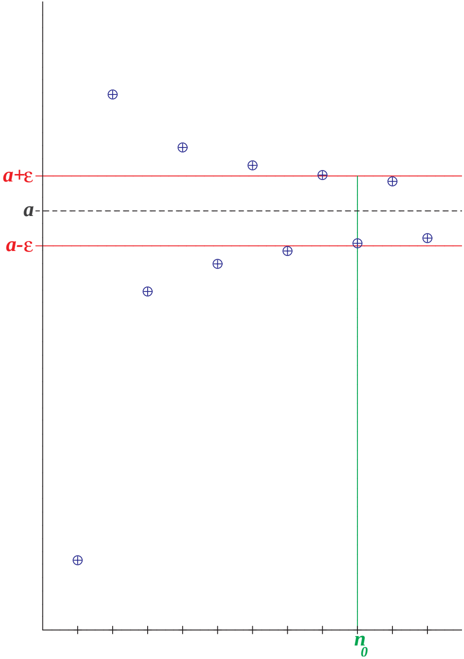

a+ε
a
a-ε
n
0

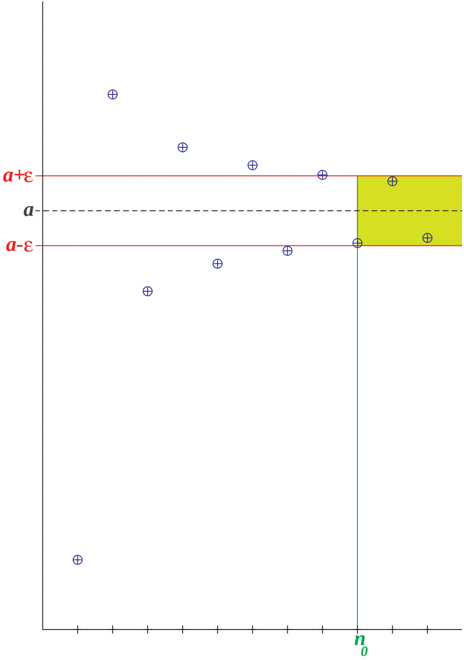

a+ε
a
a-ε


a+ε
a
a-ε
n
0


a+ε
a
a-ε
n
0


a+ε
a
a-ε


a+ε
a
a-ε
n
0


a+ε
a
a-ε
n0

Theorem 1 (uniqueness of a limit)

Every sequence has at most one limit.

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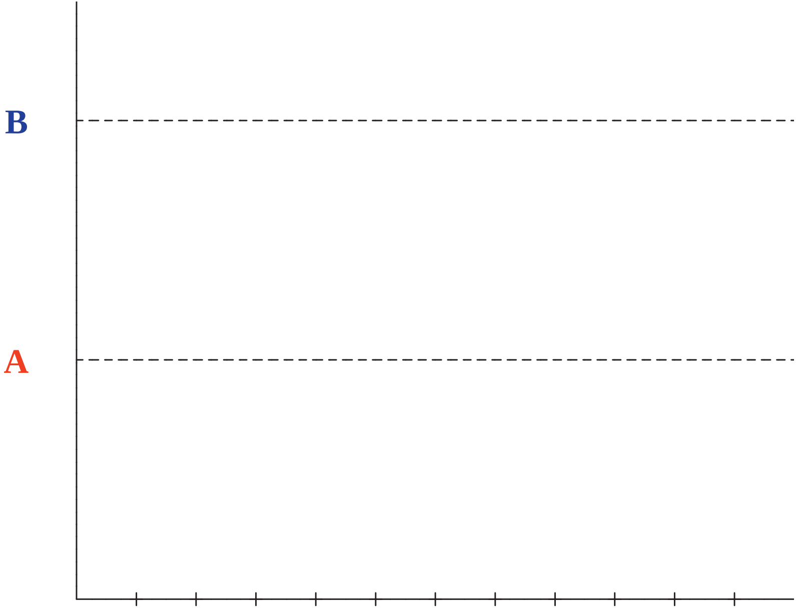
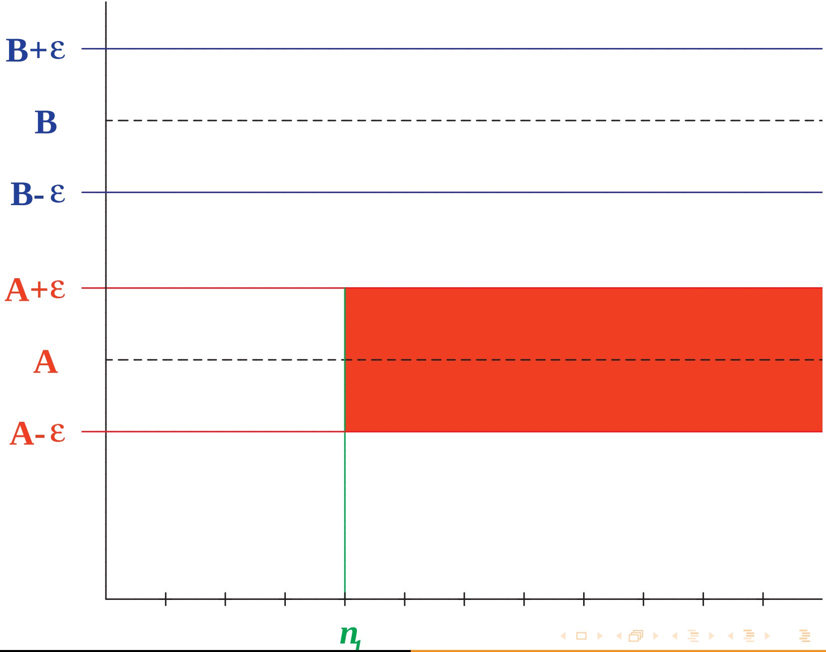
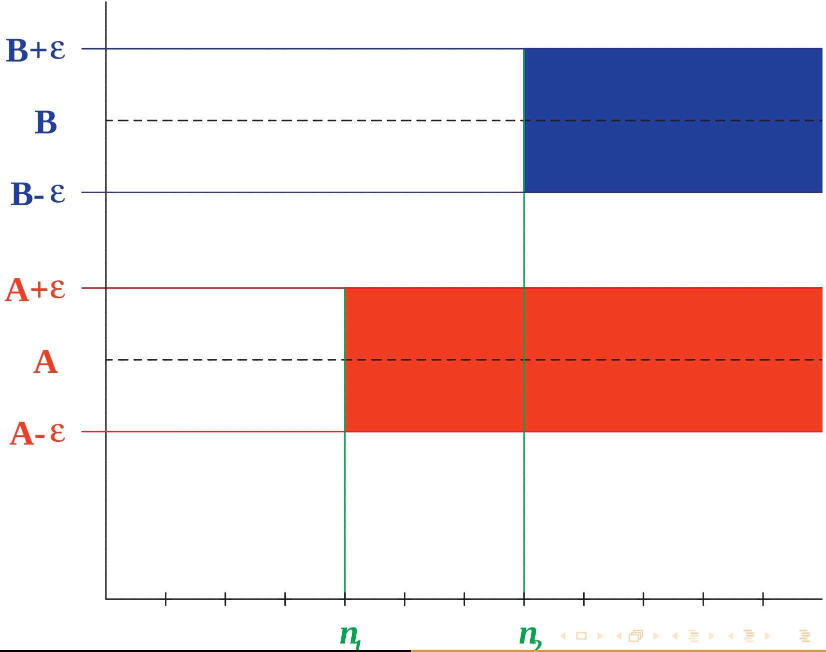
We use the notation $\lim_{n \to \infty} a_n = A$ or simply $\lim a_n = A$.


B
A

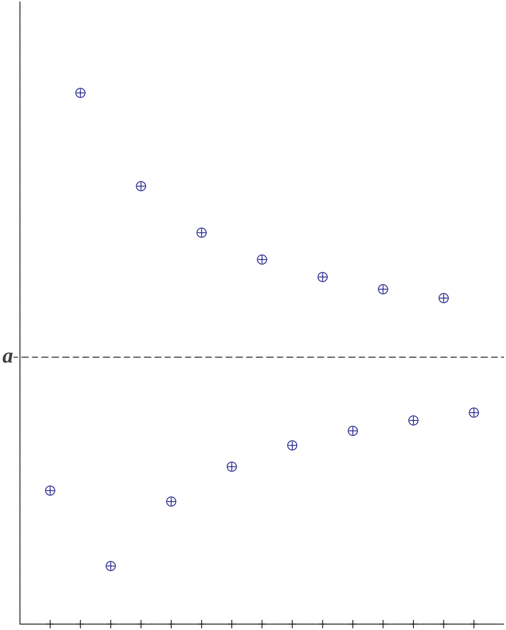

B+ε
B
B- ε
A+ε
A
A- ε

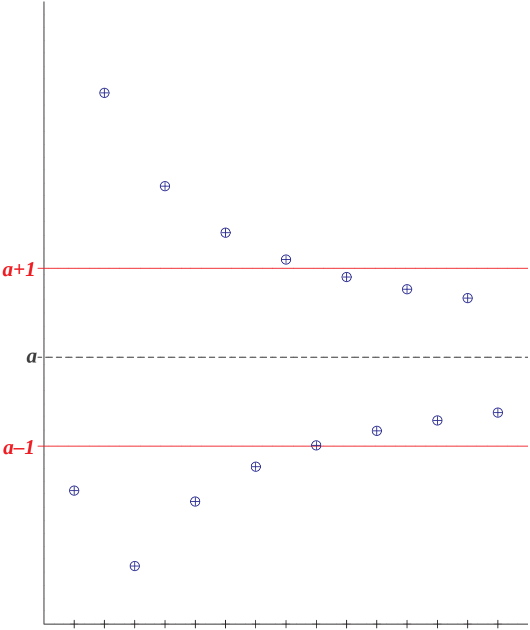

B+ε
B
B- ε
A+ε
A
A- ε
n_1

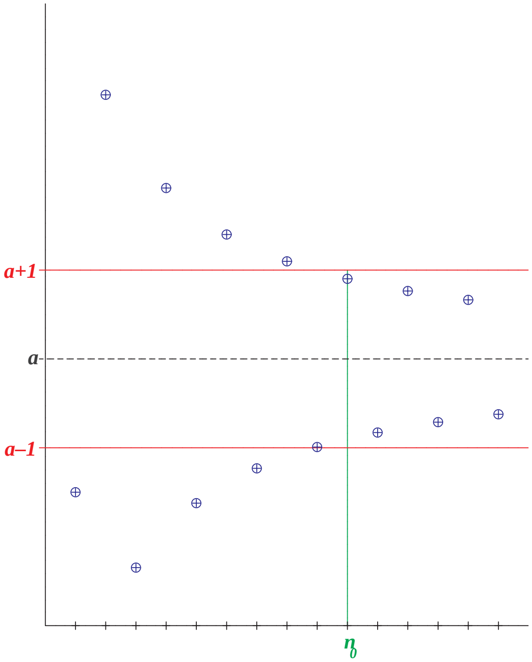

B+ε
B
B- ε
A+ε
A
A- ε
n_1
n_2

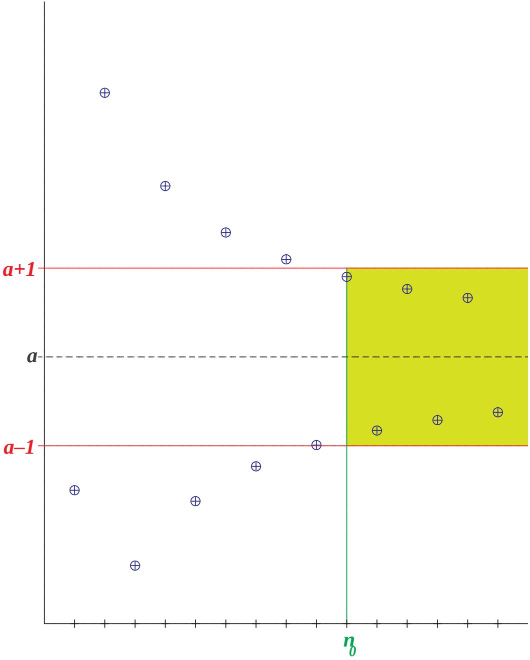
Theorem 2

Every convergent sequence is bounded.

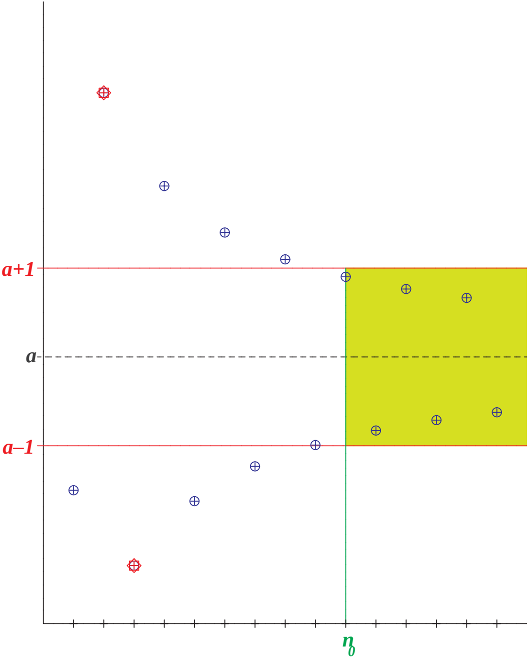

a

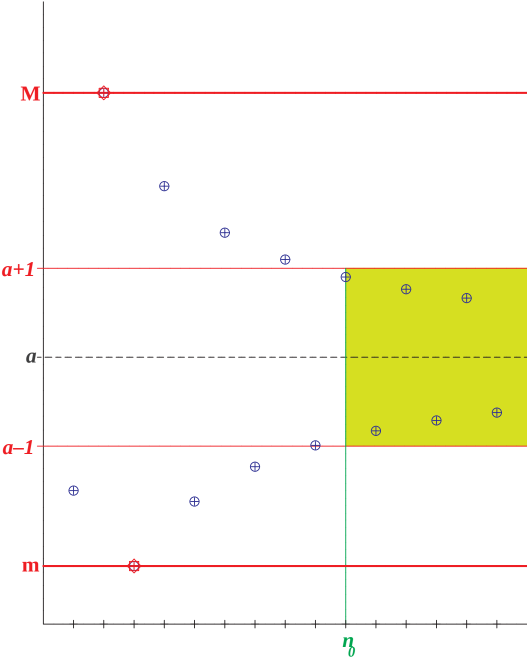

a+1
a
a–1


a+1
a
a–1
n_0



a+1
a
a–1
n_0


a+1
a
a−1
n_0


M
a+1
a
a–1
m
n0

Exercise

Find a sequence, which is

1. bounded and convergent
2. bounded and divergent
3. unbounded and convergent
4. unbounded and divergent

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

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Exercise

Let $a_n = 3, 7, 4, 1/2, \pi, -1$. Find $b_n = a_{2n}$:

A $6, 14, 8 \ldots$

B $5, 9, 6 \ldots$

C $7, 1/2, -1 \ldots$

D $4, 1/2, \pi \ldots$

By:https://www.cpp.edu/conceptests/question-library/mat116.shtm

Theorem 3 (limit of a subsequence)

Let $\{b_k\}_{k=1}^{\infty}$ *be a subsequence of* $\{a_n\}_{n=1}^{\infty}$. *If* $\lim_{n\to\infty} a_n = A \in \mathbb{R}$, *then also* $\lim_{k\to\infty} b_k = A$.

Remark

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $A \in \mathbb{R}$, $K \in \mathbb{R}$, $K > 0$. If

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 \colon \; |a_n - A| < K\varepsilon,$$

then $\lim a_n = A$.

Theorem 4 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

(i) $\lim\limits_{n\to\infty} (a_n + b_n) = A + B,$

Remark

Consider cases

1. $a_n = (-1)^n$, $b_n = (-1)^n$
2. $a_n = n$, $b_n = \frac{1}{n}$
3. $a_n = n^2$, $b_n = \frac{1}{n}$

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(ii) $\lim\limits_{n\to\infty} (a_n \cdot b_n) = A \cdot B$,

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(ii) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} (a_n / b_n) = A/B$.

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Theorem 5 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$. *Then*

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(ii) $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}(a_n/b_n) = A/B$.

Idea of the proof

Proof for $+$ follows from definition.

Proof for $\cdot$ is harder and is based on important trick of "adding and subtracting":

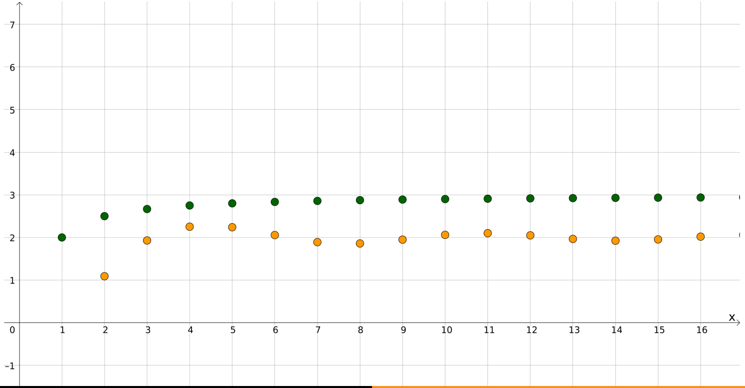
$$A \cdot B - a_n \cdot b_n = A \cdot B - A \cdot b_n + A \cdot b_n - a_n \cdot b_n$$

$$= \underbrace{A}_{|.|\leq C} \cdot \underbrace{(B - b_n)}_{|.|\leq \varepsilon} + \underbrace{(A - a_n)}_{|.|\leq \varepsilon} \cdot \underbrace{b_n}_{|.|\leq C}$$

Theorem 6 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.



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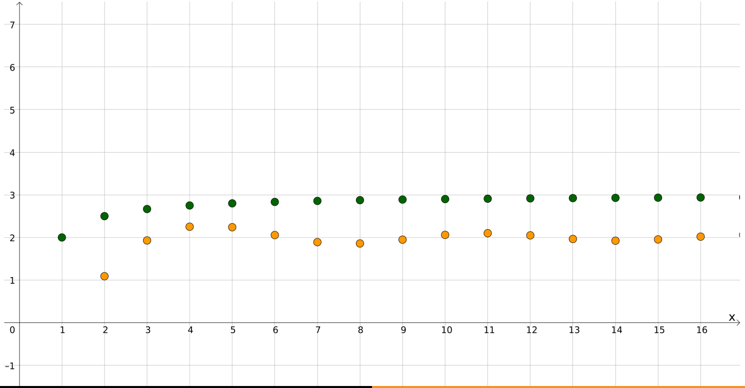
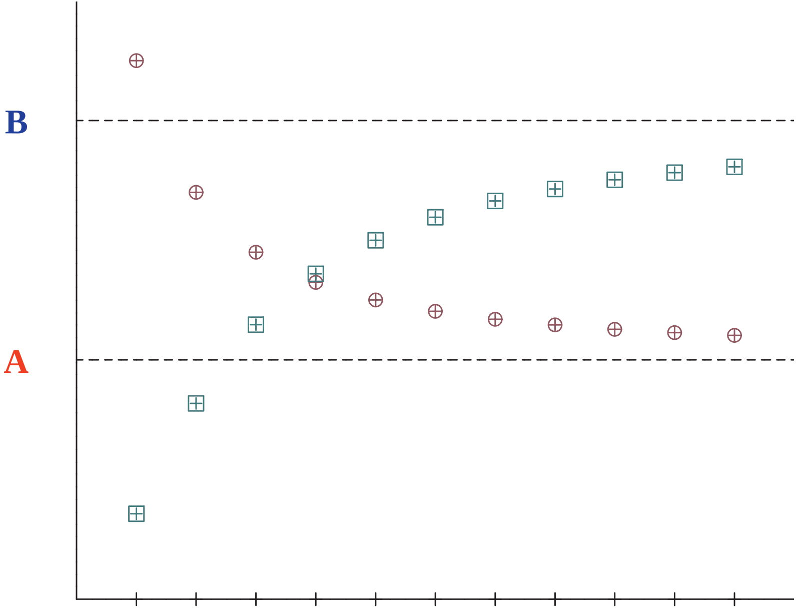
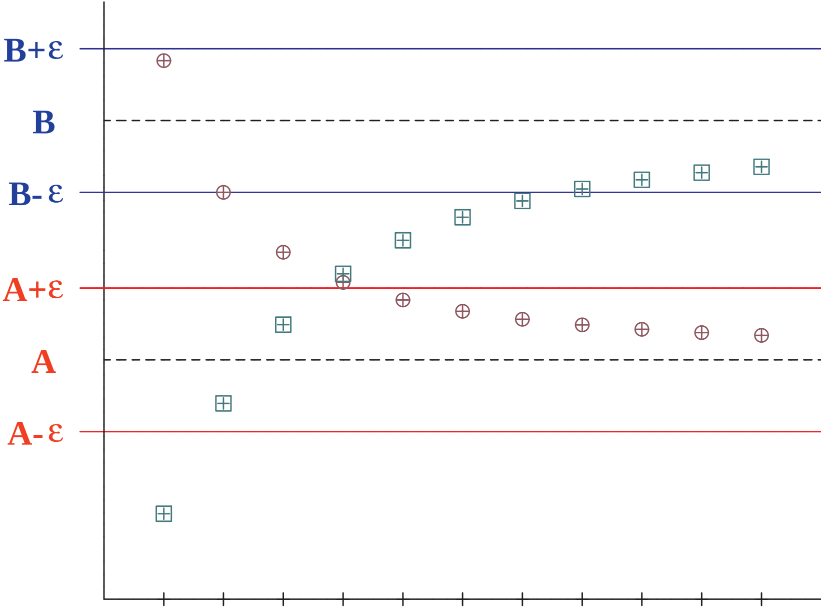
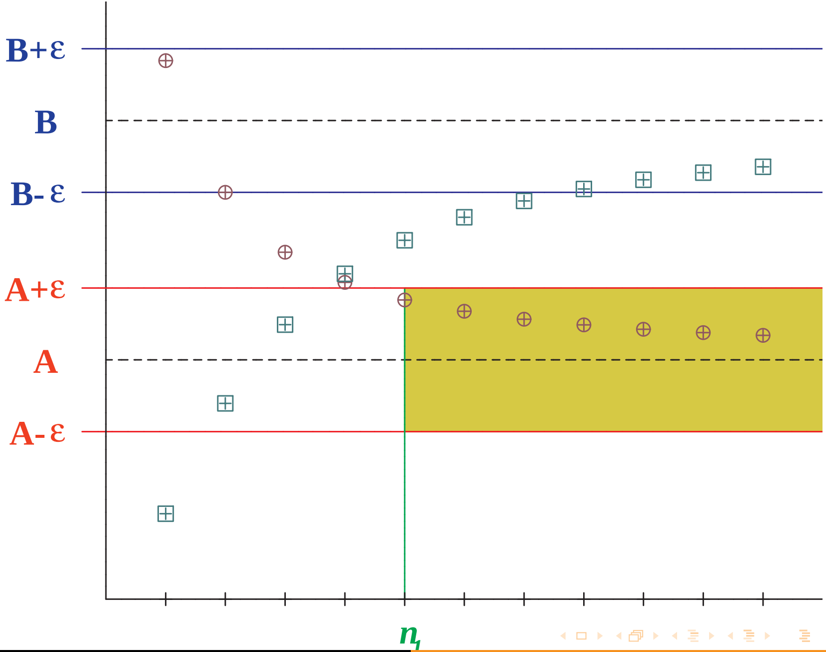
(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.

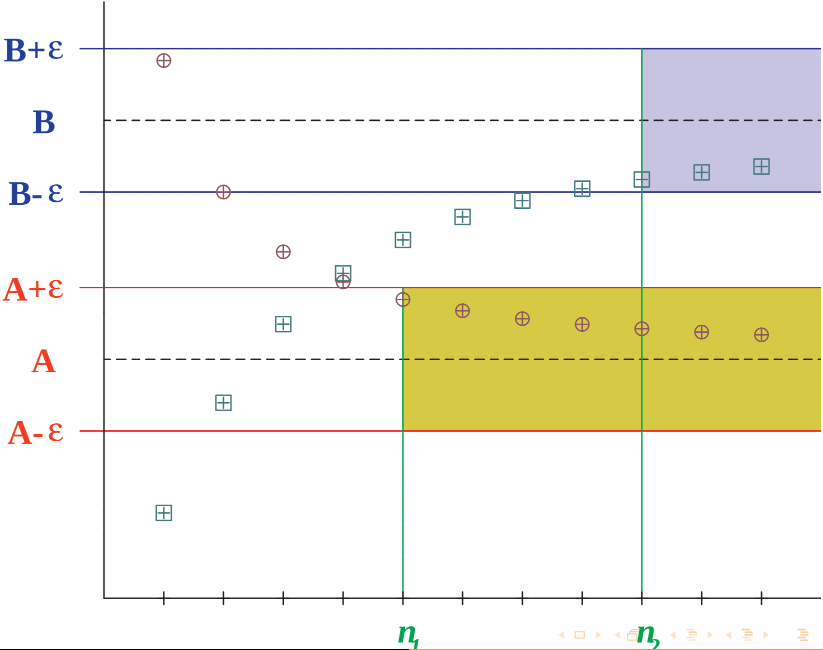
(ii) *Suppose that* $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

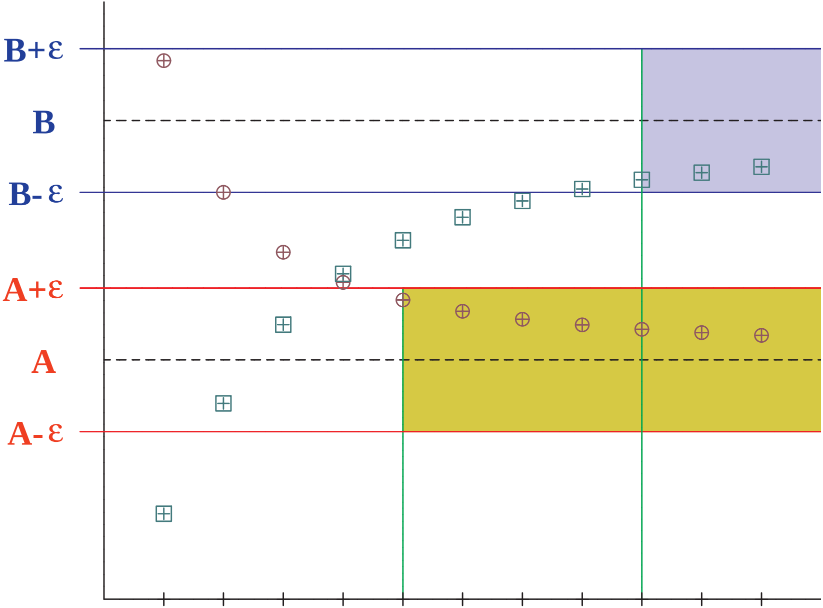


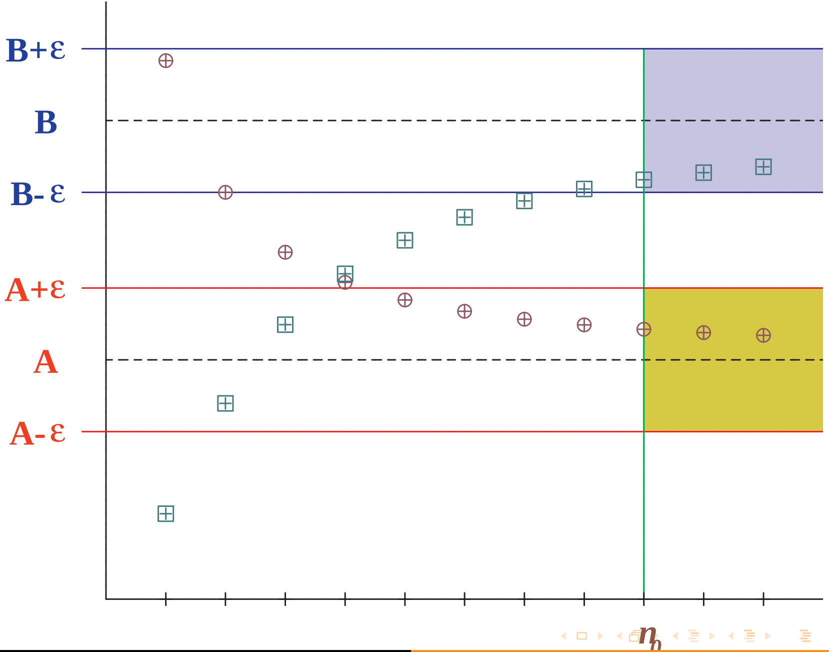

B
A


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n₁


B+ε
B
B- ε
A+ε
A
A- ε
n1
n2


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n_0

Theorem 7 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

1. *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.
2. *Suppose that* $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

Exercise (True or false)

Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$.
If $a_n < b_n$, then $A < B$.

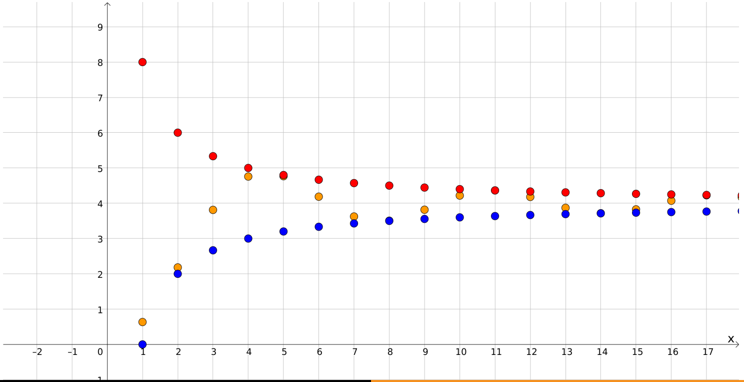
Theorem 8 (two policemen (sandwich theorem))

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 : a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.



Theorem 9 (two policemen)

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.

Exercise

Find the cops for the sequence $a_n = \dfrac{\cos n}{n}$.

Theorem 9 (two policemen)

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.

Exercise

Find the cops for the sequence $a_n = \dfrac{\cos n}{n}$.

Corollary 10

Suppose that $\lim a_n = 0$ *and the sequence* $\{b_n\}$ *is bounded. Then* $\lim a_n b_n = 0$.

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

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$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n < K.$$

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Theorem 1 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$.

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 : a_n > L.$$

We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 : a_n < K.$$

Theorem 1 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$. If $\lim a_n \in \mathbb{R}$, then we say that the limit is finite, if $\lim a_n = +\infty$ or $\lim a_n = -\infty$, then we say that the limit is infinite.

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n > L.$$

Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

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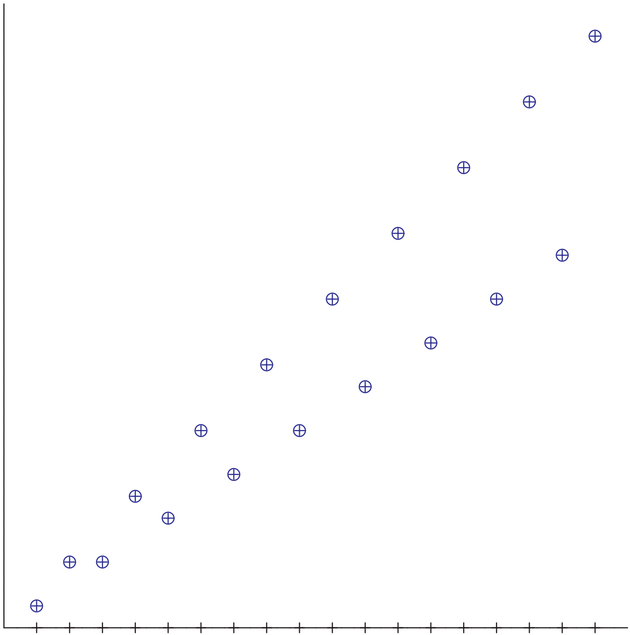
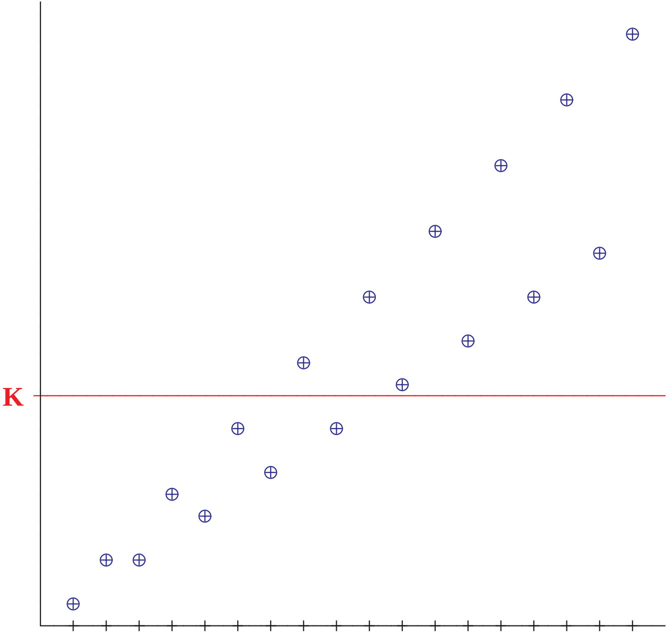
We say that a sequence $\{a_n\}$ has a limit $-\infty$ (minus infinity) if

$$\forall K \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n < K.$$

https://www.geogebra.org/calculator/cpuzsnnh

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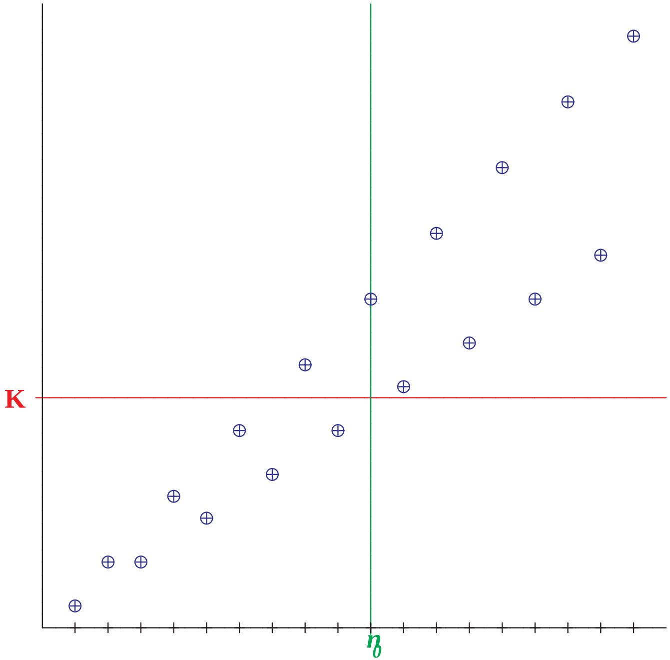
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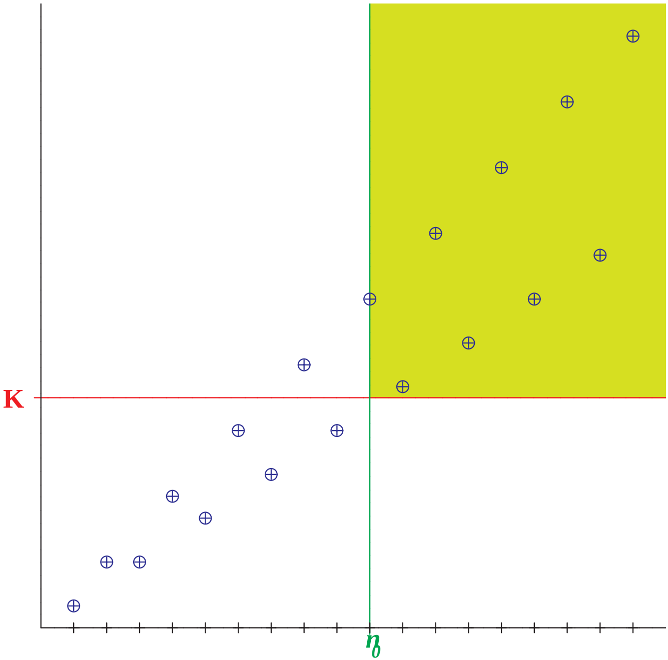
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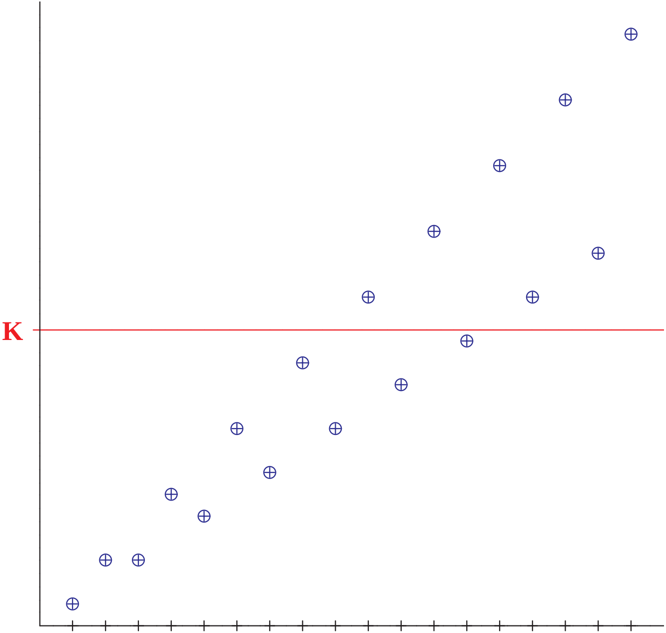
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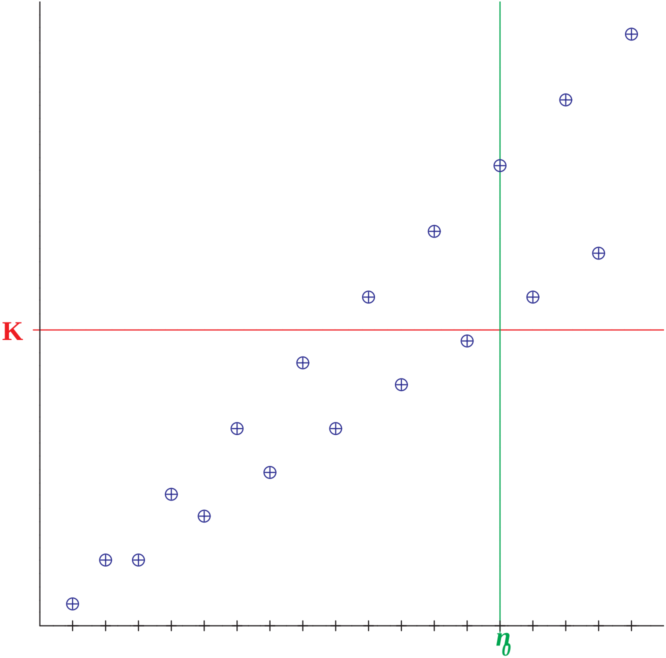
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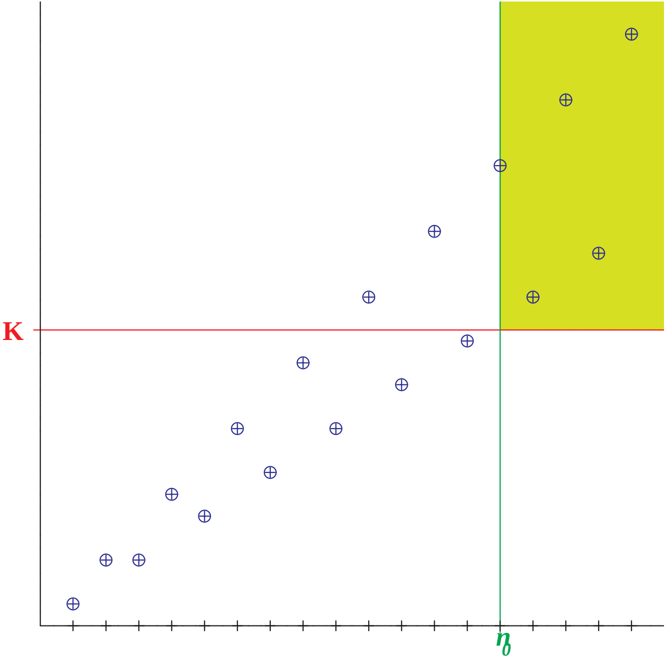
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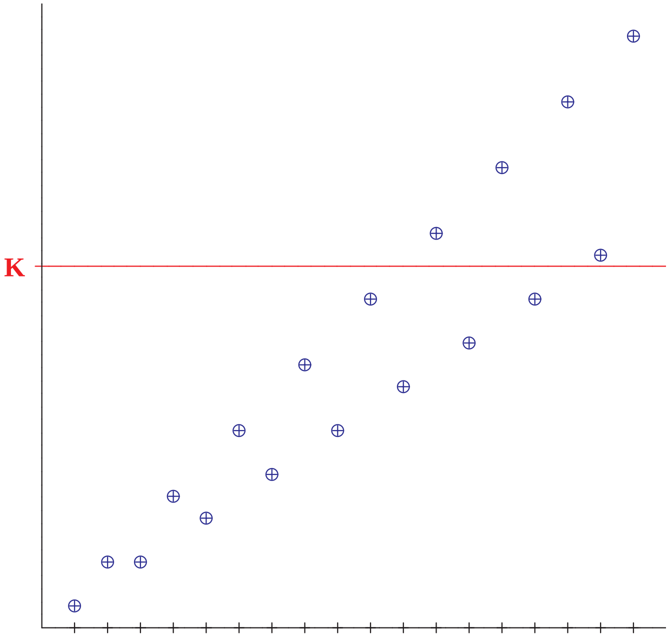
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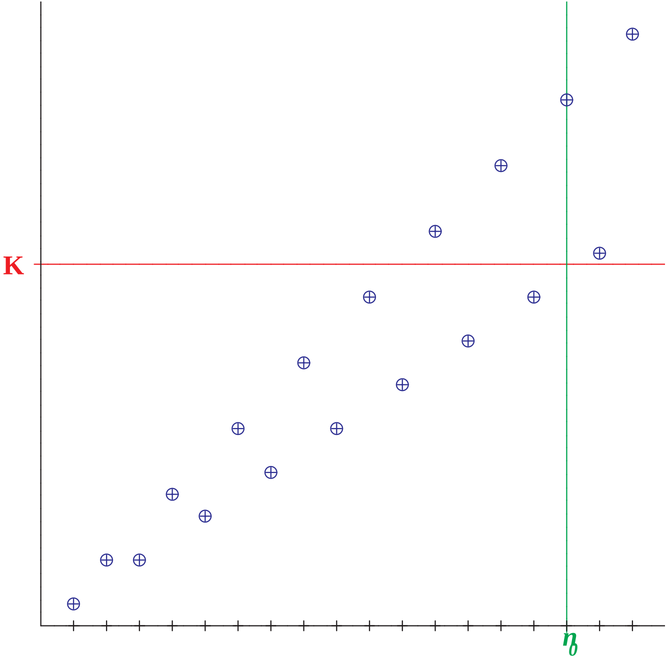
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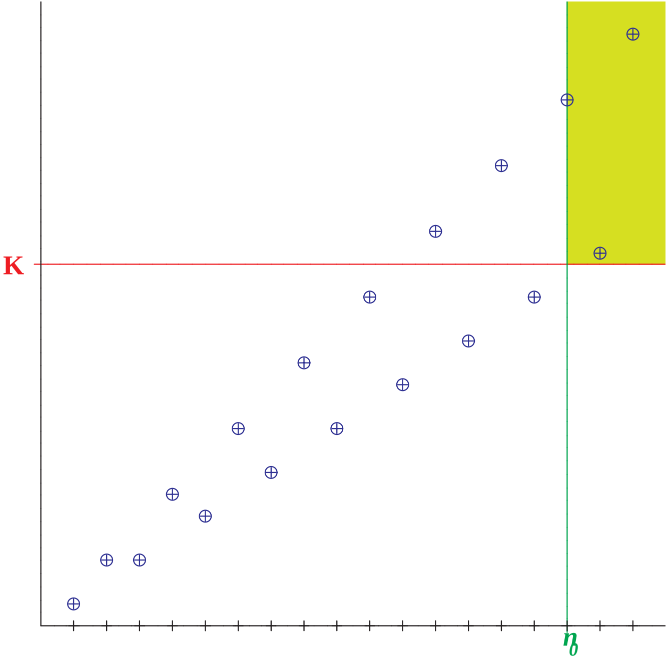
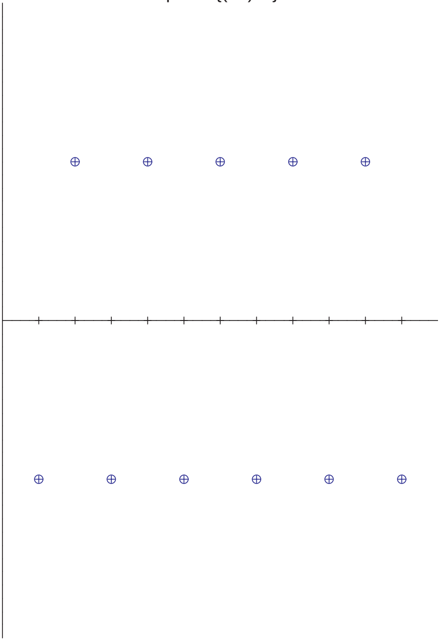
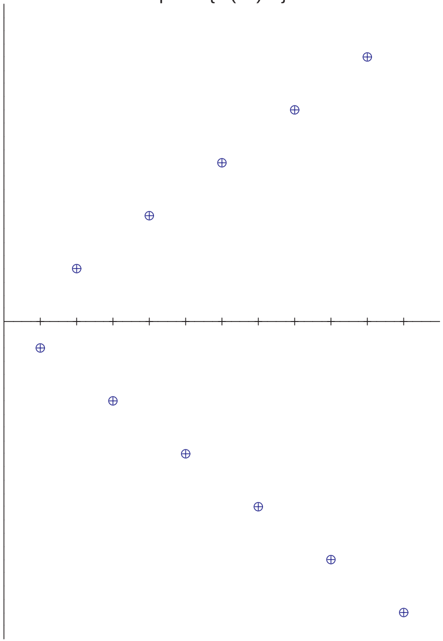
n_0

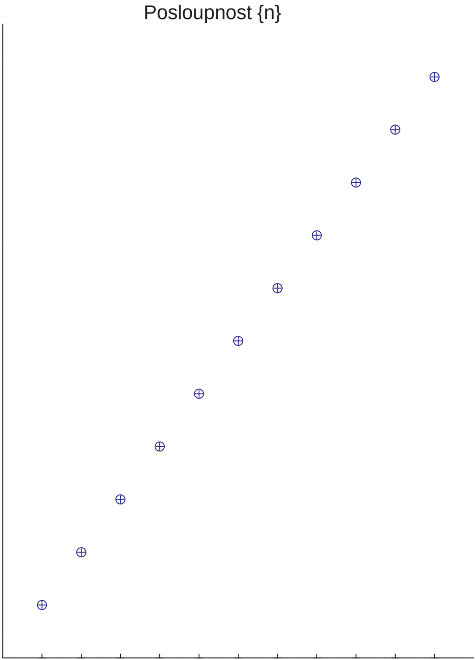

K
n_0

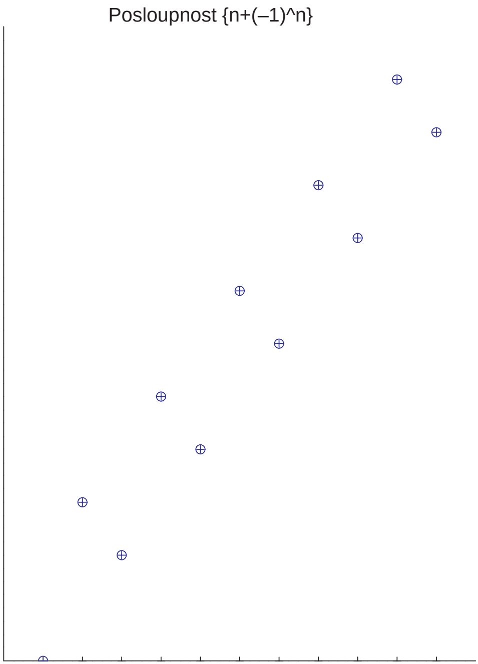
Posloupnost {(–1)^n}

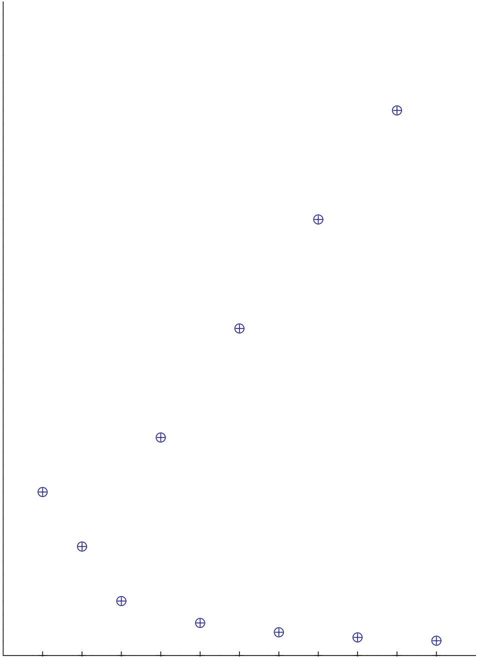


Posloupnost {n.(–1)^n}




Posloupnost {n}


Posloupnost {n+(–1)^n}



Theorem 2 does not hold for infinite limits. But:

Theorem 2'

- *Suppose that* $\lim a_n = +\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

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Exercise

Give an example of $a_n \to \infty$ and find its lower bound.

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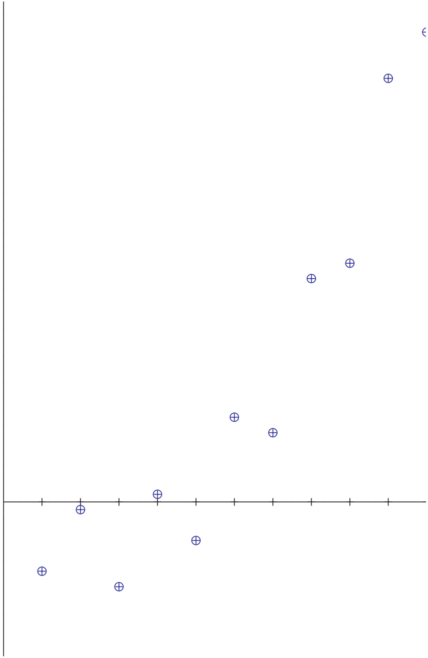
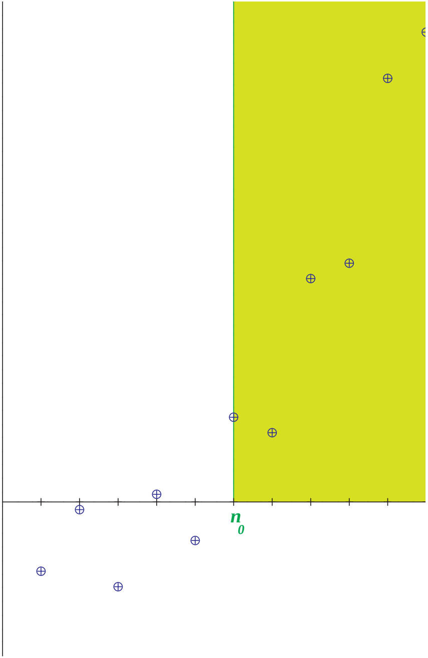
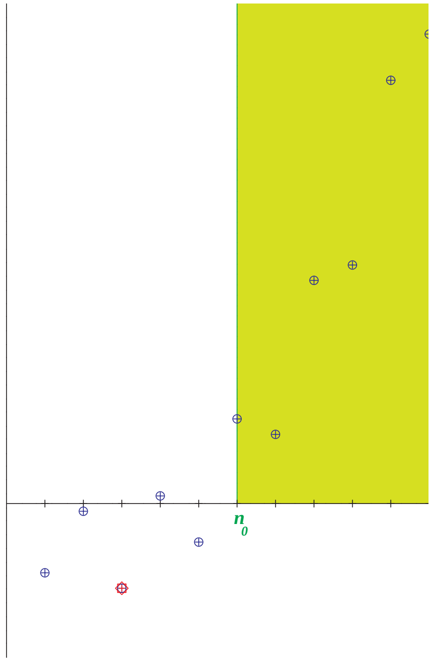
Exercise

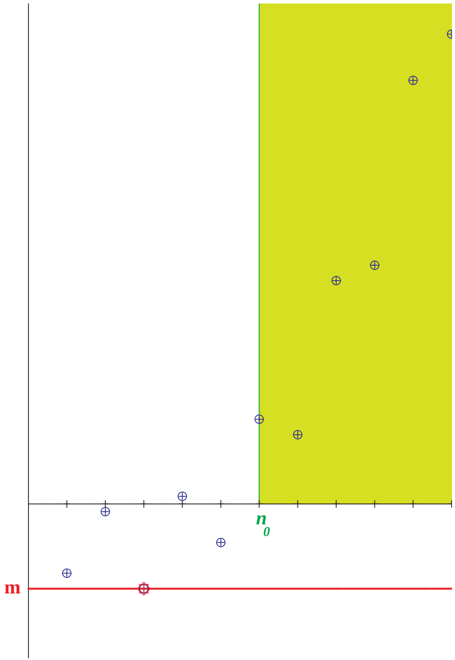
Give an example of $a_n \to \infty$ and find its lower bound.

Theorem 3 (limit of a subsequence) holds also for infinite limits.




n_0


n_0



m
n_0

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

Exercise

1. $2 + \infty$
2. $-\infty + 3$
3. $\pi\infty$
4. $-4(-\infty)$
5. -7∞
6. $\frac{\infty}{-3}$
7. $\frac{5}{\infty}$

The following operations are not defined:

- $(-\infty)+(+\infty)$, $(+\infty)+(-\infty)$, $(+\infty)-(+\infty)$, $(-\infty)-(-\infty)$,
- $(+\infty)\cdot 0$, $0\cdot(+\infty)$, $(-\infty)\cdot 0$, $0\cdot(-\infty)$,
- $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{a}{0}$ for $a\in\mathbb{R}^*$.

Theorem 5' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

Theorem 5' (arithmetics of limits)

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(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

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(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Theorem 11

Suppose that $\lim a_n = A \in \mathbb{R}^*$, $A > 0$, $\lim b_n = 0$ *and there is* $n_0 \in \mathbb{N}$ *such that we have* $b_n > 0$ *for every* $n \in \mathbb{N}$, $n \geq n_0$. *Then* $\lim a_n/b_n = +\infty$.

https://www.geogebra.org/calculator/cpuzsnnh

Theorem 7 (limits and ordering) and Theorem 9 (two cops theorem) hold also for infinite limits. Even the following modification holds:

Theorem 9' (one policeman)

Let $\{a_n\}$ *and* $\{b_n\}$ *be two sequences.*

- *If* $\lim a_n = +\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \geq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = +\infty$.
- *If* $\lim a_n = -\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \leq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = -\infty$.

Definition

Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

Definition

Let $A \subset \mathbb{R}$ be non-empty. If A is not bounded from above, then we define $\sup A = +\infty$. If A is not bounded from below, then we define $\inf A = -\infty$.

Lemma 12

Let $M \subset \mathbb{R}$ be non-empty and $G \in \mathbb{R}^*$. Then the following statements are equivalent:

(1) $G = \sup M$.

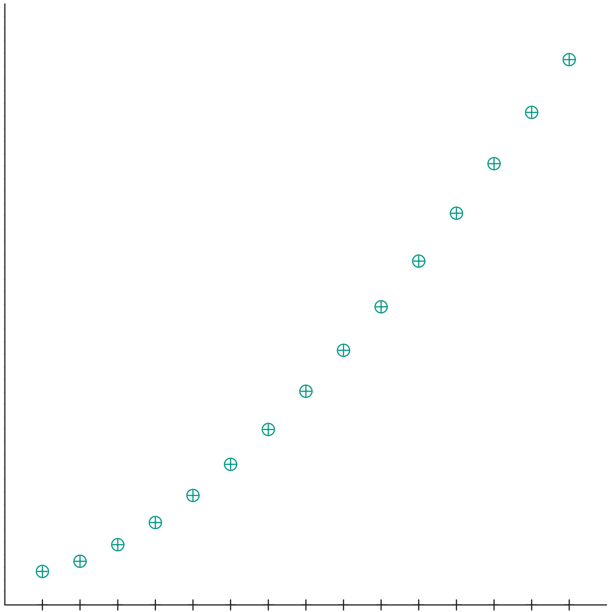
(2) *The number G is an upper bound of M and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of members of M such that $\lim x_n = G$.*

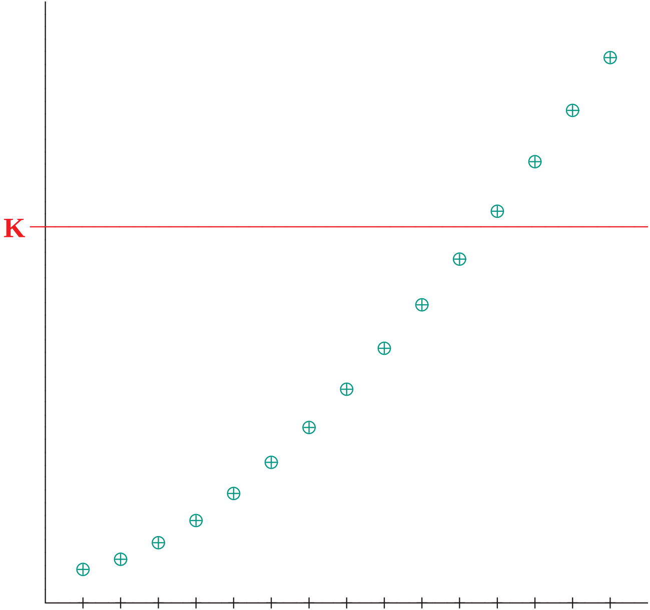
Exercise

Find a sequence $\{x_n\}$ for a set $M = [2, 5)$.

Theorem 13 (limit of a monotone sequence)

Every monotone sequence has a limit. If $\{a_n\}$ *is non-decreasing, then* $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$. *If* $\{a_n\}$ *is non-increasing, then* $\lim a_n = \inf\{a_n;\ n \in \mathbb{N}\}$.

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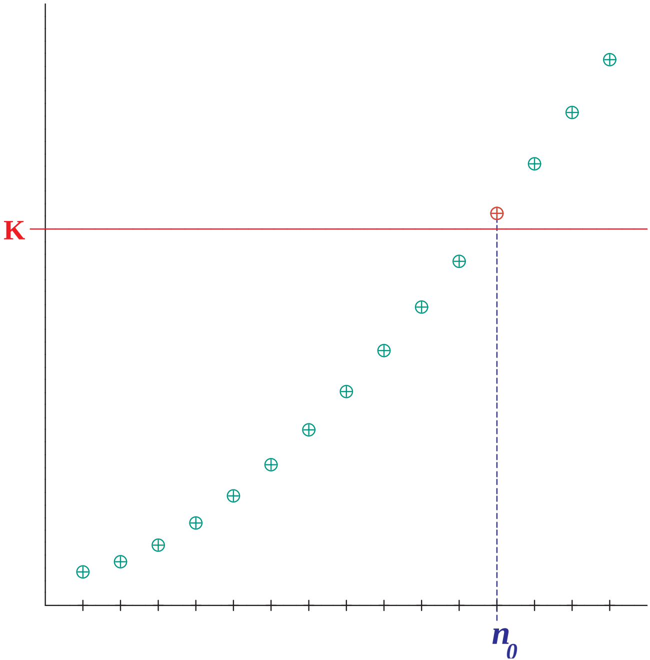
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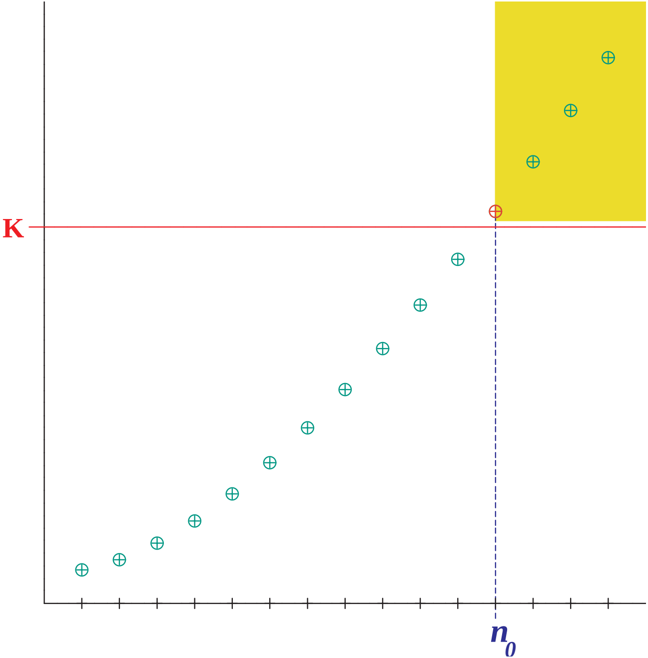
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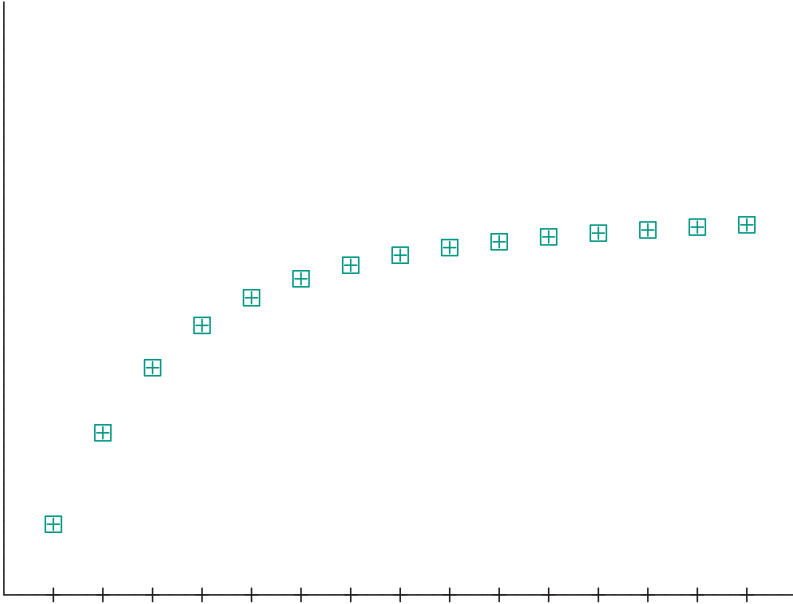

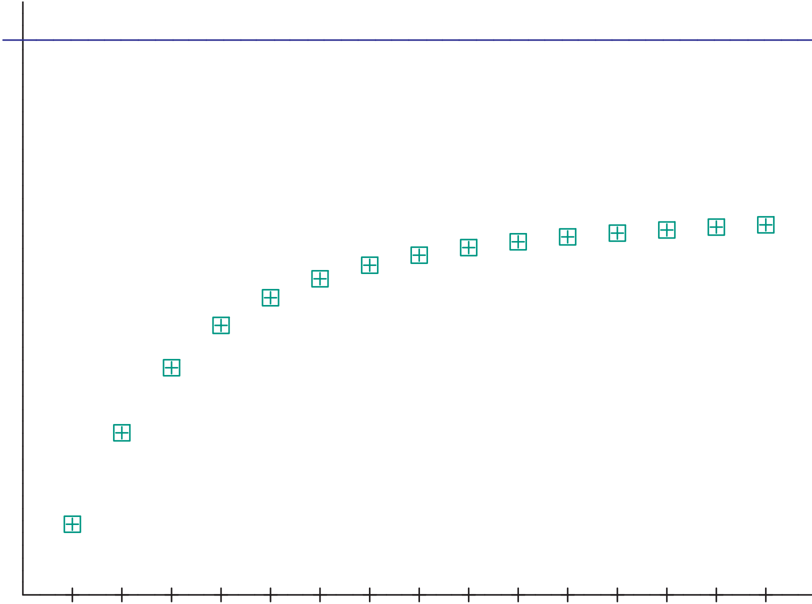

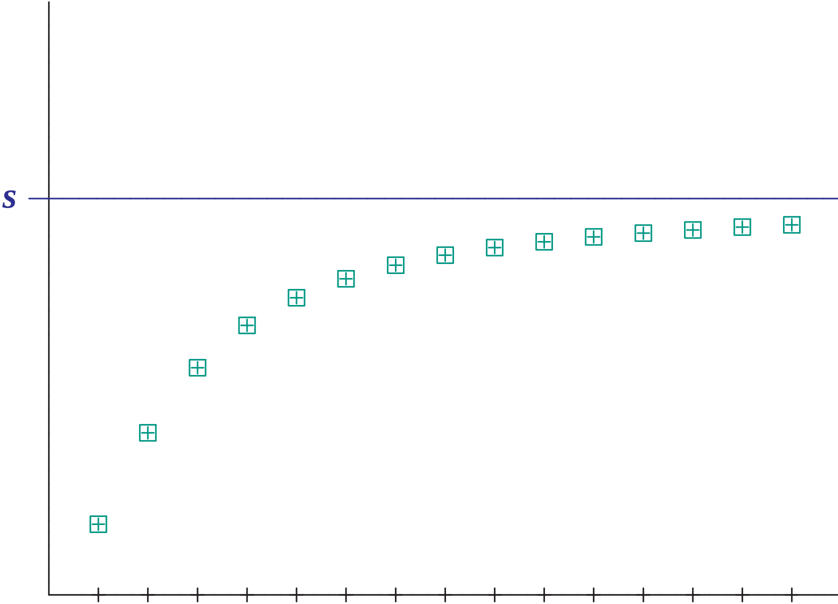
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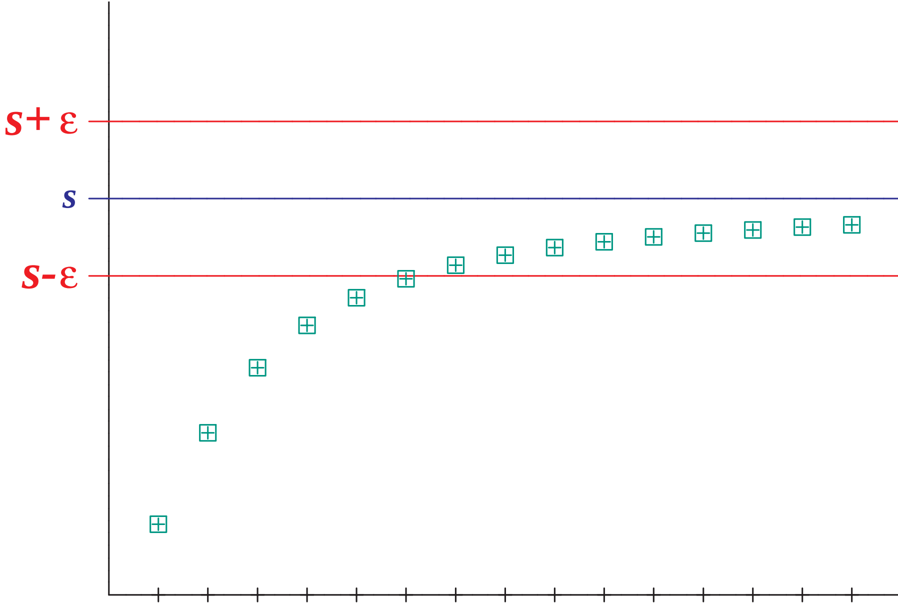
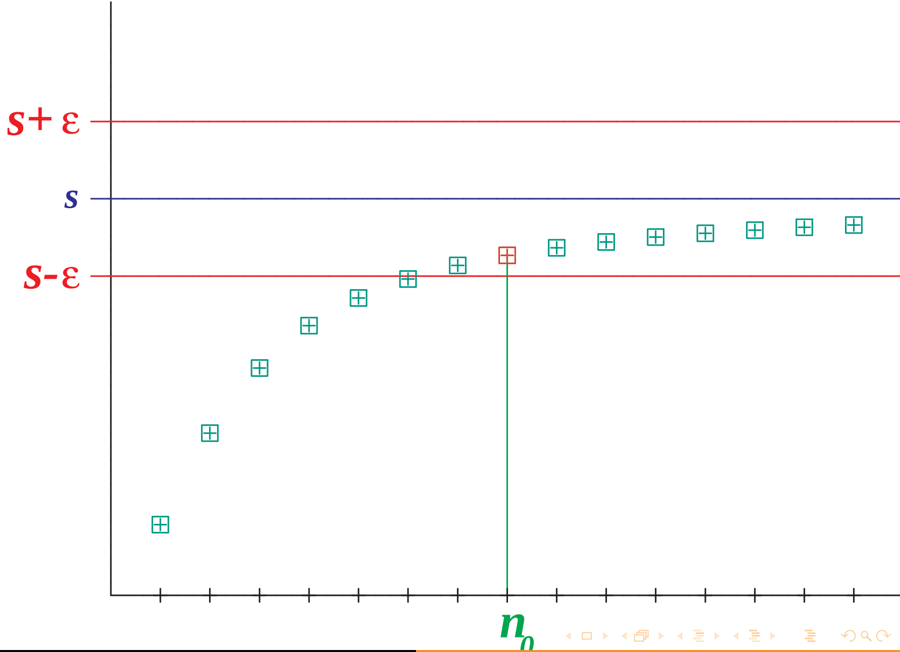
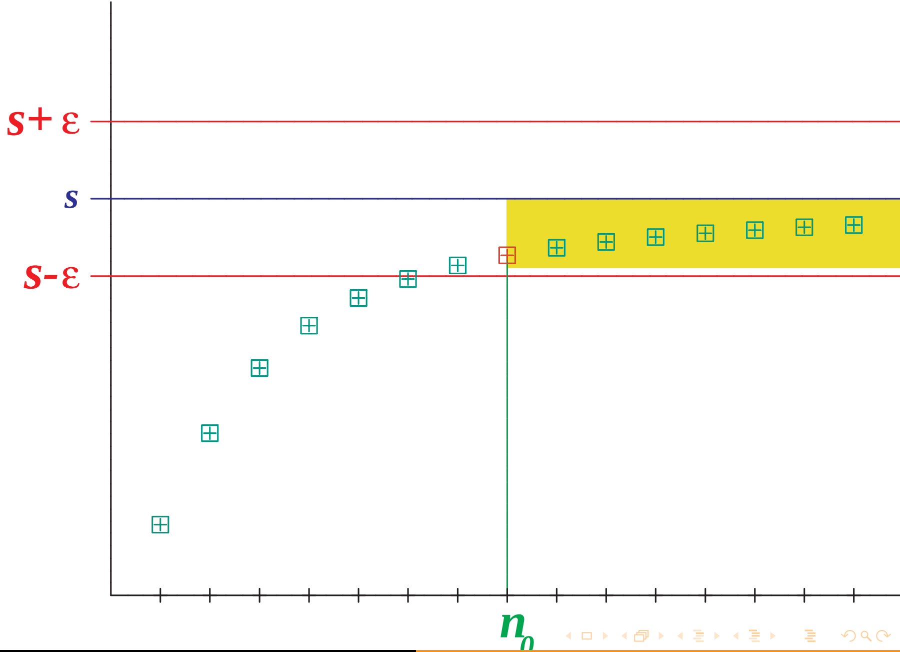


s+ ε
s
s- ε



s+ ε
s
s-ε
n0



$s+\varepsilon$
s
$s-\varepsilon$
n_0

Theorem 14 (Cauchy criteria)

$$\exists \lim_{n\to\infty} a_n \in \mathbb{R} \iff \forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall n, m \geq N : \ |a_n - a_m| < \varepsilon.$$

Theorem 14 (Cauchy criteria)

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Proof

"$\Rightarrow$" Easy: if $b = \lim\limits_{n\to\infty} a_n$, then
$\forall\varepsilon\, \exists N \in \mathbb{N}\, \forall n, m \geq N : |a_n - a_m| \leq |a_n - b| + |a_m - b| < 2\varepsilon.$

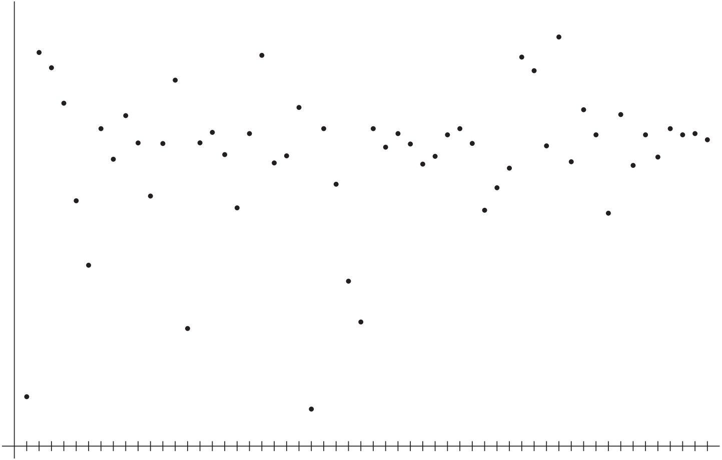
"$\Leftarrow$" Complicated: relies on the infimum axiom. Take a sequence of epsilons: $\varepsilon = \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^k}, \ldots$.
For $\varepsilon = \frac{1}{2}\ \exists N_1 \in \mathbb{N}\, \forall n, m \geq N_1 : \ |a_n - a_m| < \frac{1}{2}$. Put $m = N_1$, then for all $n \geq N_1 : \ a_n \in [A_1 := a_{N_1} - \frac{1}{2}, B_1 := a_{N_1} + \frac{1}{2}]$.

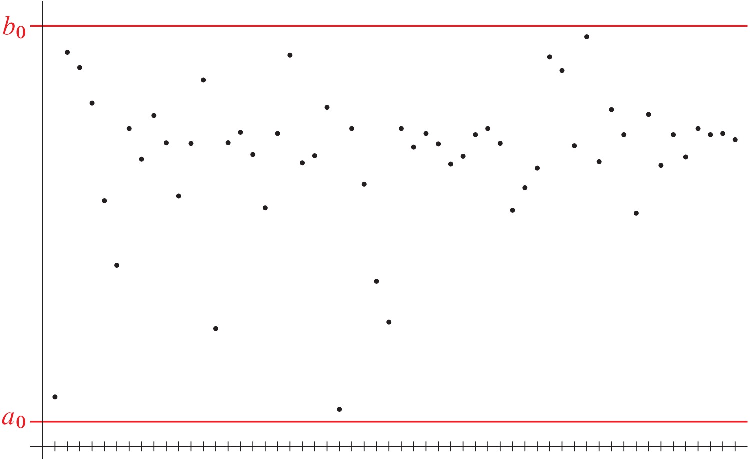
For $\varepsilon = \frac{1}{4}\ \exists \tilde{N}_2 \in \mathbb{N}\, \forall n, m \geq \tilde{N}_2 : \ |a_n - a_m| < \frac{1}{4}$. Set $m = N_2 := \max\left\{N_1, \tilde{N}_2\right\}$, then for all $n \geq \tilde{N}_2 : \ a_n \in [A_2, B_2]$, where $A_2 = \max\left\{A_1, a_{N_2} - \frac{1}{4}\right\}$, $B_2 = \min\left\{B_1, a_{N_2} + \frac{1}{4}\right\}$.

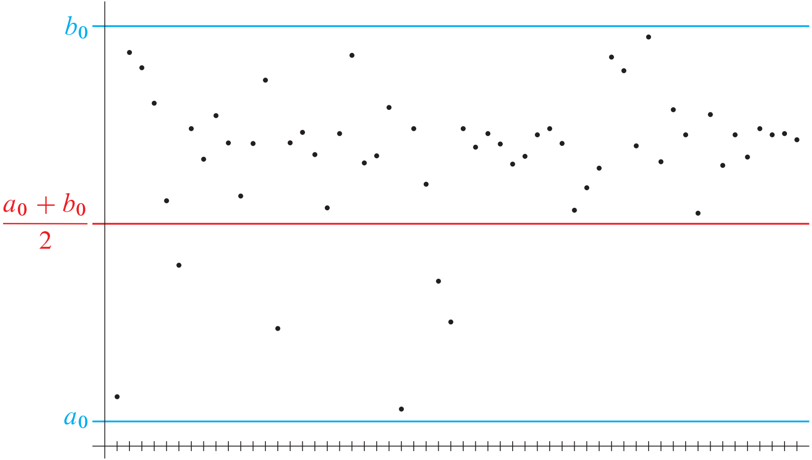
Continuing, we construct a sequence of nested contracting segments $\{[A_p, B_p,]\}$, $A_1 \leq A_2 \leq \ldots A_p \leq B_p \leq \ldots \leq B_2 \leq B_1$.

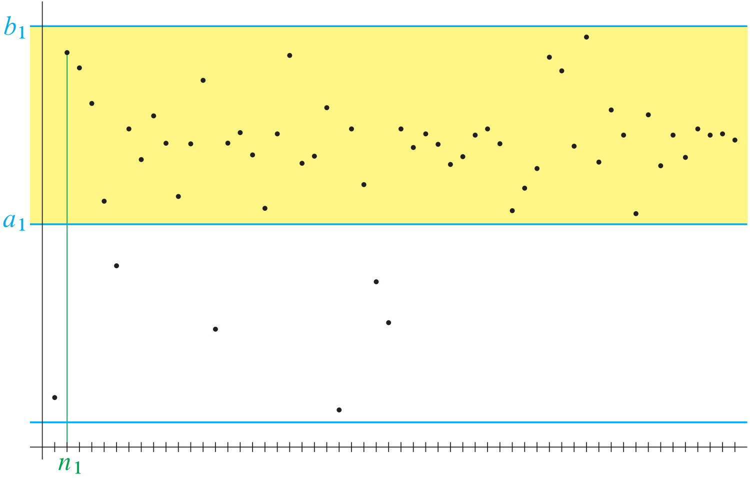
Theorem 15 (Bolzano-Weierstraß)

Every bounded sequence contains a convergent subsequence.

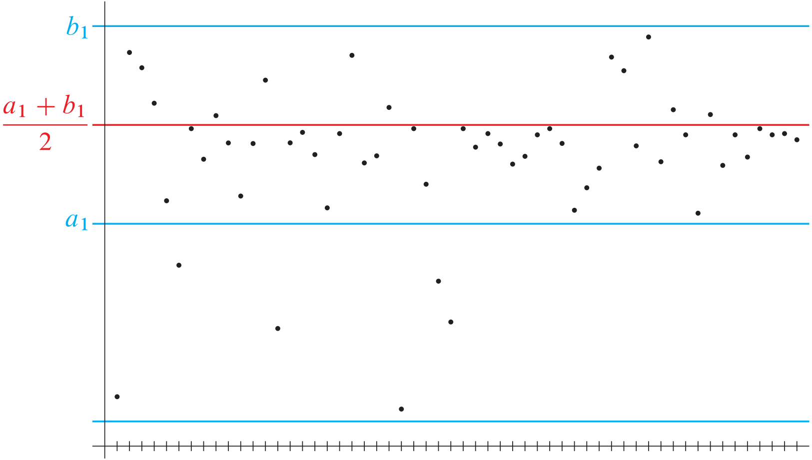


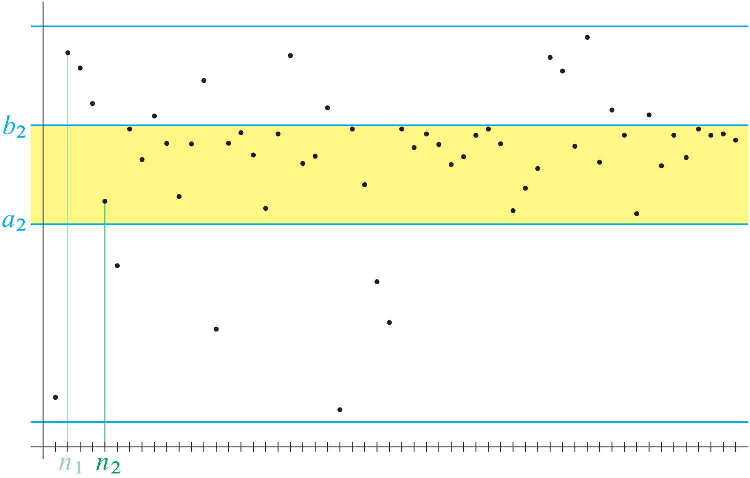

b_0
a_0


b_0
$\frac{a_0 + b_0}{2}$
a_0

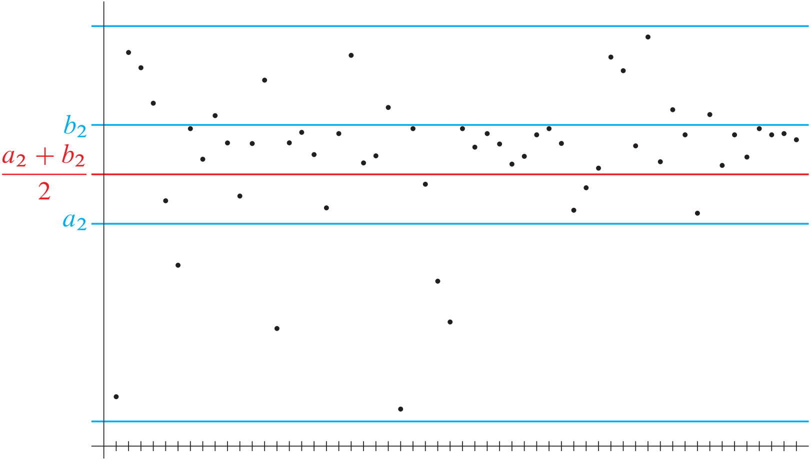


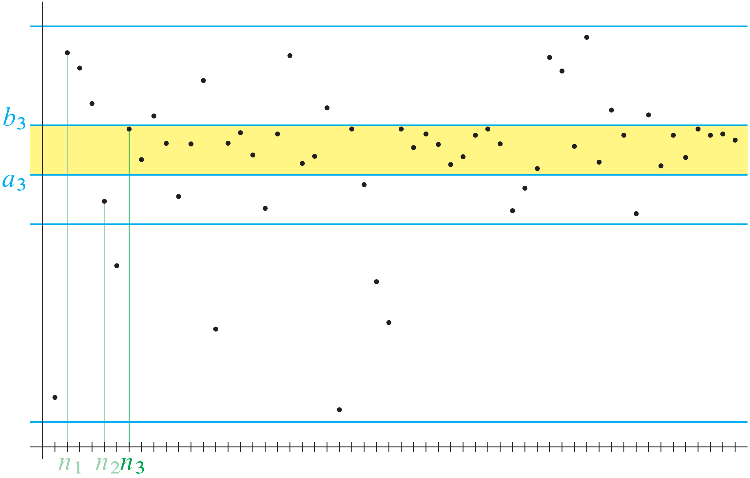
b_1
a_1
n_1


b_1
$\frac{a_1 + b_1}{2}$
a_1

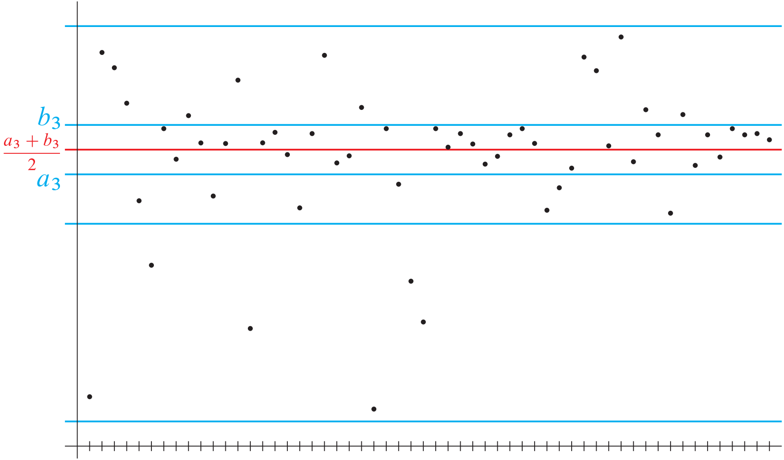


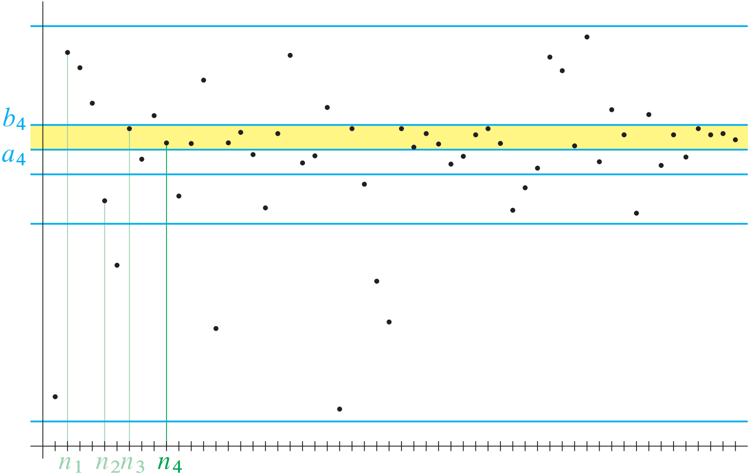
b_2
a_2
n_1
n_2


b_2
$\frac{a_2 + b_2}{2}$
a_2

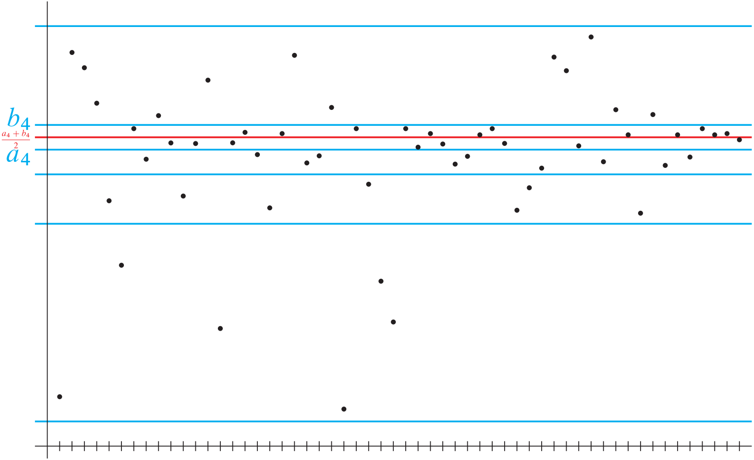


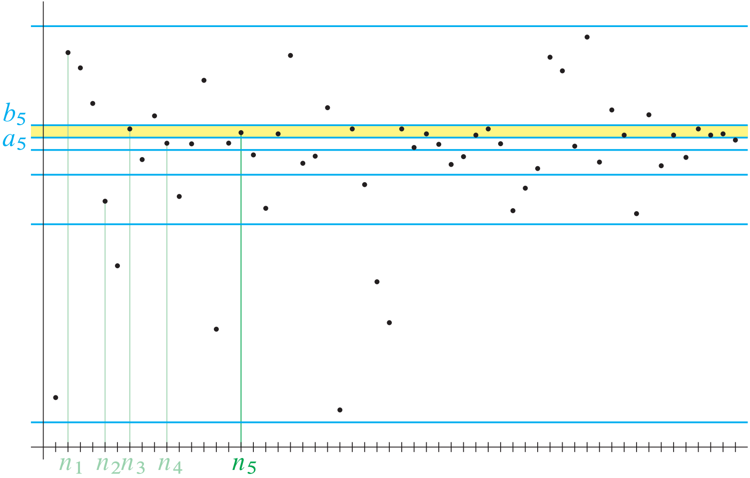
b_3
a_3
n_1
n_2
n_3


b_3
$\frac{a_3 + b_3}{2}$
a_3

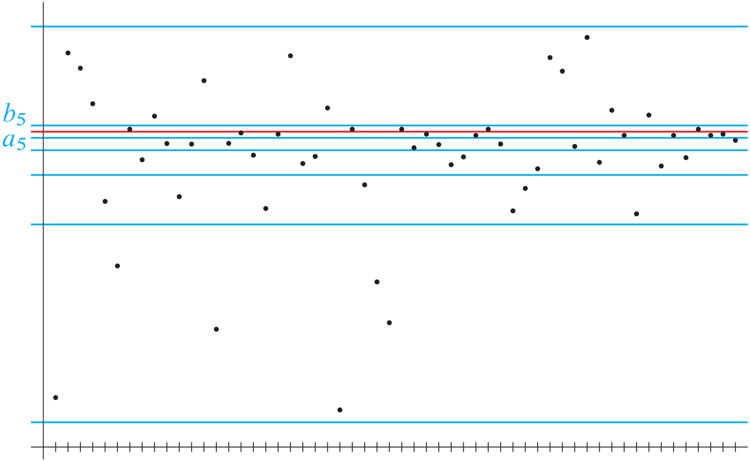


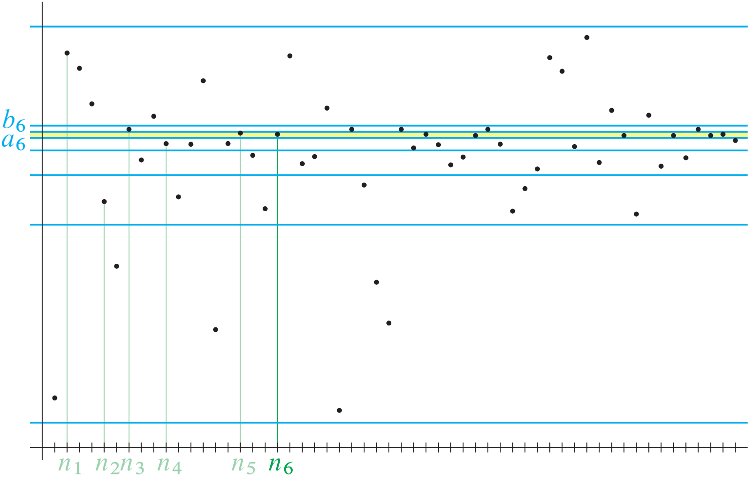
b_4
a_4
n_1
n_2
n_3
n_4


b_4
$\frac{a_4+b_4}{2}$
a_4



b_5
a_5
n_1
n_2
n_3
n_4
n_5


b_5
a_5



b_6
a_6
n_1
n_2
n_3
n_4
n_5
n_6

Theorem 16 (Bolzano-Weierstraß)

Every bounded sequence contains a convergent subsequence.

Exercise

Find a convergent subsequence:

A $a_n = (-1)^n$

B $a_n = \{0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \ldots\}$