Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

DIPLOMOVÁ PRÁCE

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Numerická simulace proudění v ložisku

Matematický ústav Univerzity Karlovy

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Studijní program: Matematika
Studijní obor: Matematické a počítačové modelování ve fyzice a v technice

Děkuji svému vedoucímu Doc. RNDr. Josefu Málkovi, CSc. za jeho pečlivé vedení a trpělivost. Stejně tak děkuji Dr. Jaroslavu Hronovi za vydatnou pomoc se vším, co souviselo s numerickými simulacemi. Oba přispěli k tomu, že mne práce začala bavit.

Upřímné díky patří mé rodině a mé přítelkyni za podporu a pomoc při studiu.

Prohlašuji, že jsem svou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

Martin Lanzendörfer

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Abstrakt: Kluzná ložiska, která se již používají po tisíce let a provázejí naši civilizaci stejně jako kolo, mohou být zobrazena jako excentrické mezikruží, vyplněné tekutinou. V této jednoduché geometrii zkoumáme proudění ne-Newtonovské kapaliny.
V prvé části popisujeme geometrii, model tekutiny, teoretické výsledky a předchozí práce, které se k tomuto problému vztahují.
V druhé části dokážeme existenci řešení zobecněných Navier-Stokesových rovnic s viskozitou závislou na tlaku a na gradientu rychlosti, opatřených nehomogenní Dirichletovou okrajovou podmínkou. Ukážeme také další výsledky existence a jednoznačnosti.
V poslední části provedeme numerické simulace proudění v ložisku za použití metody konečných prvků implementovaných v numerickém softwaru `featflow`. Srovnáme výsledky klasických Navier-Stokesových rovnic s našimi zobecněnými a budeme diskutovat parametry modelu na několika dalších příkladech.
Klíčová slova: slabé řešení pro nelineární PDR, ne-Newtonovské tekutiny, závislost viskozity na gradientu rychlosti, závislost viskozity na tlaku, nehomogenní Dirichletova okrajová podmínka, numerické simulace, metoda konečných prvků, kluzné ložisko

Title: Numerical simulations of the flow in the journal bearing
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Abstract: Journal bearings that have been used for thousands of years and that go along with our civilization as well as the wheel, could be imagined as two eccentric cylinders, separated by fluid. Within this simple geometry we investigate the flow of non-Newtonian fluid.
In the first part, we describe the geometry, the fluid model, related theoretical results and previous investigations.
In the second part, we establish the existence of solution of the generalized Navier-Stokes equations with both the pressure- and the shear- dependent viscosity, completed with the non-homogeneous Dirichlet condition. We also present other existence and uniqueness results.
In the third part, we provide numerical simulations of the flow within the journal bearing using the finite element software package `featflow`. We compare the classical Navier-Stokes model and the generalized one and provide several example simulations discussing the parameters of the model.
Keywords: weak solution for nonlinear PDEs, non-Newtonian fluids, shear dependent viscosity, pressure dependent viscosity, non-homogeneous Dirichlet boundary condition, numerical simulation, finite elements method, journal bearing

1 Introduction

Lubrication generally, and the journal bearings as well, have been helping mankind for thousands of years. Basic laws of friction were first correctly deduced by da Vinci (1519), who was interested in the music made by the friction of the heavenly spheres. The scientific study of lubrication began with Rayleigh, who, together with Stokes, discussed the feasibility of a theoretical treatment of film lubrication.

The journal bearings are heavily used in these days, and they are designed and studied on the mathematical basis and by numerical computations for a long time. Even by browsing the Internet you can find web sites where simple computational simulations are provided by an automatic software for free. (Mostly based on the Reynolds approximation.)

This thesis does not aspire to present any kind of directly applicable numerical result or method at all. The intentions of this work are rather to follow one of the lines of today's investigation; to study mathematically one of the recent generalizations of the Navier-Stokes model of fluid motion and present it in the context of journal bearing lubrication problem.

The considered generalized Navier-Stokes model, as it is in more details described in sections 2.3 and 3.3, is based on the assumption that the viscosity depends both on the pressure and the shear rate. We note that theoretical results concerning the existence of solutions for such a class of fluids are rare. This work mostly follows the results by Franta, Málek, Rajagopal [1], where the existence for the homogeneous Dirichlet condition is established. Herein, we generalize this statement for the non-homogeneous Dirichlet condition in two dimensions. We do so without any "smallness" restriction, just incorporating another result from Kaplický, Málek, Stará [2] applied to models with shear-dependent viscosities under the assumption that there is no inflow and outflow through the boundary.

In the second part, in section 4, several numerical simulations are provided for the fluid model that meets the condition assumed in the theoretical part. We use the software package `featflow` initially developed as a solver for Navier-Stokes equations and modified in order to solve the flow of non-Newtonian fluids. We show both the pressure-thickening and the shear-thinning capability of the chosen viscosity form and we compare the obtained results with those for the classical Navier-Stokes model.

2 Description of the investigated problem

Friction, Lubrication†

If two solid bodies, in direct or indirect surface contact, are made to slide relative to one another, there is always a resistance to the motion called friction. Friction can be beneficial in many instances, however, in other cases it is energy consuming and we endeavor to decrease it, although it may be never eliminated entirely.

Friction is present in all machinery, and it converts part of the useful kinetic energy to heat, thus decreasing the overall efficiency of the machine. About 30% of the power in an automobile is wasted through friction. In 1951, G. Vogelpohl estimated that one-third to one-half of the world's energy production is consumed by friction ([12]). Friction could be represented by the coefficient of friction $f = \frac{F}{W}$, F being the resisting force (parallel to direction of motion) and W being the applied load (the force perpendicular to surfaces).

Lubrication is used to reduce/prevent wear and lower friction. The behavior of sliding surfaces is strongly modified with the introduction of a lubricant between them. When the minimum film thickness exceeds, say, 2.5μm, the coefficient of friction $f = \frac{F}{W}$ is small, (on the contrary to the case of lower film thickness), and depends on no other material property of the lubricant than its viscosity. (For a lightly loaded journal bearing the Petroff's law $f \sim \mu N/P$ is approximately obeyed, N being the shaft speed, $P = W/LD$ the specific load, L is the length of journal/bearing and D is the diameter of the journal, see [12].) This type of lubrication is called thick-film lubrication and it is in many respects the simplest and most desirable kind of lubrication to have.

Fluid film bearing†

Bearings are machine elements whose function is to promote smooth relative motion at low friction between two solid surfaces. The lubricant film separating surfaces can be liquid, gaseous or solid.

When there is a continuous fluid film separating the solid surfaces we speak of fluid film bearings. There are two principal ways of creating and maintaining a load-carrying film between solid surfaces in relative motion. We call a bearing *self-acting*, and say that it operates in the *hydrodynamic mode* of lubrication, when the film is generated and maintained by the viscous drag of the surfaces themselves, as they are sliding relative to one another. The film could be also created and maintained by an external pump that forces the lubricant between solid surfaces, then we call the bearing *externally pressurized*, operating in the *hydrostatic mode*; but we are not going to study this case here.

The oil required for hydrodynamic lubrication can be fed from an oil reservoir under gravity, it may be supplied from a sump by rings, discs, or wicks. The bearing might be even made of a porous metal impregnated with oil, which "bleeds" oil to the bearing surface as the journal rotates.

Hydrodynamic bearings vary enormously both in their size and in the load they support. At the low end of the specific-load scale we find bearings used by the jeweler, and at the high end we find the journal bearings of a large turbine generator set, which might be 0.8m in diameter and carry a specific load of 3MPa, or the journal bearings of a rolling mill, for which a specific load of 30MPa is not uncommon.

2.1 Geometry

Journal bearing†

If the motion which the bearing must accommodate is rotational and the load vector is perpendicular to the axis of rotation, the hydrodynamic bearing employed is *journal bearing*. In their simplest form, a journal and its bearing consist of two eccentric, rigid,

† Many of what is written in these paragraphs can be found in the *Fluid film lubrication* book by A. Z. Szeri, [12].

cylinders. The outer cylinder (bearing) is usually held stationary while the inner cylinder (journal) is made to rotate at an angular velocity ω.

2.1.1 Restriction to two dimensions

If the bearing is "infinitely" long, there is no pressure relief in the axial direction. Axial flow is therefore absent and changes in shear flow must be balanced by changes in circumferential pressure flow alone. This condition will also apply in first approximation to finite bearings, leading to the so-called *long-bearing* theory (see the Reynolds Equation, see e. g. [9] or [12]) if the length/diameter ratio $L/D > 2$. We remark that the aspect ratio of industrial bearings is customarily in the range $0.25 < L/D < 1.5$; neither the short-bearing (see [12]) nor the long-bearing approximation apply to these bearings. Yet, in this work, we follow this assumption, which allows us to restrict our further considerations to two-dimensional plane perpendicular to the axial direction. We do so for several reasons:

- The CPU time required for simulations in three dimensions would not allowed us to perform so many numerical experiments.
- We have in our disposal the finite element method software package **featflow** (visit `www.featflow.de`), developed as an efficient multigrid solver for the incompressible Navier-Stokes problem. It includes also the modification for solving two-dimensional equations with viscosity depending on the symmetric part of the velocity gradient $\boldsymbol{D}(\boldsymbol{v})$ and on the pressure.
- In two dimensions, we will show the existence of a solution to the generalized Navier-Stokes equations with both the pressure- and the shear- dependent viscosity, without any "small data" restriction, assuming that only the tangential velocity is prescribed on the boundary and the velocity in normal direction is held to be zero.

We thus consider the geometry as it can be seen in figure 1. The domain of the flow is an eccentric annular ring, the outer circle with the radius R_B, the inner circle radius being R_J, the distance between their centres is denoted by e. The inner circle rotates around its centre with (clock-wise) rotational speed ω, or we can say, with tangential velocity v_0.

It is customary to define the radial clearance $C = R_B - R_J$. As the possible values of e are in the range $e \in \langle 0, C\rangle$ we denote $\varepsilon = e/C$, $\varepsilon \in \langle 0, 1\rangle$ the eccentricity ratio. Hereafter, we shall say "eccentricity" talking about ε. We can clearly set $R_B = 1$ such that the geometry of our problem is described by two characteristic numbers ε and R_J.

2.2 Basic equations

2.2.1 Steady-state problem

In practice, the journal is not fixed at all but flows in the lubricant, driven by the applied load on one hand, and by the forces caused by the lubricant on the other hand. Therefore, in the time dependent case the geometry would not be fixed, the journal axis would observe some non-trivial trajectory in the neighbourhood of the bearing axis. The simulation would then look somehow as follows: we could set all fluid parameters, the radii of both the bearing and the journal cylinders, prescribe the velocity of rotation and the load applied on the journal (the load could also be changing with time) and then we could study the trajectory of journal axis in time. Such an approach could be seen e. g. in [10] with many important outcomes concerning the operational regime. One of these observations is that in some cases the motion of journal axis can cease and can become stable in some "equilibrium" position. The position of course depends on the applied load.

In the steady-case approach, which we will present in this work, the position of journal is prescribed and from the solution of lubricant motion we compute the force applied to

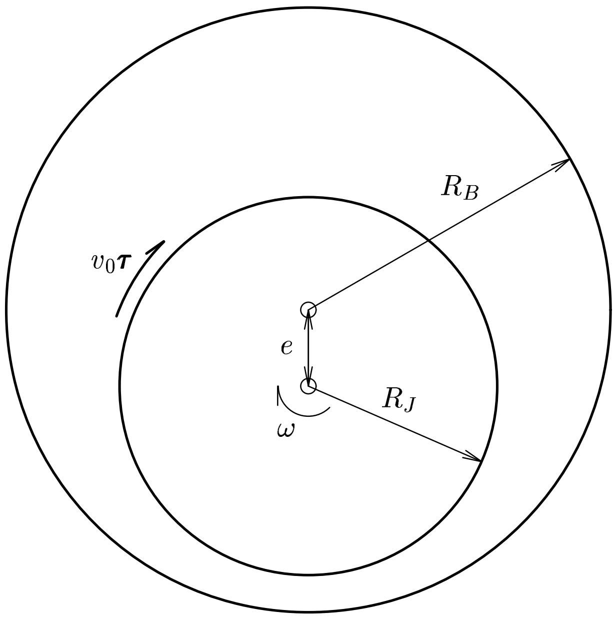


Figure 1: Simplified geometry of the journal bearing.

the journal by the fluid. By this procedure we obtain the reaction force depending on the eccentricity of cylinders, without performing the complex and more time consuming time-dependent simulations. Thus we can effectively study the influence of both geometrical and fluid parameters on the resulting operational regime. The disadvantage of this approach is that knowing the position of the journal and the corresponding reaction force, we still do not know anything about the stability of such a configuration. In other words, we do not know whether such a case could happen in reality or not. Anyway, this questions are out of the scope of this work.

2.2.2 Notation

Hereafter, we use the following notation in the text:

Ω ... bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a boundary $\partial\Omega$;

$\boldsymbol{x}$ ... spatial coordinates in $\mathbb{R}^d$, $\boldsymbol{x} = (x_1, \dots, x_d)$;

$\boldsymbol{v}$ ... velocity field, $\boldsymbol{v} = (v_1, \dots, v_d)$;

p ... pressure;

$\boldsymbol{T}$ ... Cauchy stress tensor;

ρ ... density of the fluid, here ρ is a positive constant;

$\boldsymbol{b}$ ... specific body force (force acting on a mass unit).

Since we deal with time-independent problem all quantities as $\boldsymbol{v}$, p, $\boldsymbol{T}$ and $\boldsymbol{b}$ are functions of the actual position $\boldsymbol{x}$.

We denote the gradient of some vector field, say, $\boldsymbol{\zeta} \in \mathbb{R}^d$ by $\nabla\boldsymbol{\zeta}$, i. e.

$$(\nabla\boldsymbol{\zeta})_{ij} = \frac{\partial \zeta_i}{\partial x_j}, \quad i, j = 1, \dots, d.$$

The symmetric part of the gradient is defined through

$$\boldsymbol{D}(\boldsymbol{\zeta}) = \frac{1}{2}\left(\nabla\boldsymbol{\zeta} + (\nabla\boldsymbol{\zeta})^T\right),$$

where $(\nabla\boldsymbol{\zeta})^T$ means the transposed matrix to $\nabla\boldsymbol{\zeta}$. For $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ the symbol $|\boldsymbol{A}|$ is used to define the euclidean norm of $\boldsymbol{A}$, i. e.

$$|\boldsymbol{A}|^2 \equiv \sum_{i,j=1}^{d} |A_{ij}|^2.$$

2.2.3 Constitutive equations

Hereafter, we consider a motion of a homogeneous incompressible fluid in a bounded domain Ω in $\mathbb{R}^2$ with boundary $\partial\Omega$. We do not consider any cavitation in the model, treating only full film of lubricant. The circumstances and effects of cavitation can be found e. g. in [10]. The motion is described by the equations expressing the balance of mass (recall that ρ is a constant)

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2.1}$$

and the balance of momentum

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho \sum_{i=1}^{2} v_i \frac{\partial \boldsymbol{v}}{\partial x_i} = \operatorname{div} \boldsymbol{T} + \rho \boldsymbol{b} \quad \text{in } \Omega.$$

As we have decided to study the steady-state problem, the balance of momentum takes the form

$$\rho \sum_{i=1}^{2} v_i \frac{\partial \boldsymbol{v}}{\partial x_i} = \operatorname{div} \boldsymbol{T} + \rho \boldsymbol{b} \quad \text{in } \Omega. \tag{2.2}$$

We can see that as soon as we would consider low velocities of the motion, the other terms would dominate to the convective term $v_i \frac{\partial \boldsymbol{v}}{\partial x_i}$. Together with the fact that the nonlinear convective term makes the analysis more difficult, this motivates us to simplify the problem and study the system

$$\boldsymbol{0} = \operatorname{div} \boldsymbol{T} + \rho \boldsymbol{b} \quad \text{in } \Omega, \tag{2.3}$$

where the convective term is neglected.

As we have established these equations for the steady flow of an incompressible fluid, the crucial step is to set the model for the Cauchy stress tensor $\boldsymbol{T}$ and then to complete the system with boundary conditions.

2.3 Fluids with shear- and pressure- dependent viscosity

A fluid is called Newtonian if the dependence of the stress tensor on the spatial variation of velocity is linear. This model was introduced by Stokes in 1844* (see [8]), and already Stokes remarked that the model may be applicable to fluid flows at normal conditions. For instance, while the dependence of the viscosity on the pressure does not show up in certain common flows, it can have a significant effect when the pressure becomes very high.

As the lubricant in journal bearing is forced through a very narrow region, of order of micrometers, the pressure becomes sometimes so high that the fluid obtains a "glassy" state. Moreover, since the shear-rate becomes also high, the viscosity of lubricant does not suffice to be considered constant with respect to the shear-rate.

Another generalization of the Navier-Stokes fluid goes by the name Stokesian fluid. (In fact, Stokes derived a more general model and after that made simplification to obtain the popular Navier-Stokes model.) In such a fluid the material moduli can depend on the

* model was earlier introduced also by Navier and Poisson

symmetric part of the velocity gradient through its principal invariants $\mathsf{I}_{\boldsymbol{D}}$, $\mathsf{II}_{\boldsymbol{D}}$, and $\mathsf{III}_{\boldsymbol{D}}$, defined as

$$\mathsf{I}_{\boldsymbol{D}} = \operatorname{tr} \boldsymbol{D}, \quad \mathsf{II}_{\boldsymbol{D}} = \frac{1}{2}[(\operatorname{tr} \boldsymbol{D})^2 - \operatorname{tr} \boldsymbol{D}^2] = -\frac{1}{2} \operatorname{tr} \boldsymbol{D}^2, \quad \text{and } \mathsf{III}_{\boldsymbol{D}} = \det \boldsymbol{D}.$$

This model can describe both shear-thinning and shear-thickening fluids, it is customary to use the shear-thinning fluids in the context of journal bearings.

Incorporating the pressure- and the shear- dependence of the viscosity into the lubricant model could have a significant impact on the dynamics, and hence on the load bearing capacity, of a journal bearing. One of the cases can be seen e. g. in [10] where is, among others, demonstrated the stabilization effect of the piezoviscous lubricant on the journal motion in a contrast to the constant-viscosity case.

In this work we consider a model that takes into account both types of generalization discussed above, i. e. the material moduli depend on the symmetric part of the velocity gradient as well as the pressure. Since we talk about incompressible fluids only, we require the constraint

$$\operatorname{tr} \boldsymbol{D} = \operatorname{div} \boldsymbol{v} = 0$$

to be met in all motions of the fluid. In accordance with the representation theorem, the Cauchy stress $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \alpha_1(p, \mathsf{II}_{\boldsymbol{D}}, \mathsf{III}_{\boldsymbol{D}})\boldsymbol{D} + \alpha_2(p, \mathsf{II}_{\boldsymbol{D}}, \mathsf{III}_{\boldsymbol{D}})\boldsymbol{D}^2, \tag{2.4}$$

where $-p\boldsymbol{I}$ is the indeterminate part of the stress due to the constraint of incompressibility. We assume that the constraint response ensures that the incompressibility is met, therefore the material moduli depend also on the Lagrange multiplier, i. e. α_1 and α_2 depend upon p.

Note that due to $\operatorname{tr} \boldsymbol{D} = 0$ there holds $p = -\frac{1}{3} \operatorname{tr} \boldsymbol{T}$ and p has thus the meaning of mean normal stress.

Since there is no experimental work for fluids that would support the presence of the term $\alpha_2(p, \mathsf{II}_{\boldsymbol{D}}, \mathsf{III}_{\boldsymbol{D}})\boldsymbol{D}^2$, we restrict ourselves to a subclass of models of (2.4), namely

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu(p, |\boldsymbol{D}|^2)\boldsymbol{D}, \tag{2.5}$$

where

$$|\boldsymbol{D}|^2 = \operatorname{tr} \boldsymbol{D}^2 = -2\mathsf{II}_{\boldsymbol{D}}.$$

2.3.1 Viscosity models in practise

The dependence of the viscosity on the pressure has been studied for quite a long time. For instance in the magisterial treatise of Bridgman (1931) ([13]) there is a discussion of the studies up to 1931. Andrade suggested (on the basis of experiments), see [13], the dependence of the viscosity μ on the density ρ, the pressure p and the temperature ϑ, of the form

$$\mu(\rho, \vartheta, p) = A\rho^{\frac{1}{2}} \exp\left((p + \rho^2 r)\frac{s}{\vartheta}\right),$$

A, r and s being constants. This approximation however works well only for a certain temperature range and it is not clear that it works for all liquids (see [1]). Passing over the dependence on the density ρ, as the variation in the densities is indeed not very large, we can come to the form

$$\mu = B \exp\left(\frac{Cp}{\vartheta}\right),$$

where B and C are constants. The popular model used in lubrication theory is the Vogel's formula

$$\mu = \mu_0 \exp\left(\frac{a}{b+\vartheta}\right),$$

a, b are constants.

The dependence of the viscosity on the pressure is almost at all events considered to be exponential, simple form

$$\mu = \exp(\gamma p)$$

is also often used. In quite a recent work of Gwynllyw, Davies and Phillips (1996) [10] on the dynamics of a journal bearing with the piezoviscous lubricant there is considered the model

$$\mu = \left(\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (K\sqrt{2\,\mathrm{tr}\,\boldsymbol{D}^2})^m}\right) \times \exp(\alpha p),$$

where K is a function of the pressure

$$K = K(p) = \exp(\bar{\alpha} p + E),$$

μ_0, μ_∞, m, α, $\bar{\alpha}$ and E are material parameters estimated by best-fitting the experimental data. (The parameters are said to be taken from [11] and [14].)

As a representative of models where the viscosity depends only on the shear-rate we cannot forget the power-law model

$$\mu = \mu_0 |\boldsymbol{D}|^{p-2}, \quad p \in (1,2).$$

In this work, we are going to take into account models described above keeping the form (2.5). Nevertheless, we will introduce some different viscosity formulas, in order to be able to show the existence of the solution to our system of equations, which is the main aim of this work. More details concerning the specific forms of μ are provided in section 3.

2.4 Boundary conditions

2.4.1 Dirichlet boundary condition

Having the fluid motion equations (2.1) and (2.2) and the specification of the stress tensor (2.5), we need to complete the system of governing equations by the suitable set of boundary conditions.

As we have proposed in section 2.1, we consider a flow in a two-dimensional domain that can be viewed as an eccentric annular ring. Each circle then means a fixed wall, the outer wall being fixed meanwhile the inner one rotates around its own axis. On both walls we set the no-slip condition such that the resulting Dirichlet boundary condition is prescribed:

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{0} \quad \text{on } \Gamma_O \subset \partial\Omega \text{ (the outer circle)}, \\ \boldsymbol{v} &= v_0 \boldsymbol{\tau} \quad \text{on } \Gamma_I \subset \partial\Omega \text{ (the inner circle)}, \end{aligned} \tag{2.6}$$

where v_0 is given and $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{x})$ is the (clock-wise) unit tangential vector to the inner circle Γ_I.

We notice that there is no inflow or outflow, i. e. the normal part of velocity $\boldsymbol{n}.\boldsymbol{v}$ is equal to zero everywhere on the boundary $\partial\Omega$. We will strongly use this fact when proving the existence of solution to the Navier-Stokes-like problem, referring to the result by Kaplický, Málek, Stará [2]. This will allow us to establish the existence without any restrictions on the greatness of v_0.

However, since we present also other theoretical results such as existence of solutions to the Stokes-like problem or the uniqueness of solution, in the section 3 we consider Dirichlet boundary condition of the more general form

$$\boldsymbol{v} = \boldsymbol{\varphi} \quad \text{on } \partial\Omega,$$

where $\boldsymbol{\varphi}$ will be specified.

2.4.2 Mean value of the pressure

There is a quite important difference between analysis of the equations governing the flow of incompressible fluid with constant viscosity or with the viscosity depending only on the $\boldsymbol{D}(\boldsymbol{v})$ on the one hand, and analysis of the equations with the viscosity depending also on the pressure, on the other hand.

In the first case the solution is never unique considering the values of pressure, since the pressure can be somehow 'shifted' by an arbitrary constant. Even if the boundary conditions include the pressure values, these can be changed by some constant and the nature of the solution (namely the velocity field) will be exactly the same. It is a direct consequence of the fact that there is only ∇p in the equations. However, nobody is confused prescribing this constant in order to obtain a physically suitable solution because there is no need to care about that. In order to compare the values of the pressure field with experimental data the pressure field can be arbitrarily increased or decreased after the computation, so the common manner is e. g. to fix the meanvalue to be a zero (by clear numerical reasons).

On the contrary, considering the viscosity depending on the pressure this approach changes totally. The pressure have to be somehow fixed even in the sense of a constant and giving different values, e. g. prescribing the meanvalue of the pressure in the whole domain or in some of its part, we can obtain significantly different solutions. To give an example, let us take the viscosity of the form $\nu = \nu_0 \exp(\beta p)$ in (2.5) and assume that we have a solution $(\boldsymbol{v}, p)$ to equations (2.1) and (2.2) (in fact, we have no theory about the existence for such a problem, but it is the simplest example) meeting the condition $\int_\Omega p \,\mathrm{d}x = 0$, and a solution $(\tilde{\boldsymbol{v}}, \tilde{p})$, of same equations but meeting $\int_\Omega \tilde{p} \,\mathrm{d}x = p_0 \neq 0$. Then writing the equations with $\tilde{p} - p_0$ we see that it fulfills the condition of meanvalue being zero as in the first case and, moreover, it meets the same equations as soon as we set $\nu_0 \cdot \exp(-p_0)$ instead of ν_0. In this simple case, to prescribe a different meanvalue of the pressure has the same effect on the velocity field as to change the constant ν_0 in the viscosity term.

We notice that, in a real journal bearing, there is often an inflow of the lubricant provided by some channel or groove. We do not reflect this inflow, since the flow is negligible, and moreover because such a detail would make our considerations quite more difficult. For example, we assume that bearing infinitely long and thus the flow two-dimensional where, in fact, as soon as there is the inflow, there must be also the outflow, by most provided by the ends of a bearing which are free. In such a case the pressure should be probably best prescribed being equal to some value at the inflow and being equal to zero (or, say, to the atmospherical pressure) at the ends of journal bearing. This is no more a long bearing approximation and it is no more two-dimensional conception.

On the other hand, there are some consequences of e. g. the position of the inflow channel, which should be important. Let us consider a small inflow channel in the outer wall somewhere close to the narrow gap between the eccentrical cylinders. It is easy to imagine (and it will be seen in numerical simulations below) that the pressure of the fluid after it has got through the narrow gap is significantly lower than it is upstream the gap. Maintaining some pressure level at the inflow channel we can obtain entirely different flow solutions in the case when the channel is located downstream, in comparison to the case when it is located upstream to the narrow gap.

However, in this work we prescribe the pressure level by setting the meanvalue over whole domain. This suffices to provide interesting numerical experiments and to show the role of dependence of the viscosity on the pressure. Moreover, we will avoid possible troubles concerning the proof of existence.

We thus complete our system of equations by the mathematically natural condition

$$\frac{1}{|\Omega|}\int_\Omega p\,\mathrm{d}x = p_0. \tag{2.7}$$

2.5 Governing equations of the investigated problem

In our theoretical considerations all functions will act on an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\partial\Omega$. As we have explained in the previous sections, we focus on a fluid whose Cauchy stress is of the form

$$\boldsymbol{T} = -p\boldsymbol{I} + \rho\nu\boldsymbol{D} \quad \text{with} \quad \nu = \nu\left(\frac{p}{\rho}, |\boldsymbol{D}|^2\right), \tag{2.8}$$

(we give $\rho\nu(\frac{p}{\rho}, |\boldsymbol{D}|^2)$ instead of $\mu(p, |\boldsymbol{D}|^2)$ in (2.5)). Hereafter, we shall write only p instead of $\frac{p}{\rho}$ since in our considerations ρ is a constant, but remember that originally the viscosity term depends on $\frac{p}{\rho}$ in fact.

The balances of mass (2.1) and momentum (2.2) give the equations

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2.9}$$

$$v_i\frac{\partial\boldsymbol{v}}{\partial x_i} + \nabla p - \operatorname{div}[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}] = \boldsymbol{b} \quad \text{in } \Omega, \tag{2.10}$$

while neglecting the convective term such as in (2.3) we write

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega \tag{2.11}$$

$$\nabla p - \operatorname{div}[\nu(p, |\boldsymbol{D}|^2)\boldsymbol{D}] = \boldsymbol{b} \quad \text{in } \Omega. \tag{2.12}$$

We complete the equations by the non-homogeneous Dirichlet boundary condition

$$\boldsymbol{v} = \boldsymbol{\varphi} \quad \text{on } \partial\Omega \tag{2.13}$$

and finally, we shall suppose that the pressure p meets

$$\frac{1}{|\Omega|}\int_\Omega p\,\mathrm{d}x = p_0, \tag{2.14}$$

where $p_0 \in \mathbb{R}$ is given and $|\Omega|$ denotes the d-dimensional Lebesgue measure of Ω. As I will discuss later, we can choose $p_0 = 0$ without any restriction.

We shall denote the system of equations (2.9)-(2.10),(2.13)-(2.14) by (P) and the system (2.11)-(2.12),(2.13)-(2.14) by ($\mathrm{P_S}$). It is not surprising that problem ($\mathrm{P_S}$) is much easier to solve and the existence to ($\mathrm{P_S}$) is proved under more general conditions on the viscosity ν in comparison to the conditions needed for the existence proof to (P).

3 Theoretical results

In this section we present our existence and uniqueness results concerning the steady flow of fluid with both the pressure- and the shear- dependent viscosity, with the non-homogeneous Dirichlet boundary condition prescribed. The main result is included in Theorem 3.13 that establishes the existence of a weak solution to equations (P) under the assumption that there is no flow through the boundary (the normal component of velocity at the boundary is zero) meanwhile the tangential velocity is prescribed and could be arbitrary large (in the chosen functional space). Our result is a generalization of the result by Franta, Málek, Rajagopal [1] where the homogeneous Dirichlet condition problem was solved and the result by Kaplický, Málek, Stará [2] where the shear-dependent fluid model with nonzero tangential component of the velocity on the boundary is treated. We also present the existence theorem for the Stokes-like system ($\mathrm{P_S}$) where we do not need the condition on the normal-component of the velocity on the boundary nor the condition of two-dimensionality. Next, also the uniqueness of a weak solution is proved, in the case of (P) only for small data.

First of all, we introduce notations, definitions and present several useful lemmas.

3.1 More notation, preliminaries

We introduce a notation of function spaces. Let $X(\Omega)$ be a Banach space of scalar functions defined on Ω, equipped with the norm $\|\cdot\|_X$. By $(X(\Omega))^*$ we denote its dual space, while the brackets $\langle\cdot,\cdot\rangle$ mean the corresponding duality pairing. For vector functions spaces we use the notation $X(\Omega)^d := \{\boldsymbol{u} : \Omega \to \mathbb{R}^d; u_i \in X(\Omega), i = 1,\dots,d\}$ and similarly $X(\Omega)^{d\times d} := \{\boldsymbol{T} : \Omega \to \mathbb{R}^{d\times d}; T_{ij} \in X(\Omega), i,j = 1,\dots,d\}$.

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary $\partial\Omega$. Then $\mathcal{D}(\Omega)$ denotes the space of smooth $\mathcal{C}^\infty$-functions with a compact support in Ω and $\mathcal{D}^*(\Omega)$ denotes the space of distributions. We define $\frac{\partial f}{\partial x_i} \in \mathcal{D}^*(\Omega)$, the distributional derivative for $f \in \mathcal{D}^*(\Omega)$, by the identity

$$\langle \frac{\partial f}{\partial x_i}, \phi\rangle = -\langle f, \frac{\partial \phi}{\partial x_i}\rangle, \quad \forall \phi \in \mathcal{D}(\Omega).$$

We then define operators grad and div in the sense of distributions

- $u \in \mathcal{D}^*(\Omega)$

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}\right) \in \mathcal{D}^*(\Omega)^d$$

- $\boldsymbol{u} \in \mathcal{D}^*(\Omega)^d$

$$\operatorname{div} \boldsymbol{u} = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} \in \mathcal{D}^*(\Omega).$$

Let $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, be a multiindex, $|\alpha| = \sum_{i=1}^d \alpha_i$. We then define operator D^α in the sense of distributions

- $u \in \mathcal{D}^*(\Omega)$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d}, \quad i,j = 1,\dots,d.$$

For $r \in \langle 1, \infty)$, we set $\|f\|_r = \left(\int_\Omega |f(\boldsymbol{x})|^r \,\mathrm{d}x\right)^{\frac{1}{r}}$ and for $r = \infty$, $\|f\|_\infty = \operatorname{ess\,sup}_{\boldsymbol{x}\in\Omega} |u(\boldsymbol{x})|$. The Lebesgue spaces are defined as

$$\mathbf{L}^r(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ is measurable on } \Omega, \|f\|_r < \infty\},$$

and the Sobolev spaces are then defined as

$$\mathbf{W}^{k,r}(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ is measurable on } \Omega, \|f\|_{k,r} < \infty\},$$

for $r \geq 1$ and $k \in \mathbb{N}$, where we set $\|f\|_{k,r} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_r^r\right)^{\frac{1}{r}}$.

For $r \in \langle 1, \infty)$ there exists a bounded linear operator (trace) $\mathrm{Tr} : \mathbf{W}^{1,r}(\Omega) \to \mathbf{L}^r(\partial\Omega)$ such that

$$\mathrm{Tr}(u) = u|_{\partial\Omega} \quad \text{if } u \in \mathbf{W}^{1,r}(\Omega) \cap \mathcal{C}(\bar{\Omega}).$$

Hereafter, we rather write

$$F = f \quad \text{on } \partial\Omega$$

instead of $\mathrm{Tr}(F) = f$.

We introduce the zero-trace space $\mathbf{W}_0^{1,r}(\Omega) := \{u \in \mathbf{W}^{1,r}(\Omega);\ \mathrm{Tr}(u) = 0 \text{ at } \partial\Omega\}$, the space of divergence-free functions $\mathbf{W}_{\mathrm{div}}^{1,r}(\Omega)^d := \{\boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega)^d; \operatorname{div} \boldsymbol{u} = 0 \text{ a. e. in } \Omega\}$ and, finally, the dual space to $\mathbf{W}_0^{1,r}(\Omega)^d$, $(\mathbf{W}^{-1,r'}(\Omega)^d, \|\cdot\|_{-1,r'}) := (\mathbf{W}_0^{1,r}(\Omega)^d)^*$ where $r' = \frac{r}{r-1}$.

In what follows, we use sometimes the notation $(\boldsymbol{v}, p)$ for the ordered pair of the velocity- and the pressure- part of solution, another time we denotes (a, b) an open interval in $\mathbb{R}$, but most often $(\boldsymbol{f}, \boldsymbol{g})$ means

$$(\boldsymbol{f}, \boldsymbol{g}) := \int_\Omega \boldsymbol{f}(\boldsymbol{x}).\boldsymbol{g}(\boldsymbol{x})\, \mathrm{d}x,$$

providing that $\boldsymbol{f}.\boldsymbol{g} \in \mathbf{L}^1(\Omega)$. We believe that this polyvalence might not lead to any misunderstanding.

Next, we introduce some standard lemmas. See for example Evans [18] or Lions [17].

Lemma 3.1 (Gauss-Green Theorem) *Suppose $u \in \mathcal{C}^1(\bar{\Omega})$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i}\, \mathrm{d}x = \int_{\partial\Omega} u n_i\, \mathrm{d}S \quad (i = 1, \ldots, d),$$

where $\boldsymbol{n} = (n_1, \ldots, n_d)$ is the outer unit normal vector to $\partial\Omega$.

Lemma 3.2 (Hölder's inequality) *Let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p$ and $q < \infty$ or $p = 1$ and $q = \infty$. Then, for $u \in \mathbf{L}^p(\Omega)$ and $v \in \mathbf{L}^q(\Omega)$*

- *$uv \in \mathbf{L}^1(\Omega)$,*
- *$\|uv\|_1 \leq \|u\|_p \|v\|_q$.*

Lemma 3.3 (Korn's inequality) *Let $p \in (1, \infty)$, then there exists $k_p = k_p(\Omega)$ such that*

$$\|\boldsymbol{u}\|_{1,p} \leq k_p \|\boldsymbol{D}(\boldsymbol{u})\|_p, \quad \textit{for all } \boldsymbol{u} \in \mathbf{W}_0^{1,p}(\Omega)^d.$$

Lemma 3.4 (Vitali's theorem) *Let Ω be a bounded domain in $\mathbb{R}^d$ and $f^n : \Omega \to \mathbb{R}$ be integrable for every $n \in \mathbb{N}$. Assume that*

- *$\lim_{n\to\infty} f^n(\boldsymbol{x})$ exists and is finite for almost all $\boldsymbol{x} \in \Omega$,*
- *for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_n \int_Q |f^n(\boldsymbol{x})|\, \mathrm{d}x < \varepsilon \quad \forall Q \subset \Omega, |Q| < \delta. \tag{3.15}$$

Then

$$\lim_{n\to\infty} \int_\Omega f^n(\boldsymbol{x})\, \mathrm{d}x = \int_\Omega \lim_{n\to\infty} f^n(\boldsymbol{x})\, \mathrm{d}x.$$

Lemma 3.5 (Imbeddings) *Let $1 \le p < d$, then there holds an imbedding*

$$\mathbf{W}^{1,p}(\Omega) \hookrightarrow \mathbf{L}^q(\Omega) \quad \text{for all } 1 \le q \le \frac{dp}{d-p}$$

and a compact imbedding (Ω is a bounded set)

$$\mathbf{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathbf{L}^q(\Omega) \quad \text{for all } 1 \le q < \frac{dp}{d-p}.$$

Lemma 3.6 (Brouwer's Fixed Point Theorem) *Assume*

$$\mathcal{M} : B_1(\mathbf{0}) \to B_1(\mathbf{0})$$

is continuous, where $B_1(\mathbf{0})$ denotes the closed unit ball in $\mathbb{R}^n$. Then $\mathcal{M}$ has a fixed point; that is, there exists a point $\boldsymbol{c} \in B_1(\mathbf{0})$ such that $\mathcal{M}(\boldsymbol{c}) = \boldsymbol{c}$.

In section 3.7 we establish a small modification of this theorem.

3.2 Weak formulation, definition of the problem

Let $\boldsymbol{b}$ satisfy

$$\boldsymbol{b} \in \left(\mathbf{W}_0^{1,r}(\Omega)^d\right)^* \tag{3.16}$$

and the Dirichlet boundary condition (2.13) be given by

$$\boldsymbol{\varphi} = \mathrm{Tr}(\boldsymbol{\Phi}), \quad \boldsymbol{\Phi} \in \mathbf{W}^{1,r}(\Omega)^d, \quad \mathrm{div}\,\boldsymbol{\Phi} = 0 \text{ in } \Omega. \tag{3.17}$$

Then we use the folowing definitions:

Definition 3.7 (Weak solution of ($\mathbf{P}_\mathrm{S}$)) *A pair $(\boldsymbol{v}, p)$ is called the weak solution to the problem (P_S) if $(\boldsymbol{v}, p)$ fulfills*

$$\begin{aligned} &\boldsymbol{v} \in \mathbf{W}_{\mathrm{div}}^{1,r}(\Omega)^d, && \boldsymbol{v} = \boldsymbol{\varphi} \text{ on } \partial\Omega, \\ &p \in \mathbf{L}^{r'}(\Omega), \quad r' = \tfrac{r}{r-1}, && \textstyle\int_\Omega p \,\mathrm{d}x = 0 \end{aligned} \tag{3.18}$$

and

$$\big(\nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{\psi})\big) - (p, \mathrm{div}\,\boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \qquad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d.$$

Definition 3.8 (Weak solution of (P)) *A pair $(\boldsymbol{v}, p)$ is called the weak solution to the problem (P) if $(\boldsymbol{v}, p)$ fulfills*

$$\begin{aligned} &\boldsymbol{v} \in \mathbf{W}_{\mathrm{div}}^{1,r}(\Omega)^d, && \boldsymbol{v} = \boldsymbol{\varphi} \text{ on } \partial\Omega, \\ &p \in \mathbf{L}^{r'}(\Omega), \quad r' = \tfrac{r}{r-1}, && \textstyle\int_\Omega p \,\mathrm{d}x = 0 \end{aligned} \tag{3.19}$$

and

$$\Big(v_i \frac{\partial \boldsymbol{v}}{\partial x_i}, \boldsymbol{\psi}\Big) + \big(\nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{\psi})\big) - (p, \mathrm{div}\,\boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle, \qquad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d.$$

3.3 Structure of the viscosity

Following the results in [1], [3], [4], etc., we shall consider the viscosities meeting the following general conditions:

(1) For a given $r \in (1,2)$, there are positive constants C_1 and C_2 such that for all symmetric linear transformations $\boldsymbol{B}$, $\boldsymbol{D}$ and all $p \in \mathbb{R}$

$$C_1(1+|\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2 \le \frac{\partial[\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial \boldsymbol{D}} \cdot (\boldsymbol{B}\otimes\boldsymbol{B}) \le C_2(1+|\boldsymbol{D}|^2)^{\frac{r-2}{2}}|\boldsymbol{B}|^2,$$

where $(\boldsymbol{B}\otimes\boldsymbol{B})_{ijkl} = \boldsymbol{B}_{ij}\boldsymbol{B}_{kl}$.

(2) For all symmetric linear transformations $\boldsymbol{D}$ and for all $p \in \mathbb{R}$

$$\left|\frac{\partial[\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}]}{\partial p}\right| \le \gamma_0(1+|\boldsymbol{D}|^2)^{\frac{r-2}{4}} \le \gamma_0,$$

$$\text{with } \gamma_0 < \frac{1}{C_{\mathrm{div},2}}\frac{C_1}{C_1+C_2} < \frac{1}{C_{\mathrm{div},2}}$$

Now we just refer two useful lemmas, both presented for example in [1]:

Lemma 3.9 *Let (1) and (2) hold. For arbitrary $\boldsymbol{D}^1$, $\boldsymbol{D}^2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and p^1, $p^2 \in \mathbb{R}$ we set*

$$I^{1,2} := \int_0^1 (1+|\boldsymbol{D}^2 + s(\boldsymbol{D}^1-\boldsymbol{D}^2)|^2)^{\frac{r-2}{2}}|\boldsymbol{D}^1-\boldsymbol{D}^2|^2\,\mathrm{d}s.$$

Then

$$\frac{C_1}{2}I^{1,2} \le [S(p^1,\boldsymbol{D}^1) - S(p^2,\boldsymbol{D}^2)] : (\boldsymbol{D}^1-\boldsymbol{D}^2) + \frac{\gamma_0^2}{2C_1}|p^1-p^2|^2.$$

Lemma 3.10 *Let (1) holds for $r \in (1,2)$. Then for all $p \in \mathbb{R}$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$*

$$\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}\cdot\boldsymbol{D} \ge \frac{C_1}{2r}(|\boldsymbol{D}|^r - 1) \tag{3.20}$$

and

$$|\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}| \le \frac{C_2}{1-(2-r)\lambda}(1+|\boldsymbol{D}|)^{1-(2-r)\lambda} \tag{3.21}$$

$$\textit{for all } \lambda:\ 0\le\lambda\le 1.$$

In this paper (3.21) is used only with $\lambda = 1$, i. e.

$$|\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}| \le \frac{C_2}{r-1}(1+|\boldsymbol{D}|)^{r-1}. \tag{3.22}$$

3.4 Survey of known results

Although the fluid models with the pressure- and/or the shear- dependent viscosities are studied and used at least from the first third of the last century, mathematical results concerning the existence of solutions are rare. To our knowledge (see e. g. [1]) there is no global-in-time existence theory available for the case that the viscosity depends only on the pressure. In recent studies by Renardy (1986), Gazzola (1997) and Gazzola & Secchi (1998) (see [19], [20] and [21]) either the kinematical viscosity satisfies

$$\frac{\nu(p)}{p} \to 0 \quad \text{as} \quad p\to\infty,$$

a condition contradicting by experiments, or authors established only local-in-time existence of smooth solutions for small data on very restrictive "smallness" conditions both on $\boldsymbol{b}$ and the initial data.

Recently, the global-in-time existence of solutions for a class of fluids with the viscosity depending not only on the pressure but also on the shear rate was established – see Málek et al. (2002) [15] and [16], and Hron et al. (2002) [4]. These results have been established under a quite artificial assumption that the flow is spatially periodic.

The existence of solutions for the steady flows of fluids with the pressure- and the shear- dependent viscosities, meeting the assumptions **(1)** and **(2)** stated in section 3.3, for homogeneous Dirichlet condition is presented in Franta, Málek, Rajagopal [1]. Here, dealing with the two-dimensional model, we generalize this result to the non-homogeneous Dirichlet condition, provided that only a tangential component of the velocity is nonzero on the boundary, i. e. under the condition that

$$\boldsymbol{v}.\boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{3.23}$$

($\boldsymbol{n}$ means a normal vector to $\partial\Omega$).

3.5 Existence of solutions

The main result of this work is the proof of existence and uniqueness of weak solution to the problem (P), i. e. to the equations (2.9)-(2.10) governing the flow of fluid with both the pressure- and the shear- dependent viscosity (meeting **(1)** and **(2)**). The system is completed by the non-homogeneous Dirichlet boundary condition (2.13) and by the condition concerning the pressure level

$$\frac{1}{|\Omega|}\int_\Omega p\,\mathrm{d}x = p_0.$$

It is easy to see that as asoon as we prove the existence of solution to the case $p_0 = 0$, we can accept this result for arbitrary $p_0 \in \mathbb{R}$ at once. We just need to see, that there is no constraint on the value of the pressure in conditions **(1)** and **(2)** but there is only constraint on the derivative of the viscosity with respect to the pressure. Seeking for the solution with the non-zero pressure meanvalue we can just write $p - p_0$ everywhere and consider $\tilde{\nu}(p, |\boldsymbol{D}(\boldsymbol{v})|^2) = \nu(p - p_0, |\boldsymbol{D}(\boldsymbol{v})|^2)$, which fulfills the conditions **(1)** and **(2)** in the same way as $\nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)$.

In this section, we first prove the existence of weak solution to the system ($\mathrm{P_S}$), where the convective term is neglected. The reason is to show more clearly the technique used to cope with the non-homogeneous boundary condition in a context of the chosen form of stress tensor and, additionally, to establish the existence theorem under more general conditions than we will obtain for the problem (P).

As the next step, an important lemma introduced in Kaplický, Málek, Stará [2] is stated and the existence to the Navier-Stokes-like system (P) is proved in two dimensions, provided that (3.23) holds but without any "smallness" restriction concerning the tangential velocity prescribed on $\partial\Omega$. Finally, the uniqueness of solutions to both (P) and ($\mathrm{P_S}$) is proved.

In order to prove the existence of a solution to (P) or ($\mathrm{P_S}$) we use the approximate systems of equations in Ω

$$-\varepsilon\Delta p^\varepsilon + \varepsilon p^\varepsilon + \operatorname{div}\boldsymbol{v}^\varepsilon = 0 \tag{3.24}$$

$$v_i^\varepsilon \frac{\partial \boldsymbol{v}^\varepsilon}{\partial x_i} + \nabla p^\varepsilon - \operatorname{div}(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)) = \boldsymbol{b} \tag{3.25}$$

or

$$-\varepsilon\Delta p^\varepsilon + \varepsilon p^\varepsilon + \operatorname{div}\boldsymbol{v}^\varepsilon = 0 \tag{3.26}$$

$$\nabla p^\varepsilon - \operatorname{div}(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)) = \boldsymbol{b} \tag{3.27}$$

subjected to the boundary conditions

$$\frac{\partial p^\varepsilon}{\partial \boldsymbol{n}} = \mathbf{0} \quad \text{and} \quad \boldsymbol{v}^\varepsilon = \boldsymbol{\varphi} \quad \text{on } \partial\Omega. \tag{3.28}$$

From (3.24) or (3.26), (3.28) and Gauss theorem it follows that

$$\frac{1}{|\Omega|}\int_\Omega p^\varepsilon \, \mathrm{d}x = 0.$$

We shall denote the system of equations (3.24),(3.25) and (3.28) by (P^ε) and (3.26),(3.27) and (3.28) by ($\mathrm{P}^\varepsilon_\mathrm{S}$).

3.5.1 Existence of solutions for the generalized Stokes system

Theorem 3.11 (Existence of solutions for the system ($\mathbf{P}_\mathrm{S}$)) *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set with the Lipschitz boundary $\partial\Omega$, $d = 2$ or 3. Let the assumptions **(1)** and **(2)** be satisfied with r fulfilling*

$$\frac{2d}{d+2} < r < 2 \tag{3.29}$$

and let (3.16) *and* (3.17) *hold.*

Then there is at least one weak solution $(\boldsymbol{v}, p)$ to the problem (P_S) in the sence of Definition 3.7.

Proof. The structure of the proof is following: we recall the problem ($\mathrm{P}^\varepsilon_\mathrm{S}$) and assume that it has a solution. We derive the energy estimates and estimates for the pressure p^ε uniform with respect to ε. Then for some sequence $\varepsilon_n \to 0$ we find weakly converging subsequence $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$ to the limit $(\boldsymbol{v}, p)$ in the spaces stated in (3.18) and, in addition to that, we show the strong convergence of $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$. Finally, we prove the existence of weak solutions to the approximate problem and thus vindicate our assumption.

Weak solution of ($\mathbf{P}^\varepsilon_\mathrm{S}$)

We suppose that for r fulfilling (3.29) and all $\varepsilon > 0$ there is a weak solution $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ of the problem ($\mathrm{P}^\varepsilon_\mathrm{S}$) such that

$$\boldsymbol{v}^\varepsilon - \boldsymbol{\Phi} \in \mathbf{W}^{1,r}_0(\Omega)^d \quad \text{and} \quad p^\varepsilon \in \mathbf{W}^{1,2}(\Omega) \tag{3.30}$$

satisfying

$$\varepsilon(\nabla p^\varepsilon, \nabla\xi) + \varepsilon(p^\varepsilon, \xi) + (\operatorname{div}\boldsymbol{v}^\varepsilon, \xi) = 0 \quad \text{for all } \xi \in \mathbf{W}^{1,2}(\Omega) \tag{3.31}$$

and

$$(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\psi})) - (p^\varepsilon, \operatorname{div}\boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \quad \text{for all } \boldsymbol{\psi} \in \mathbf{W}^{1,r}_0(\Omega)^d. \tag{3.32}$$

Note that all integrals in our weak formulation are finite: From Hölder inequality we see it for (3.31) as soon as $r > \frac{2d}{d+2}$, since $\operatorname{div}\boldsymbol{v}^\varepsilon \in \mathbf{L}^r(\Omega)$ and $\xi \in \mathbf{W}^{1,2}(\Omega) \hookrightarrow \mathbf{L}^{r'}(\Omega)$. (We have made things easier by assuming $r > \frac{2d}{d+2}$: without that assumption we should need $\xi \in \mathbf{W}^{1,2}(\Omega) \cap \mathbf{L}^{r'}(\Omega)$ in (3.31) and we could come to problems when we try to set $\xi := p^\varepsilon$ where we would need $p^\varepsilon \in \mathbf{L}^{r'}(\Omega)$.) The viscous term is finite from (3.22).

The existence of solution $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ fulfilling (3.30)-(3.32) for $\varepsilon > 0$ fixed will be proved in the end of this section.

Energy estimates and their direct consequences

We shall define $\boldsymbol{u}^\varepsilon$ by

$$\boldsymbol{v}^\varepsilon = \boldsymbol{\Phi} + \boldsymbol{u}^\varepsilon,$$

where according to the boundary condition and (3.17) clearly

$$\boldsymbol{u}^\varepsilon \in \mathbf{W}^{1,r}_{0,\mathrm{div}}(\Omega)^d.$$

Let us set $\xi := p^\varepsilon$ in (3.31) and $\boldsymbol{\psi} := \boldsymbol{u}^\varepsilon$ in (3.32), then it follows

$$\begin{aligned} \varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2 + (\operatorname{div}\boldsymbol{v}^\varepsilon, p^\varepsilon) \;&=\; 0 \\ (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^\varepsilon)) - (p^\varepsilon, \operatorname{div}\boldsymbol{u}^\varepsilon) \;&=\; \langle \boldsymbol{b}, \boldsymbol{u}^\varepsilon\rangle. \end{aligned}$$

Note that such a ξ is a possible test function due to (3.30). Summing these equations and using the assumption $\operatorname{div}\boldsymbol{\Phi} = 0$ (3.17) we find

$$\varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2 + (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^\varepsilon)) = \langle \boldsymbol{b}, \boldsymbol{u}^\varepsilon\rangle. \tag{3.33}$$

Since

$$\begin{aligned} (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^\varepsilon)) \;&=\; (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{v}^\varepsilon)) - \\ &-\; (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\Phi})), \end{aligned}$$

we obtain (applying (3.20) and (3.22)):

$$\begin{aligned} (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{v}^\varepsilon)) &\overset{(3.20)}{\geq} \frac{C_1}{2r}\int_\Omega (|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^r - 1)\,\mathrm{d}x = \\ &= \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - \frac{C_1}{2r}|\Omega|, \end{aligned}$$

and

$$\begin{aligned} (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\Phi})) &\overset{(3.22)}{\leq} \frac{C_2}{r-1}\int_\Omega (1 + |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|)^{r-1}|\boldsymbol{D}(\boldsymbol{\Phi})|\,\mathrm{d}x \leq \\ &\overset{\text{Hölder}}{\leq} \frac{C_2}{r-1}\|1 + \boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{r-1}\|\boldsymbol{D}(\boldsymbol{\Phi})\|_r \leq \\ &\leq \frac{C_2}{r-1}\|\boldsymbol{D}(\boldsymbol{\Phi})\|_r(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1}, \end{aligned}$$

i. e.

$$(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^\varepsilon)) \geq \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - \frac{C_1}{2r}|\Omega| - \frac{C_2}{r-1}\|\boldsymbol{D}(\boldsymbol{\Phi})\|_r(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1}. \tag{3.34}$$

Using then the inequality $|\langle \boldsymbol{b}, \boldsymbol{u}^\varepsilon\rangle| \leq \|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{u}^\varepsilon\|_{1,r} \leq c_1\|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{D}(\boldsymbol{u}^\varepsilon)\|_r$, due to the Korn's inequality, we conclude from (3.34) and (3.33):

$$\begin{aligned} &\varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2 + \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - \frac{C_1}{2r}|\Omega| - \\ &\quad - \frac{C_2}{r-1}\|\boldsymbol{D}(\boldsymbol{\Phi})\|_r(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1} \leq c_1\|b\|_{-1,r'}(\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r + \|\boldsymbol{D}(\boldsymbol{\Phi})\|_r), \end{aligned}$$

which implies the estimate

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + C\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r \leq C < \infty, \tag{3.35}$$

or the equivalent one (using the Korn's inequality)

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + C\|\nabla \boldsymbol{u}^\varepsilon\|_r^r \le C < \infty. \tag{3.36}$$

(Here C denotes generally different, positive constants).

Using again (3.22)

$$\|\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_{r'}^{r'} \overset{(3.22)}{\le} \frac{C_2}{r-1}\int_\Omega (1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|)^r \,\mathrm{d}x \le \frac{C_2}{r-1}(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{u}^\varepsilon)\|_r + \|\boldsymbol{D}(\boldsymbol{\Phi})\|_r)^r$$

we obtain

$$\|\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_{r'} \le C < \infty. \tag{3.37}$$

In order to obtain the estimates for p^ε independent of ε we set $\boldsymbol{\psi} := \boldsymbol{\psi}^\varepsilon$ in (3.32) where $\boldsymbol{\psi}^\varepsilon$ solves:

$$\begin{aligned} \operatorname{div}\boldsymbol{\psi}^\varepsilon &= |p^\varepsilon|^{r'-2}p^\varepsilon - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon|^{r'-2}p^\varepsilon \,\mathrm{d}x =: h^\varepsilon \quad \text{in } \Omega \\ \boldsymbol{\psi}^\varepsilon &= \mathbf{0} \quad \text{on } \Omega \\ \|\boldsymbol{\psi}^\varepsilon\|_{1,s} &\le C_{\mathrm{div},s}\|h^\varepsilon\|_s \quad \text{for all } s \in (1,\infty), \end{aligned} \tag{3.38}$$

in particular, for $s = r$

$$\|\boldsymbol{\psi}^\varepsilon\|_{1,r} \le C_{\mathrm{div},r}\|p^\varepsilon\|_{r'}^{\frac{1}{r-1}}, \tag{3.39}$$

as it is showed in [1] and [3]. The existence of $\boldsymbol{\psi}^\varepsilon$ is to be seen e. g. in [6] or [7]. We can then conclude, using the fact that $\int_\Omega p^\varepsilon \,\mathrm{d}x = 0$,

$$\begin{aligned} \|p^\varepsilon\|_{r'}^{r'} &= (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)) - \langle \boldsymbol{b}, \boldsymbol{\psi}^\varepsilon\rangle \le \\ &\overset{(3.22)}{\le} \frac{C_2}{r-1}\int_\Omega (1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|)^{r-1}|\boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)| \,\mathrm{d}x + \|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{\psi}^\varepsilon\|_{1,r} \le \\ &\overset{\text{Hölder}}{\le} c_3\|1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|\|_r^{r-1}\|\boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)\|_r + \|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{\psi}^\varepsilon\|_{1,r} \le \\ &\underset{(3.35)}{\overset{(3.39)}{\le}} c_4\|p^\varepsilon\|_{r'}^{\frac{r'}{r}} \end{aligned}$$

which gives us (as $r > 1$)

$$\|p^\varepsilon\|_{r'} \le C < \infty. \tag{3.40}$$

Letting ε tend to zero, the estimates (3.35), (3.36), (3.37) and (3.40) allow us to find a sequence $\varepsilon \searrow 0$, $\{(\boldsymbol{v}^\varepsilon, p^\varepsilon)\}$ and $(\boldsymbol{v}, p) \in \mathbf{W}^{1,r}_{\mathrm{div}}(\Omega)^d \times \mathbf{L}^{r'}(\Omega)$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) &\rightharpoonup \boldsymbol{D}(\boldsymbol{v}) && \text{weakly in } \mathbf{L}^r(\Omega)^{d\times d}, \\ \nabla\boldsymbol{v}^{\varepsilon_n} &\rightharpoonup \nabla\boldsymbol{v} && \text{weakly in } \mathbf{L}^r(\Omega)^{d\times d}, \\ p^{\varepsilon_n} &\rightharpoonup p && \text{weakly in } \mathbf{L}^{r'}(\Omega), \\ \nu(p^{\varepsilon_n}, |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|^2)\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) &\rightharpoonup \boldsymbol{\chi} && \text{weakly in } \mathbf{L}^{r'}(\Omega)^{d\times d}, \end{aligned} \tag{3.41}$$

and moreover, from compact imbedding we conclude

$$\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} \quad \text{strongly in } \mathbf{L}^s(\Omega)^d \text{ for all } s\text{: } 1 \le s < \frac{dr}{d-r}. \tag{3.42}$$

Let us note that from (3.31) and (3.35) it directly follows that

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{a. e. in } \Omega, \tag{3.43}$$

and we can also pass to the limit in (3.32): we need to show is that for $n \to \infty$

$$\int_\Omega \nu(p^{\varepsilon_n}, |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|^2)\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x \to \int_\Omega \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x. \qquad (3.44)$$

at least for all $\boldsymbol{\psi} \in \mathcal{C}^\infty(\Omega)$.

In order to see (3.44), it is enough to show at least for a subsequence the convergences:

$$p^{\varepsilon_n} \to \quad p \qquad \text{a. e. in } \Omega, \qquad (3.45)$$

$$\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) \to \boldsymbol{D}(\boldsymbol{v}) \qquad \text{a. e. in } \Omega. \qquad (3.46)$$

Once we prove (3.45) and (3.46) the limit (3.44) follows using the Vitali's theorem (formulated in lemma 3.4) as soon as (3.15) holds: but for any $Q \subset \Omega$, $|Q| < \delta$ we can write

$$\int_Q |\nu(p^{\varepsilon_n}, |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|^2)\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) : \boldsymbol{D}(\boldsymbol{\psi})|\,\mathrm{d}x \overset{(3.22)}{\leq} \|\boldsymbol{D}(\boldsymbol{\psi})\|_\infty C \int_Q (1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|)^{r-1}\,\mathrm{d}x \underset{(3.35)}{\overset{\text{Hölder}}{\leq}} C|Q|^{1/r} < C\delta^{1/r}$$

and (3.15) then follows.

Strong convergence of $\{p^{\varepsilon_n}\}_{n=1}^\infty$ and $\{\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})\}_{n=1}^\infty$.

We will put to use Lemma 3.9, denoting

$$Y^n := \int_\Omega \int_0^1 \left(1 + |\boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))|^2\right)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}x \qquad (3.47)$$

it impies that

$$\frac{C_1}{2}Y^n \leq \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2, \quad (3.48)$$

where $\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) = \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})$. It can be seen in Lemma 3.17 below, that our estimates $\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})\|_r, \|\boldsymbol{D}(\boldsymbol{v})\|_r \leq K$ imply

$$\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq C(K)Y^n. \qquad (3.49)$$

If we set $\boldsymbol{\psi} := \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v} = \boldsymbol{u}^{\varepsilon_n} - \boldsymbol{u}$ in (3.32) it gives us

$$\int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x =$$
$$= -\int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x + \int_\Omega p^{\varepsilon_n}\,\mathrm{div}(\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v})\,\mathrm{d}x + \langle \boldsymbol{b}, \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\rangle. \quad (3.50)$$

Using that $\mathrm{div}\,\boldsymbol{v} = 0$ a. e. and neglecting $\int_\Omega p^{\varepsilon_n}\,\mathrm{div}\,\boldsymbol{v}^{\varepsilon_n}\,\mathrm{d}x = -\|p^{\varepsilon_n}\|_2^2 - \|\nabla p^{\varepsilon_n}\|_2^2 \leq 0$ it follows from (3.50), (3.49) and (3.48) that

$$\alpha\frac{C_1}{2}Y^n + (1-\alpha)\frac{C_1}{2}\frac{1}{C(K)}\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2 + \langle \boldsymbol{b}, \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}\rangle -$$
$$- \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x$$

for any $\alpha \in (0,1)$. Considering the weak convergences (3.41) we get

$$\alpha\frac{C_1}{2}Y^n + (1-\alpha)C\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2 + \delta_1(\varepsilon_n), \qquad (3.51)$$

where $\delta_i(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$. In order to handle the term $\|p^{\varepsilon_n} - p\|_2$ we test (3.32) with $\boldsymbol{\psi} = \boldsymbol{\psi}^n$ where $\boldsymbol{\psi}^n$ solves

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi}^n &= p^{\varepsilon_n} - p \quad \text{in } \Omega \\ \boldsymbol{\psi}^n &= \mathbf{0} \quad \text{on } \partial\Omega \\ \|\boldsymbol{\psi}^n\|_{1,q} &\leq C_{\mathrm{div},q} \|p^{\varepsilon_n} - p\|_q \quad \text{for all } q \in (1,\infty). \end{aligned} \tag{3.52}$$

Note that $\int_\Omega (p^{\varepsilon_n} - p)\,\mathrm{d}x = 0$ and from $p^{\varepsilon_n} \rightharpoonup p$ weakly in $\mathbf{L}^{r'}(\Omega)$ it follows that

$$\begin{aligned} &\boldsymbol{\psi}^n \rightharpoonup 0 \quad \text{weakly in } \mathbf{W}^{1,r'}(\Omega)^d \\ &\boldsymbol{\psi}^n \to 0 \quad \text{strongly in } \mathbf{L}^q(\Omega)^d,\ \forall q \in \langle 1, \frac{dr'}{r'-d}). \end{aligned}$$

This gives us

$$\begin{aligned} \|p^{\varepsilon_n} - p\|_2^2 = & -(p, p^{\varepsilon_n} - p) - \langle \boldsymbol{b}, \boldsymbol{\psi}^n \rangle + \\ & + \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : \boldsymbol{D}(\boldsymbol{\psi}^n)\,\mathrm{d}x + \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{\psi}^n)\,\mathrm{d}x = \\ = & \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : \boldsymbol{D}(\boldsymbol{\psi}^n)\,\mathrm{d}x + \delta_2(\varepsilon_n). \end{aligned}$$

Let us denote $p^s := p + s(p^{\varepsilon_n} - p)$, $\boldsymbol{D}^s := \boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))$ and consider

$$\begin{aligned} [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] = & \int_0^1 \frac{\partial \boldsymbol{S}(p^s, \boldsymbol{D}^s)}{\partial \boldsymbol{D}} (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}s + \\ & + \int_0^1 \frac{\partial \boldsymbol{S}(p^s, \boldsymbol{D}^s)}{\partial p} (p^{\varepsilon_n} - p)\,\mathrm{d}s \\ =: & \ I_1 + I_2. \end{aligned}$$

Since the derivatives of $\boldsymbol{S}$ are supposed to fulfil **(1)** and **(2)** we conclude

$$|I_1| \overset{(\mathbf{2})}{\leq} C_2 \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|\,\mathrm{d}s$$

and

$$|I_2| \overset{(\mathbf{1})}{\leq} \gamma_0 |p^{\varepsilon_n} - p|,$$

which implies

$$\begin{aligned} \|p^{\varepsilon_n} - p\|_2^2 \leq & \ \gamma_0 \int_\Omega |p^{\varepsilon_n} - p||\boldsymbol{D}(\boldsymbol{\psi}^n)|\,\mathrm{d}x + \\ & + C_2 \int_\Omega \int_0^1 (1 + |\boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})||\boldsymbol{D}(\boldsymbol{\psi}^n)|\,\mathrm{d}s\,\mathrm{d}x + \\ & + \delta_2(\varepsilon_n). \end{aligned}$$

Recalling the definition of Y^n from (3.47) and using the fact that $(1+\omega^2)^{\frac{r-2}{2}} \leq (1+\omega^2)^{\frac{r-2}{4}}$ as $r < 2$, the Hölder inequality (once used on the integral over $\Omega \times (0,1)$) gives

$$\|p^{\varepsilon_n} - p\|_2^2 \leq \gamma_0 \|p^{\varepsilon_n} - p\|_2 \|\nabla \boldsymbol{\psi}^n\|_2 + C_2 \sqrt{Y^n} \|\nabla \boldsymbol{\psi}^n\|_2 + \delta_2(\varepsilon_n).$$

The estimate in (3.52) with $q = 2$ then leads to

$$\|p^{\varepsilon_n} - p\|_2^2 \leq \gamma_0 C_{\mathrm{div},2} \|p^{\varepsilon_n} - p\|_2^2 + C_2 C_{\mathrm{div},2} \sqrt{Y^n} \|p^{\varepsilon_n} - p\|_2 + \delta_2(\varepsilon_n),$$

i. e.

$$(1 - \gamma_0 C_{\mathrm{div},2}) \|p^{\varepsilon_n} - p\|_2 \leq C_2 C_{\mathrm{div},2} \sqrt{Y^n} + \delta_3(\varepsilon_n). \tag{3.53}$$

Putting this estimate into (3.51) we obtain

$$\alpha\frac{C_1}{2}Y^n + (1-\alpha)C\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \le \frac{\gamma_0^2}{2C_1}\cdot\frac{C_2^2 C_{\mathrm{div},2}^2}{(1-\gamma_0 C_{\mathrm{div},2})^2}Y^n + \delta_4(\varepsilon_n).$$

Recalling $\gamma_0 < \frac{1}{C_{\mathrm{div},2}}\frac{C_1}{C_2+C_1}$ from the assumption **(2)** we can choose $\alpha \in (0,1)$ such that $\gamma_0 < \frac{1}{C_{\mathrm{div},2}}\frac{\sqrt{\alpha}C_1}{C_2+\sqrt{\alpha}C_1}$, which is enough to show that

$$\alpha\frac{C_1}{2} - \frac{\gamma_0^2}{2C_1}\frac{C_2^2 C_{\mathrm{div},2}^2}{(1-\gamma_0 C_{\mathrm{div},2})^2} > 0,$$

thus we can conclude

$$\lim_{n\to\infty}\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 = 0 \tag{3.54}$$

and

$$\lim_{n\to\infty} Y^n = 0. \tag{3.55}$$

The last fact and (3.53) then imply that

$$\lim_{n\to\infty}\|p^{\varepsilon_n} - p\|_2 = 0. \tag{3.56}$$

The convergences (3.45) and (3.46) then follow and the proof is complete.

Existence of approximations

In this section we show that for $\varepsilon > 0$ fixed there is a solution $(\boldsymbol{v}, p) = (\boldsymbol{v}^\varepsilon, p^\varepsilon)$ fulfilling (3.30)-(3.32). Since $\varepsilon > 0$ is fixed the dependence of the quantities on ε is not designated in what follows. The proof is via Galerkin approximations, following step by step the proof given in [1]

Let $\{\alpha^k\}_{k=1}^\infty$ be a basis in $\mathbf{W}^{1,2}(\Omega)$ and $\{\boldsymbol{a}^k\}_{k=1}^\infty$ be a basis of $\mathbf{W}_0^{1,r}(\Omega)^d$. We look for approximations p^N and $\boldsymbol{v}^N$ in the form

$$p^N = \sum_{k=1}^N c_k^N \alpha_k \quad \text{and} \quad \boldsymbol{v}^N = \boldsymbol{\Phi} + \sum_{k=1}^N d_k^N \boldsymbol{a}^k,$$

where $\boldsymbol{c}^N = (c_1, \ldots, c_N)$ and $\boldsymbol{d}^N = (d_1, \ldots, d_N)$ solve the Galerkin system

$$\varepsilon(\nabla p^N, \nabla\alpha^k) + \varepsilon(p^N, \alpha^k) - (\boldsymbol{v}^N, \nabla\alpha^k) = 0, \quad k = 1, \ldots, N, \tag{3.57}$$

$$\int_\Omega \nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N) : \boldsymbol{D}(\boldsymbol{a}^l)\,\mathrm{d}x + \int_\Omega \nabla p^N \cdot \boldsymbol{a}^l\,\mathrm{d}x = \langle \boldsymbol{b}, \boldsymbol{a}^l\rangle, \quad l = 1, \ldots, N. \tag{3.58}$$

This is a system of $2N$ nonlinear algebraic equations with $2N$ unknowns. The solvability follows from the Brouwer fixed point theorem formulated below and the apriori estimates which we are going to derive just now. Let us multiply the k-th equation in (3.57) by c_k^N and sum all equations for $k = 1, \ldots, N$, then multiply l-th equation in (3.58) by d_l^N and sum them over $l = 1, \ldots, N$. We obtain

$$\varepsilon\|\nabla p^N\|_2^2 + \varepsilon\|p^N\|_2^2 + (\operatorname{div}\boldsymbol{v}^N, p^N) = 0 \tag{3.59}$$

$$(\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)), \boldsymbol{D}(\boldsymbol{u}^N)) - (\operatorname{div}\boldsymbol{u}^N, p^N) = \langle \boldsymbol{b}, \boldsymbol{u}^N\rangle. \tag{3.60}$$

Summing these two equations we conclude (as $\operatorname{div}\boldsymbol{\Phi} = 0$)

$$\varepsilon\|p^N\|_{1,2}^2 + (\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)), \boldsymbol{D}(\boldsymbol{u}^N)) = \langle \boldsymbol{b}, \boldsymbol{u}^N\rangle, \tag{3.61}$$

from which it follows in the exactly same way as on page 21, using **(1)**, **(2)**, Korn's and Young's inequalities, that

$$\varepsilon\|p^N\|_{1,2}^2 + \|\nabla \boldsymbol{v}^N\|_r^r \leq C < \infty. \tag{3.62}$$

From (3.22) it follows that

$$\|\nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N)\|_{r'} \leq C < \infty. \tag{3.63}$$

We can therefore find a subsequence (which we denote same as the original sequence) such that

$$\begin{aligned}
\boldsymbol{v}^N &\rightharpoonup \boldsymbol{v} && \text{weakly in } \mathbf{W}^{1,r}(\Omega)^d && \text{(3.64a)}\\
\boldsymbol{v}^N &\to \boldsymbol{v} && \text{strongly in } \mathbf{L}^q(\Omega)^d \text{ for } q < \frac{dr}{d-r} && \text{(3.64b)}\\
p^N &\rightharpoonup p && \text{weakly in } \mathbf{W}^{1,2}(\Omega) && \text{(3.64c)}\\
p^N &\to p && \text{strongly in } \mathbf{L}^2(\Omega) && \text{(3.64d)}\\
\nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N) &\rightharpoonup \boldsymbol{\chi} && \text{weakly in } \mathbf{L}^{r'}(\Omega)^{d\times d}, && \text{(3.64e)}
\end{aligned}$$

which allows us to pass to the limit in (3.57)-(3.58). We obtain

$$\varepsilon(\nabla p, \nabla \xi) + \varepsilon(p, \xi) + (\operatorname{div} \boldsymbol{v}, p^N) = 0 \qquad \forall \xi \in \mathbf{W}^{1,2}(\Omega) \tag{3.65}$$

$$(\boldsymbol{\chi}, \boldsymbol{D}(\boldsymbol{\psi})) - (p, \operatorname{div} \boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \qquad \forall \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d. \tag{3.66}$$

In particular, testing $\xi := p$ in (3.65) and $\boldsymbol{\psi} := \boldsymbol{u}$ in (3.66) and summing the equations we obtain

$$\varepsilon\|p\|_{1,2}^2 + (\boldsymbol{\chi}, \boldsymbol{D}(\boldsymbol{u})) = \langle \boldsymbol{b}, \boldsymbol{u}\rangle. \tag{3.67}$$

All we need in order to complete the proof is to identify $\boldsymbol{\chi}$ as $\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))$. We do it as soon as we show that

$$\boldsymbol{D}(\boldsymbol{v}^N) \to \boldsymbol{D}(\boldsymbol{v}) \quad \text{and} \quad p^N \to p \quad \text{a.e. in } \Omega \tag{3.68}$$

at least for a subsequence. From Vitali's theorem (formulated in lemma 3.4) we again conclude

$$\int_\Omega \nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x \to \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x = \int_\Omega \boldsymbol{\chi} : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x.$$

In order to conclude (3.68) it is enough to show, at least for a subsequence, that

$$\lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r = 0.$$

Since we know (3.64d), (3.68) then follows.

Let us recall (3.48) and (3.49) with $p^{\varepsilon_n} = p^N$ and $\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) = \boldsymbol{D}(\boldsymbol{v}^N)$. We have

$$\begin{aligned}
&C\|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \\
&\leq \int_\Omega [\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 = \\
&= \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u}^N)\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x - \\
&\quad - \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 = \\
&\overset{(3.61)}{=} \langle \boldsymbol{b}, \boldsymbol{u}^N\rangle - \varepsilon\|p^N\|_2^2 - \varepsilon\|\nabla p^N\|_2^2 + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 - \\
&\quad - \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x.
\end{aligned}$$

Using $\lim_{N\to\infty} \|\nabla p^N\|_2^2 \geq \liminf_{N\to\infty} \|\nabla p^N\|_2^2 \geq \|\nabla p\|_2^2$ we obtain

$$\varepsilon\|\nabla p\|_2^2 + C \cdot \lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \langle \boldsymbol{b}, \boldsymbol{u}\rangle - \varepsilon\|p\|_2^2 - \int_\Omega \boldsymbol{\chi} : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x.$$

Then from (3.67) directly follows

$$C \cdot \lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq 0$$

which implies (3.68) and the proof is thus complete. □

3.5.2 Existence of solutions for the generalized Navier-Stokes system

In order to handle the convective term we introduce the following Lemma, which can be found in Kaplický, Málek, Stará [2]. It strongly takes an advantage of that the boundary condition $\boldsymbol{\varphi}$ fulfills $\boldsymbol{\varphi}.\boldsymbol{n} = 0$ at $\partial\Omega$ and gives an extension of $\boldsymbol{\varphi}$ which could be arbitrary small. Moreover, it gives a control of its gradient.

Lemma 3.12 (An extension of boundary conditions) *Let $r \in (1,2\rangle$, $\partial\Omega \in \mathcal{C}^3$, $\boldsymbol{\varphi} = \mathrm{Tr}(\boldsymbol{\Phi})$, $\boldsymbol{\Phi} \in \mathbf{W}^{3,q}(\Omega)$ for some $q > 2$, and $\boldsymbol{\varphi}$ satisfy*

$$\boldsymbol{\varphi}\cdot\boldsymbol{n} = 0 \quad \textit{at } \partial\Omega \tag{3.69}$$

(where $\boldsymbol{n}$ denotes the outer normal vector to $\partial\Omega$).

Then for each $\eta > 0$ there exists $\boldsymbol{\Phi}^\eta \in \mathbf{W}^{1,\infty}(\Omega)$ such that

$$\operatorname{div}\boldsymbol{\Phi}^\eta = 0 \ \textit{ in } \Omega \quad \textit{and} \quad \mathrm{Tr}(\boldsymbol{\Phi}^\eta) = \boldsymbol{\varphi} \ \textit{ at } \partial\Omega, \tag{3.70}$$

$$\left|\int_\Omega u_i \frac{\partial \Phi_j^\eta}{\partial x_i} u_j\,\mathrm{d}x\right| < \eta^{\frac{3r-4}{r}} \|\boldsymbol{u}\|_{1,r}^2 \quad \forall \boldsymbol{u} \in \mathbf{W}_0^{1,r}(\Omega) \tag{3.71}$$

$$\begin{array}{l} \|\boldsymbol{\Phi}^\eta\|_q < C\eta^{\frac{1}{q}} \\ \|\boldsymbol{\Phi}^\eta\|_{1,q} < C\eta^{\frac{1}{q}-1} \end{array} \quad \textit{for all } q \in \langle 1,\infty\rangle \tag{3.72}$$

<u>PROOF</u>. A proof of this lemma is provided in [2].

I believe that Lemma 3.12 could be in some special cases (as axial symmetry) generalized into three dimensions, so far as $\boldsymbol{\varphi}$ retains its two-dimensional nature. Such an extension is a future project.

The construction of $\boldsymbol{\Phi}^{\boldsymbol{\eta}}$ is not difficult in such a simple geometry as the case of eccentric annular rings with constant tangential velocity v_0 prescribed on the inner circle. Let us define the function $\boldsymbol{\Theta}^\eta$ which fulfills both (3.70) and the estimate (3.72) in a few following lines:

Let $\Omega \subset \mathbb{R}^2$ be the eccentric annulus such as we can see in figure 1. For a while, let us consider the cartesian coordinates $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ with the origin located in the centre of the inner circle. Denote $r = r(\boldsymbol{x}) = \|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2}$ the distance from the centre, the unit "tangential" vector field $\boldsymbol{\tau} := \boldsymbol{\tau}(\boldsymbol{x}) = \frac{1}{r}(x_2, -x_1)$ and the unit "axial" vector field $\boldsymbol{r} := \boldsymbol{r}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{1}{r}(x_1, x_2)$. It is easy to see that any function $\boldsymbol{\Theta}^\eta$ of the form

$$\boldsymbol{\Theta}^\eta(\boldsymbol{x}) = f(r(\boldsymbol{x}))\boldsymbol{\tau}(\boldsymbol{x})$$

meets the equation

$$\operatorname{div}\boldsymbol{\Theta}^\eta = 0, \quad \text{for } \|\boldsymbol{x}\| \geq R_J.$$

We define

$$f(r) := \begin{cases} 0, & \text{for } \ r \geq R_J + \eta \\ \frac{v_0}{\eta^2}(R_j + \eta - r)^2 & \text{for } R_J \leq r \leq R_J + \eta. \end{cases}$$

We then see, for $\eta > 0$ small enough, that the form $\boldsymbol{\Theta}^\eta \in \mathcal{C}^\infty(\bar{\Omega})$ satisfy the boundary condition $\boldsymbol{\varphi} = v_0\boldsymbol{\tau}$ on the inner circle (and $\boldsymbol{\varphi} = \mathbf{0}$ on the outer circle) such that (3.70) is fulfilled. It follows from definition that for $r < R_J + \eta$

$$\begin{aligned}\frac{\partial \boldsymbol{\Theta}^\eta}{\partial \boldsymbol{\tau}}(\boldsymbol{x}) &= 0,\\ \frac{\partial \boldsymbol{\Theta}^\eta}{\partial \boldsymbol{r}}(\boldsymbol{x}) &= -\frac{2v_0}{\eta^2}(R_J + \eta - r),\end{aligned}$$

meanwhile $\boldsymbol{\Theta}^\eta = \mathbf{0}$ for $r \geq R_J + \eta$. We thus easily obtain $|\boldsymbol{\Theta}^\eta| \leq v_0$ and $|\nabla \boldsymbol{\Theta}^\eta| \leq C(R_J, v_0)\frac{1}{\eta}$. Realizing the fact that $\boldsymbol{\Theta}^\eta$ is non-zero only on the area of dimension $\pi(R_J + \eta)^2 - \pi R_J^2 \leq C\eta$ we conclude that (3.72) also holds.

Theorem 3.13 (Existence of solutions for the system (P)) *Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded set with the boundary $\partial\Omega \in \mathcal{C}^3$, $d = 2$. Let the assumptions **(1)** and **(2)** be satisfied with r fulfilling*

$$\frac{3}{2} = \frac{3d}{d+2} < r < 2 \tag{3.73}$$

and let (3.16) *and* (3.17) *hold together with*

$$\boldsymbol{\Phi} \in \mathbf{W}^{3,q}(\Omega)^d \quad \text{for some } q > 2 \tag{3.74}$$

$$\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \quad \text{at } \partial\Omega, \tag{3.75}$$

where $\boldsymbol{n}$ denotes the outer normal vector to $\partial\Omega$.

Then there is at least one weak solution $(\boldsymbol{v}, p)$ to the problem (P) in the sence of Definition 3.8.

<u>PROOF</u>.

The proof follows the same steps as the proof for Stokes system above. The only difference concerns the convective term, but in order to show the whole proof clearly and without question, I have decided to repeat it all, not only the differences.

We recall the problem (P^ε) and assume that it has a solution. We derive the energy estimates and estimates for the pressure p^ε uniform with respect to ε. Then for some sequence $\varepsilon_n \to 0$ we find weakly converging subsequence $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$ to the limit $(\boldsymbol{v}, p)$ in the spaces stated in (3.19) and, in addition to that, we show the strong convergence of $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$. Finally, we proof the existence of weak solutions to the approximate problem and thus vindicate our assumption.

Weak solution of $(\mathbf{P}^\varepsilon)$

We suppose that for r fulfilling (3.73) and all $\varepsilon > 0$ there is a weak solution $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ to the problem (P^ε) such that

$$\boldsymbol{v}^\varepsilon - \boldsymbol{\Phi} \in \mathbf{W}_0^{1,r}(\Omega)^d \quad \text{and} \quad p^\varepsilon \in \mathbf{W}^{1,2}(\Omega) \tag{3.76}$$

satisfying

$$\varepsilon(\nabla p^\varepsilon, \nabla \xi) + \varepsilon(p^\varepsilon, \xi) + (\operatorname{div} \boldsymbol{v}^\varepsilon, \xi) = 0 \quad \text{for all } \xi \in \mathbf{W}^{1,2}(\Omega) \tag{3.77}$$

and

$$\begin{aligned}&\left(v_i^\varepsilon \frac{\partial \boldsymbol{v}^\varepsilon}{\partial x_i}, \boldsymbol{\psi}\right) + \frac{1}{2}\left((\operatorname{div} \boldsymbol{v}^\varepsilon)\boldsymbol{v}^\varepsilon, \boldsymbol{\psi}\right) + \\ &+(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\psi})) - (p^\varepsilon, \operatorname{div} \boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \\ &\qquad\qquad\qquad\qquad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d.\end{aligned} \tag{3.78}$$

Let us note that all integrals in our weak formulation are finite: from Hölder inequality we see it for (3.77) as soon as $r > \frac{2d}{d+2}$, the viscous term is finite from (3.22). For the finiteness in the convective term we need the assumption $r > \frac{3d}{d+2}$.

The existence of solution $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ fulfilling (3.76)-(3.78) for $\varepsilon > 0$ fixed will be proved in the section 3.5.2.

Energy estimates and their direct consequences

In order to handle the convective term, we define $\boldsymbol{u}^{\varepsilon,\eta}$ by

$$\boldsymbol{v}^\varepsilon = \boldsymbol{\Phi}^\eta + \boldsymbol{u}^{\varepsilon,\eta}, \tag{3.79}$$

where $\boldsymbol{\Phi}^\eta$ is taken from the Lemma 3.12; it fulfills $\operatorname{div} \boldsymbol{\Phi}^\eta = 0$ and can be estimated by

$$\begin{aligned} \|\boldsymbol{\Phi}^\eta\|_q &< C\eta^{\frac{1}{q}}, \\ \|\nabla\boldsymbol{\Phi}^\eta\|_q &< C\eta^{\frac{1}{q}-1}, \end{aligned} \qquad \text{for all } q \in \langle 1, \infty \rangle. \tag{3.80}$$

Clearly, due to (3.70)

$$\boldsymbol{u}^{\varepsilon,\eta} \in \mathbf{W}^{1,r}_{0,\operatorname{div}}(\Omega)^d.$$

Let us set $\xi := p^\varepsilon$ in (3.77) (note that such a ξ is a possible test function as soon as $p^\varepsilon \in \mathbf{W}^{1,2}(\Omega)$) and $\boldsymbol{\psi} := \boldsymbol{u}^{\varepsilon,\eta}$ in (3.78), then it follows

$$\begin{aligned} \varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2 + (\operatorname{div}\boldsymbol{v}^\varepsilon, p^\varepsilon) &= 0 \\ \left(v_i^\varepsilon \frac{\partial \boldsymbol{v}^\varepsilon}{\partial x_i}, \boldsymbol{u}^{\varepsilon,\eta}\right) + \frac{1}{2}\left((\operatorname{div}\boldsymbol{v}^\varepsilon)\boldsymbol{v}^\varepsilon, \boldsymbol{u}^{\varepsilon,\eta}\right) + & \\ + \left(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^{\varepsilon,\eta})\right) - (p^\varepsilon, \operatorname{div}\boldsymbol{u}^{\varepsilon,\eta}) &= \langle \boldsymbol{b}, \boldsymbol{u}^{\varepsilon,\eta}\rangle. \end{aligned}$$

Summing these equations and using the assumption $\operatorname{div}\boldsymbol{\Phi}^\eta = 0$ we find

$$\begin{aligned} I_C + \varepsilon\|\nabla p^\varepsilon\|_2^2 + \varepsilon\|p^\varepsilon\|_2^2 + \left(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^{\varepsilon,\eta})\right) &= \langle \boldsymbol{b}, \boldsymbol{u}^{\varepsilon,\eta}\rangle, \\ \text{where } I_C = \left(v_i^\varepsilon \frac{\partial \boldsymbol{v}^\varepsilon}{\partial x_i}, \boldsymbol{u}^{\varepsilon,\eta}\right) + \frac{1}{2}\left((\operatorname{div}\boldsymbol{v}^\varepsilon)\boldsymbol{v}^\varepsilon, \boldsymbol{u}^{\varepsilon,\eta}\right). & \end{aligned} \tag{3.81}$$

Let us estimate I_C. Recalling (3.79), we have

$$I_C = \int_\Omega v_i^\varepsilon \frac{\partial \Phi_j^\eta}{\partial x_i} u_j^{\varepsilon,\eta}\,dx + \int_\Omega v_i^\varepsilon \frac{\partial u_j^{\varepsilon,\eta}}{\partial x_i} u_j^{\varepsilon,\eta}\,dx + \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon) v_j^\varepsilon u_j^{\varepsilon,\eta}\,dx.$$

Thus, Green's theorem (as $\boldsymbol{u}^{\varepsilon,\eta} = \boldsymbol{0}$ on $\partial\Omega$) implies $\int_\Omega v_i^\varepsilon \frac{\partial}{\partial x_i}(\frac{|\boldsymbol{u}^{\varepsilon,\eta}|^2}{2}) = -\frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)|\boldsymbol{u}^{\varepsilon,\eta}|^2$. Then,

$$\begin{aligned} I_C &= \int_\Omega v_i^\varepsilon \frac{\partial \Phi_j^\eta}{\partial x_i} u_j^{\varepsilon,\eta}\,dx + \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta u_j^{\varepsilon,\eta}\,dx = \\ &\overset{\text{Green}}{=} -\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta u_j^{\varepsilon,\eta}\,dx - \int_\Omega v_i^\varepsilon \Phi_j^\eta \frac{\partial u_j^{\varepsilon,\eta}}{\partial x_i}\,dx + \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta u_j^{\varepsilon,\eta}\,dx = \\ &= -\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta v_j^\varepsilon\,dx - \int_\Omega v_i^\varepsilon \Phi_j^\eta \frac{\partial v_j^\varepsilon}{\partial x_i}\,dx + \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta v_j^\varepsilon\,dx + \\ &\quad + \int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)|\boldsymbol{\Phi}^\eta|^2\,dx + \int_\Omega v_i^\varepsilon \frac{\partial}{\partial x_i}(\frac{|\boldsymbol{\Phi}^\eta|^2}{2})\,dx - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)|\boldsymbol{\Phi}^\eta|^2\,dx, \end{aligned}$$

which finally, using the Green's theorem and $\operatorname{div}\boldsymbol{\Phi}^\eta = 0$, leads to

$$I_C = -\frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\Phi_j^\eta v_j^\varepsilon\,dx - \int_\Omega v_i^\varepsilon \Phi_j^\eta \frac{\partial v_j^\varepsilon}{\partial x_i}\,dx + \frac{1}{2}\int_\Omega \Phi_i^\eta \Phi_j^\eta \frac{\partial \Phi_j^\eta}{\partial x_i}\,dx.$$

Using the Hölder inequality and the Lemma 3.12 we conclude that

$$
\begin{aligned}
|I_C| &\le \frac{3}{2}\|\nabla \boldsymbol{v}^\varepsilon\|_r \|\boldsymbol{v}^\varepsilon\|_{\frac{2r}{2-r}} \|\boldsymbol{\Phi}^\eta\|_{\frac{2r}{3r-4}} + \frac{1}{2}\|\boldsymbol{\Phi}^\eta\|_{2q}^2 \|\nabla \boldsymbol{\Phi}^\eta\|_{q'} \le \\
&\overset{(3.80)}{\le} c_1 \|\boldsymbol{v}^\varepsilon\|_{1,r}^2 \eta^{\frac{3r-4}{2r}} + C\eta^{\frac{1}{q}+\frac{1}{q'}-1} = \\
&\overset{\text{Korn}}{\le} d_2(\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2)\eta^{\frac{3r-4}{2r}} + C,
\end{aligned}
\tag{3.82}
$$

where C, d_2, d_4 are some (positive) constants depending on Ω, r.

Considering the viscous term, using (3.20) and (3.22) we conclude analogously as in the Stokes case (3.34)

$$
(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^{\varepsilon,\eta})) \ge \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - C - \frac{C_2}{r-1}\|\nabla \boldsymbol{\Phi}^\eta\|_r (|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1},
\tag{3.83}
$$

and from (3.80) we obtain

$$
(\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{u}^{\varepsilon,\eta})) \ge \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - C - d_1\eta^{\frac{1-r}{r}} (|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1},
\tag{3.84}
$$

where C, d_1 are again some positive constants.

The right-hand side term can be estimated using Korn's inequality and Lemma 3.12:

$$
\begin{aligned}
|\langle \boldsymbol{b}, \boldsymbol{u}^{\varepsilon,\eta}\rangle| &\le \|\boldsymbol{b}\|_{-1,r'} \|\boldsymbol{v}^\varepsilon - \boldsymbol{\Phi}^\eta\|_{1,r} \le \\
&\overset{\text{Korn}}{\le} C\|\boldsymbol{b}\|_{-1,r'} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon - \boldsymbol{\Phi}^\eta)\|_r \le \\
&\le C\|\boldsymbol{b}\|_{-1,r'} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r + \|\boldsymbol{b}\|_{-1,r'} \|\nabla \boldsymbol{\Phi}^\eta\|_r \le \\
&\overset{(3.80)}{\le} C\|\boldsymbol{b}\|_{-1,r'} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r + d_3\|\boldsymbol{b}\|_{-1,r'} \eta^{\frac{1}{r}-1}.
\end{aligned}
\tag{3.85}
$$

From (3.81) and (3.82), (3.84),(3.85) we obtain

$$
\begin{aligned}
\varepsilon\|p^\varepsilon\|_{1,2}^2 + \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r - d_1\eta^{\frac{1-r}{r}} (|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1} - \\
- d_2\eta^{\frac{3r-4}{2r}}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^2 - d_3\|\boldsymbol{b}\|_{-1,r'}\eta^{\frac{1-r}{r}} - C \le C\|\boldsymbol{b}\|_{r'}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r,
\end{aligned}
\tag{3.86}
$$

where by C we denote generally different positive constants.

We wouldn't obtain any useful result by just setting η small enough, say $\eta \searrow 0$, because as $r > 1$ the term with $\eta^{\frac{1-r}{r}}$ would become infinitely large. But for each $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r$ (greater then some constant D which doesn't depend on ε) we can find $\eta > 0$ such that it fulfills both

$$
d_1\eta^{\frac{1-r}{r}} (|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1} + d_3\|\boldsymbol{b}\|_{-1,r'}\eta^{\frac{1-r}{r}} < \frac{1}{3}\frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r, \quad \text{and} \tag{3.87}
$$

$$
d_2\eta^{\frac{3r-4}{2r}} (\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2) < \frac{1}{3}\frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r. \tag{3.88}
$$

Indeed, denoting $\tilde{d_1} := \left(\frac{1}{3d_1}\frac{C_1}{2r}\right)^{\frac{r}{1-r}}$ and $\tilde{d_2} := \left(\frac{1}{3d_2}\frac{C_1}{2r}\right)^{\frac{2r}{3r-4}}$ we can equivalently write

$$
\eta > \tilde{d_1}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{r\frac{r}{1-r}} \left[(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r)^{r-1} + \frac{d_3}{d_1}\|\boldsymbol{b}\|_{-1,r'} \right]^{-\frac{r}{1-r}} \tag{3.87}
$$

$$
\eta < \tilde{d_2}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{r\frac{2r}{3r-4}} (\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2)^{-\frac{2r}{3r-4}}, \tag{3.88}
$$

then considering $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r \geq |\Omega|^{1/r}$ and $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r \geq \frac{1}{2}\left(\frac{d_3}{d_1}\|\boldsymbol{b}\|_{-1,r'}\right)^{\frac{1}{r-1}}$ it is enough to find

$$\eta > \tilde{d}_1 2^{\frac{r^2}{r-1}} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{\frac{r^2}{1-r}+r} = \tilde{d}_1 2^{\frac{r^2}{r-1}} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{\frac{r}{1-r}} \tag{3.89}$$

instead of (3.87), and assuming $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r \geq \sqrt{d_4}\|\boldsymbol{\Phi}\|$ it is enough to find

$$\eta < \tilde{d}_2 2^{-\frac{2r}{3r-4}} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{(r-2)\frac{2r}{3r-4}} \tag{3.90}$$

instead of (3.88). Our goal is to show that

$$\tilde{d}_1 2^{\frac{r^2}{r-1}} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^{\frac{r}{1-r}} < \tilde{d}_2 2^{-\frac{2r}{3r-4}} \|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|^{(r-2)\frac{2r}{3r-4}}. \tag{3.91}$$

But as soon as

$$r > \frac{3}{2}$$

there holds

$$\frac{r}{1-r} < (r-2)\frac{2r}{3r-4},$$

and (3.91) is thus true for all $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r > D$, D being a constant which depends on r, $\tilde{d}_1$ and $\tilde{d}_2$. Moreover, considering D large enough, we can see that always $\eta < \delta_0$ (this is needed to apply Lemma 3.12.)

We conclude that for each $\varepsilon > 0$ either

$$\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r \leq D$$

or

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + \frac{1}{3}\frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r \leq C\|\boldsymbol{b}\|_{r'}\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r + C,$$

which together lead to the estimate

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + C\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r^r \leq C < \infty, \tag{3.92}$$

or the equivalent one (using the Korn's inequality)

$$\varepsilon\|p^\varepsilon\|_{1,2}^2 + C\|\nabla\boldsymbol{v}^\varepsilon\|_r^r \leq C < \infty. \tag{3.93}$$

Using again (3.22)

$$\|\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_{r'}^{r'} \overset{(3.22)}{\leq} \frac{C_2}{r-1}\int_\Omega (1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|)^r \,\mathrm{d}x$$

we obtain

$$\|\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_{r'} \leq C < \infty. \tag{3.94}$$

We remark that estimates (3.92) and (3.93) could be observed alternatively in a different way. Proving the existence of ε-approximations (see below) we notice (3.120), i. e. that $\|\boldsymbol{D}(\boldsymbol{v}^{(\varepsilon,)N})\|_r$ and therefore also $\|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)\|_r$ (from the lower semicontinuity of the norm) are bounded by a constant independent of ε. Estimates (3.92) and (3.93) then follows directly from (3.86) without any extra fitting of η. However, the "η-procedure" provided above is used in the same way in the proof of existence of Galerkin approximations that goes before (3.120).

In order to obtain the estimates for p^ε independent of ε we set $\boldsymbol{\psi} := \boldsymbol{\psi}^\varepsilon$ in (3.78) where $\boldsymbol{\psi}^\varepsilon$ solves:

$$\begin{aligned} \operatorname{div} \boldsymbol{\psi}^\varepsilon &= |p^\varepsilon|^{r'-2}p^\varepsilon - \frac{1}{|\Omega|}\int_\Omega |p^\varepsilon|^{r'-2}p^\varepsilon \,\mathrm{d}x =: h^\varepsilon \quad \text{in } \Omega \\ \boldsymbol{\psi}^\varepsilon &= \mathbf{0} \quad \text{on } \Omega \\ \|\boldsymbol{\psi}^\varepsilon\|_{1,s} &\le C_{\mathrm{div},s}\|h^\varepsilon\|_s \quad \text{for all } s \in (1,\infty), \end{aligned} \tag{3.95}$$

in particular, for $s = r$

$$\|\boldsymbol{\psi}^\varepsilon\|_{1,r} \le C_{\mathrm{div},r}\|p^\varepsilon\|_{r'}^{\frac{1}{r-1}}, \tag{3.96}$$

as it is showed in [1] and [3]. The existence of $\boldsymbol{\psi}^\varepsilon$ is to be seen e. g. in [6] or [7]. We can then conclude, similarly as in (3.40), using the fact that $\int_\Omega p^\varepsilon \,\mathrm{d}x = 0$, Hölder inequality, Sobolev imbedding and (3.22),

$$\begin{aligned} \|p^\varepsilon\|_{r'}^{r'} &= (\nu(p^\varepsilon, |\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|^2)\boldsymbol{D}(\boldsymbol{v}^\varepsilon), \boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)) - \langle \boldsymbol{b}, \boldsymbol{\psi}^\varepsilon\rangle + \\ &+ \left(v_i^\varepsilon \frac{\partial \boldsymbol{v}^\varepsilon}{\partial x_i}, \boldsymbol{\psi}^\varepsilon\right) + \frac{1}{2}\left((\operatorname{div}\boldsymbol{v}^\varepsilon)\boldsymbol{v}^\varepsilon, \boldsymbol{\psi}^\varepsilon\right) \le \\ &\le \frac{C_2}{r-1}\int_\Omega (1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|)^{r-1}|\boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)| \,\mathrm{d}x + \|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{\psi}^\varepsilon\|_{1,r} + \\ &+ \frac{3}{2}\|\boldsymbol{v}^\varepsilon\|_{\frac{rd}{d-r}}\|\nabla \boldsymbol{v}^\varepsilon\|_r \|\boldsymbol{\psi}^\varepsilon\|_{\frac{rd}{d-r}} \le \\ &\le c_3\|1+|\boldsymbol{D}(\boldsymbol{v}^\varepsilon)|\|_r^{r-1}\|\boldsymbol{D}(\boldsymbol{\psi}^\varepsilon)\|_r + \|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{\psi}^\varepsilon\|_{1,r} + c_4\|\boldsymbol{v}^\varepsilon\|_{1,r}^2\|\boldsymbol{\psi}^\varepsilon\|_{1,r} \le \\ &\overset{(3.96)}{\le} c_5\|p^\varepsilon\|_{r'}^{\frac{r'}{r}} \end{aligned}$$

which gives us (as $r > 1$)

$$\|p^\varepsilon\|_{r'} \le C < \infty. \tag{3.97}$$

Letting ε tend to zero, the estimates (3.92), (3.93), (3.94) and (3.97) allow us to find a sequence $\varepsilon \searrow 0$, $\{(\boldsymbol{v}^\varepsilon, p^\varepsilon)\}$ and $(\boldsymbol{v}, p) \in \mathbf{W}^{1,r}_{\mathrm{div}}(\Omega)^d \times \mathbf{L}^{r'}(\Omega)$

$$\begin{aligned} \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) &\rightharpoonup \boldsymbol{D}(\boldsymbol{v}) && \text{weakly in } \mathbf{L}^r(\Omega)^{d\times d}, \\ \nabla \boldsymbol{v}^{\varepsilon_n} &\rightharpoonup \nabla \boldsymbol{v} && \text{weakly in } \mathbf{L}^r(\Omega)^{d\times d}, \\ p^{\varepsilon_n} &\rightharpoonup p && \text{weakly in } \mathbf{L}^{r'}(\Omega), \\ \nu(p^{\varepsilon_n}, |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|^2)\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) &\rightharpoonup \boldsymbol{\chi} && \text{weakly in } \mathbf{L}^{r'}(\Omega)^{d\times d}, \end{aligned} \tag{3.98}$$

and moreover, from compact imbedding we conclude

$$\boldsymbol{v}^{\varepsilon_n} \to \boldsymbol{v} \quad \text{strongly in } \mathbf{L}^s(\Omega)^d \text{ for all } s\text{: } 1 \le s < \frac{dr}{d-r}. \tag{3.99}$$

As $r > \frac{3d}{d+2}$, (3.99) and (3.78) suffice to show that

$$\left.\begin{aligned} \int_\Omega v_i^\varepsilon \frac{\partial v_j^\varepsilon}{\partial x_i}\psi_j \,\mathrm{d}x &\to \int_\Omega v_i \frac{\partial v_j}{\partial x_i}\psi_j \,\mathrm{d}x \\ \int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon) v_j^\varepsilon \psi_j \,\mathrm{d}x &\to \int_\Omega (\operatorname{div}\boldsymbol{v}) v_j \psi_j \,\mathrm{d}x \end{aligned}\right. \quad \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega). \tag{3.100}$$

Let us note that from (3.77) and (3.92) it directly follows that

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{a. e. in } \Omega, \tag{3.101}$$

and we can also pass to the limit in (3.78), only we need to show is that for $n \to \infty$

$$\int_\Omega \nu(p^{\varepsilon_n}, |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})|^2)\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) : \boldsymbol{D}(\boldsymbol{\psi}) \,\mathrm{d}x \to \int_\Omega \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{\psi}) \,\mathrm{d}x. \tag{3.102}$$

In order to see (3.102), it is enough to show at least for a subsequence the convergence:

$$p^{\varepsilon_n} \to \quad p \qquad \text{a. e. in } \Omega, \tag{3.103}$$
$$\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) \to \boldsymbol{D}(\boldsymbol{v}) \qquad \text{a. e. in } \Omega. \tag{3.104}$$

Once we prove (3.103) and (3.104) the limit (3.102) follows using the Vitali's theorem (formulated in lemma 3.4) in the same way as in the Stokes case.

Strong convergence of $\{p^{\varepsilon_n}\}_{n=1}^{\infty}$ and $\{\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})\}_{n=1}^{\infty}$.

We will put to use Lemma 3.9, denoting

$$Y^n := \int_\Omega \int_0^1 \left(1 + |\boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))|^2\right)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^2 \,\mathrm{d}s \,\mathrm{d}x \tag{3.105}$$

it says that

$$\frac{C_1}{2} Y^n \leq \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})) \,\mathrm{d}x + \frac{\gamma_0^2}{2C_1} \|p^{\varepsilon_n} - p\|_2^2, \tag{3.106}$$

where $\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) = \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})$. It can be seen in Lemma 3.17 below, that our estimates $\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})\|_r, \|\boldsymbol{D}(\boldsymbol{v})\|_r \leq K$ imply

$$\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq C(K) Y^n. \tag{3.107}$$

If we set $\boldsymbol{\psi} := \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}$ in (3.78) it gives us

$$\begin{aligned}
&\int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})) \,\mathrm{d}x = \\
&= -\int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})) \,\mathrm{d}x + \int_\Omega p^{\varepsilon_n} \operatorname{div}(\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \,\mathrm{d}x + \\
&\quad + \langle \boldsymbol{b}, \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v} \rangle - \int_\Omega v_i^{\varepsilon_n} \frac{\partial \boldsymbol{v}^{\varepsilon_n}}{\partial x_i} \cdot (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \,\mathrm{d}x - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^{\varepsilon_n}) \boldsymbol{v}^{\varepsilon_n} \cdot (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \,\mathrm{d}x.
\end{aligned} \tag{3.108}$$

Using that $\operatorname{div} \boldsymbol{v} = 0$ a. e. and neglecting $\int_\Omega p^{\varepsilon_n} \operatorname{div} \boldsymbol{v}^{\varepsilon_n} \,\mathrm{d}x = -\|p^{\varepsilon_n}\|_2^2 - \|\nabla p^{\varepsilon_n}\|_2^2 \leq 0$ it follows from (3.108), (3.107) and (3.106) that

$$\begin{aligned}
&\alpha \frac{C_1}{2} Y^n + (1-\alpha) \frac{C_1}{2} \frac{1}{C(K)} \|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \frac{\gamma_0^2}{2C_1} \|p^{\varepsilon_n} - p\|_2^2 + \langle \boldsymbol{b}, \boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v} \rangle - \\
&\quad - \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})) \,\mathrm{d}x - \int_\Omega v_i^{\varepsilon_n} \frac{\partial \boldsymbol{v}^{\varepsilon_n}}{\partial x_i} \cdot (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \,\mathrm{d}x - \\
&\quad - \int_\Omega (\operatorname{div} \boldsymbol{v}^{\varepsilon_n}) \boldsymbol{v}^{\varepsilon_n} \cdot (\boldsymbol{v}^{\varepsilon_n} - \boldsymbol{v}) \,\mathrm{d}x,
\end{aligned}$$

for any $\alpha \in (0, 1)$. Considering the weak convergence (3.98) and (3.99) we get

$$\alpha \frac{C_1}{2} Y^n + (1-\alpha) C \|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \frac{\gamma_0^2}{2C_1} \|p^{\varepsilon_n} - p\|_2^2 + \delta_1(\varepsilon_n), \tag{3.109}$$

where $\delta_k(\varepsilon_n) \to 0$ as $\varepsilon_n \to 0$. In order to handle the term $\|p^{\varepsilon_n} - p\|_2$ we test (3.78) with $\boldsymbol{\psi} = \boldsymbol{\psi}^n$ where $\boldsymbol{\psi}^n$ solves

$$\begin{aligned}
\operatorname{div} \boldsymbol{\psi}^n &= p^{\varepsilon_n} - p \quad \text{in } \Omega \\
\boldsymbol{\psi}^n &= \boldsymbol{0} \quad \text{on } \partial\Omega \\
\|\boldsymbol{\psi}^n\|_{1,q} &\leq C_{\mathrm{div},q} \|p^{\varepsilon_n} - p\|_q \quad \text{for all } q \in (1, \infty).
\end{aligned} \tag{3.110}$$

Note that $\int_\Omega (p^{\varepsilon_n} - p)\,\mathrm{d}x = 0$ and from $p^{\varepsilon_n} \rightharpoonup p$ weakly in $\mathbf{L}^{r'}(\Omega)$ it follows

$$\begin{aligned}
\boldsymbol{\psi}^n &\rightharpoonup \mathbf{0} \quad \text{weakly in } \mathbf{W}^{1,r'}(\Omega)^d \\
\boldsymbol{\psi}^n &\to \mathbf{0} \quad \text{strongly in } \mathbf{L}^q(\Omega)^d,\ q \in \langle 1, \frac{dr'}{r'-d}).
\end{aligned}$$

This gives us

$$\begin{aligned}
\|p^{\varepsilon_n} - p\|_2^2 &= -(p, p^{\varepsilon_n} - p) - \langle \boldsymbol{b}, \boldsymbol{\psi}^n \rangle + \\
&\quad + \int_\Omega v_i^{\varepsilon_n} \frac{\partial \boldsymbol{v}^{\varepsilon_n}}{\partial x_i} \cdot \boldsymbol{\psi}^n \,\mathrm{d}x + \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^{\varepsilon_n}) \boldsymbol{v}^{\varepsilon_n} \cdot \boldsymbol{\psi}^n \,\mathrm{d}x + \\
&\quad + \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : \boldsymbol{D}(\boldsymbol{\psi}^n)\,\mathrm{d}x + \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{\psi}^n) = \\
&= \int_\Omega [\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : \boldsymbol{D}(\boldsymbol{\psi}^n)\,\mathrm{d}x + \delta_2(\varepsilon_n).
\end{aligned}$$

Let us denote $p^s := p + s(p^{\varepsilon_n} - p)$, $\boldsymbol{D}^s := \boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))$ and consider

$$\begin{aligned}
[\boldsymbol{S}(p^{\varepsilon_n}, \boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] &= \int_0^1 \frac{\partial \boldsymbol{S}(p^s, \boldsymbol{D}^s)}{\partial \boldsymbol{D}} (\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}s + \\
&\quad + \int_0^1 \frac{\partial \boldsymbol{S}(p^s, \boldsymbol{D}^s)}{\partial p} (p^{\varepsilon_n} - p)\,\mathrm{d}s \\
&=: I_1 + I_2.
\end{aligned}$$

Since the derivatives of $\boldsymbol{S}$ are supposed to fulfil **(1)** and **(2)** we conclude

$$|I_1| \overset{(\mathbf{2})}{\leq} C_2 \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|\,\mathrm{d}s$$

and

$$|I_2| \overset{(\mathbf{1})}{\leq} \gamma_0 |p^{\varepsilon_n} - p|,$$

which implies

$$\begin{aligned}
\|p^{\varepsilon_n} - p\|_2^2 &\leq \gamma_0 \int_\Omega |p^{\varepsilon_n} - p| |\boldsymbol{D}(\boldsymbol{\psi}^n)|\,\mathrm{d}x + \\
&\quad + C_2 \int_\Omega \int_0^1 (1 + |\boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})||\boldsymbol{D}(\boldsymbol{\psi}^n)|\,\mathrm{d}s\,\mathrm{d}x + \\
&\quad + \delta_2(\varepsilon_n).
\end{aligned}$$

Recalling the definition of Y^n from (3.105) and using the fact that $(1+\omega^2)^{\frac{r-2}{2}} \leq (1+\omega^2)^{\frac{r-2}{4}}$ as $r < 2$, the Hölder inequality (once used on the integral over $\Omega \times (0,1)$) gives

$$\|p^{\varepsilon_n} - p\|_2^2 \leq \gamma_0 \|p^{\varepsilon_n} - p\|_2 \|\nabla \boldsymbol{\psi}^n\|_2 + C_2 \sqrt{Y^n} \|\nabla \boldsymbol{\psi}^n\|_2 + \delta_2(\varepsilon_n).$$

The estimate in (3.110) with $q = 2$ then implies

$$\|p^{\varepsilon_n} - p\|_2^2 \leq \gamma_0 C_{\mathrm{div},2} \|p^{\varepsilon_n} - p\|_2^2 + C_2 C_{\mathrm{div},2} \sqrt{Y^n} \|p^{\varepsilon_n} - p\|_2 + \delta_2(\varepsilon_n),$$

which leads to

$$(1 - \gamma_0 C_{\mathrm{div},2}) \|p^{\varepsilon_n} - p\|_2 \leq C_2 C_{\mathrm{div},2} \sqrt{Y^n} + \delta_3(\varepsilon_n). \tag{3.111}$$

Putting this estimate into (3.109) we obtain

$$\alpha \frac{C_1}{2} Y^n + (1-\alpha) C \|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq \frac{\gamma_0^2}{2C_1} \frac{C_2^2 C_{\mathrm{div},2}^2}{(1 - \gamma_0 C_{\mathrm{div},2})^2} Y^n + \delta_4(\varepsilon_n).$$

Recalling $\gamma_0 < \frac{1}{C_{\text{div},2}} \frac{C_1}{C_2+C_1}$ from the assumption **(2)** we can choose $\alpha \in (0,1)$ such that $\gamma_0 < \frac{1}{C_{\text{div},2}} \frac{\sqrt{\alpha} C_1}{C_2+\sqrt{\alpha} C_1}$, which is enough to show that

$$\alpha \frac{C_1}{2} - \frac{\gamma_0^2}{2C_1} \frac{C_2^2 C_{\text{div},2}^2}{(1-\gamma_0 C_{\text{div},2})^2} > 0,$$

thus we can conclude

$$\lim_{n\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 = 0 \tag{3.112}$$

and

$$\lim_{n\to\infty} Y^n = 0. \tag{3.113}$$

The last fact and (3.111) then imply that

$$\lim_{n\to\infty} \|p^{\varepsilon_n} - p\|_2 = 0. \tag{3.114}$$

The convergence (3.103) and (3.104) then follow and the proof is complete.

Existence of approximations

In this section we show that for $\varepsilon > 0$ fixed there is a solution $(\boldsymbol{v}, p) = (\boldsymbol{v}^\varepsilon, p^\varepsilon)$ fulfilling (3.76)-(3.78). Since $\varepsilon > 0$ is fixed, the dependence of the quantities on ε is not designated in what follows. The proof is via Galerkin approximations, similarly as in the Stokes case.

Let $\{\alpha^k\}_{k=1}^\infty$ be a basis in $\mathbf{W}^{1,2}(\Omega)$ and $\{\boldsymbol{a}^k\}_{k=1}^\infty$ be a basis of $\mathbf{W}_0^{1,r}(\Omega)^d$. We look for approximations p^N and $\boldsymbol{v}^N$ in the form

$$p^N = \sum_{k=1}^N c_k^N \alpha_k \quad \text{and} \quad \boldsymbol{v}^N = \boldsymbol{\Phi}^\eta + \sum_{k=1}^N d_k^N \boldsymbol{a}^k = \boldsymbol{\Phi}^\eta + \boldsymbol{u}^N,$$

where $\boldsymbol{\Phi}^\eta$ goes from Lemma 3.12, and $\boldsymbol{c}^N = (c_1, \dots, c_N)$ and $\boldsymbol{d}^N = (d_1, \dots, d_N)$ solve the Galerkin system

$$\varepsilon(\nabla p^N, \nabla \alpha^k) + \varepsilon(p^N, \alpha^k) - (\boldsymbol{v}^N, \nabla \alpha^k) = 0, \qquad k = 1, \dots, N, \tag{3.115}$$

$$\begin{aligned} &\int_\Omega v_i^N \frac{\partial \boldsymbol{v}^N}{\partial x_i} \cdot \boldsymbol{a}^l \, \mathrm{d}x + \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^N) \boldsymbol{v}^N \cdot \boldsymbol{a}^l + \\ &+ \int_\Omega \nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2) \boldsymbol{D}(\boldsymbol{v}^N) : \boldsymbol{D}(\boldsymbol{a}^l) \, \mathrm{d}x + \int_\Omega \nabla p^N \cdot \boldsymbol{a}^l \, \mathrm{d}x = \langle \boldsymbol{b}, \boldsymbol{a}^l \rangle, \qquad l = 1, \dots, N. \end{aligned} \tag{3.116}$$

This is a system of $2N$ nonlinear algebraic equations with $2N$ unknowns, the solvability for some η follows from the Brouwer fixed point theorem formulated below and from the following considerations.

Let us define a continuous mapping $\mathcal{P} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$:

$$\begin{aligned} \mathcal{P}_k([\boldsymbol{c}^N, \boldsymbol{d}^N]) &:= \varepsilon(\nabla p^N, \nabla \alpha^k) + \varepsilon(p^N, \alpha^k) - (\boldsymbol{v}^N, \nabla \alpha^k), \quad k = 1, \dots, N, \\ \mathcal{P}_{N+l}([\boldsymbol{c}^N, \boldsymbol{d}^N]) &:= \left(v_i^N \frac{\partial \boldsymbol{v}^N}{\partial x_i}, \boldsymbol{a}^l \right) + \frac{1}{2} \left((\operatorname{div} \boldsymbol{v}^N) \boldsymbol{v}^N, \boldsymbol{a}^l \right) + \left(\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)), \boldsymbol{D}(\boldsymbol{a}^l) \right) + \\ &\quad + (\nabla p^N, \boldsymbol{a}^l) - \langle \boldsymbol{b}, \boldsymbol{a}^l \rangle, \quad l = 1, \dots, N. \end{aligned}$$

Then we see that

$$\mathcal{P}([\boldsymbol{c}^N, \boldsymbol{d}^N]) \cdot [\boldsymbol{c}^N, \boldsymbol{d}^N] = I_C + \varepsilon \|p^N\|_{1,2}^2 + \left(\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)), \boldsymbol{D}(\boldsymbol{u}^N) \right) - \langle \boldsymbol{b}, \boldsymbol{u}^N \rangle, \tag{3.117}$$

$$\text{where } I_C = \left(v_i^N \frac{\partial \boldsymbol{v}^N}{\partial x_i}, \boldsymbol{u}^N \right) + \frac{1}{2} \left((\operatorname{div} \boldsymbol{v}^N) \boldsymbol{v}^N, \boldsymbol{u}^N \right).$$

We can derive the same estimates as in section 3.5.2 and conclude

$$|I_C| \leq d_2(\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2)\eta^{\frac{3r-4}{2r}} + C,$$
$$\big(\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)), \boldsymbol{D}(\boldsymbol{v}^N)\big) \geq \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r^r - C - d_1(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^N)\|_r)^{r-1}\eta^{\frac{1-r}{r}},$$
$$|\langle \boldsymbol{b}, \boldsymbol{u}^N\rangle| \leq C\|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r + d_3\|\boldsymbol{b}\|_{-1,r'}\eta^{\frac{1-r}{r}},$$

and thus

$$\begin{aligned}
&\mathcal{P}([\boldsymbol{c}^N, \boldsymbol{d}^N]) \cdot [\boldsymbol{c}^N, \boldsymbol{d}^N] \geq \\
&\geq \varepsilon\|p^N\|_{1,2}^2 + \frac{C_1}{2r}\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r^r - d_1(|\Omega|^{1/r} + \|\boldsymbol{D}(\boldsymbol{v}^N)\|_r)^{r-1}\eta^{\frac{1-r}{r}} - \\
&\quad -d_2(\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2)\eta^{\frac{3r-4}{2r}} - C\|\boldsymbol{b}\|_{-1,r'}\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r - d_3\|\boldsymbol{b}\|_{-1,r'}\eta^{\frac{1-r}{r}} - C.
\end{aligned}$$

(C being some generally different constants, which don't depend on η). Similarly as in section 3.5.2 we can see that for each $\rho > \rho_0$, $\rho_0 > 0$ great enough we can find such an $\eta > 0$ such that

$$d_1(|\Omega|^{1/r} + |\rho|)^{r-1}\eta^{\frac{1-r}{r}} + d_2(|\rho|^2 + d_4\|\boldsymbol{\Phi}\|_{1,r}^2)\eta^{\frac{3r-4}{2r}} + d_3\|\boldsymbol{b}\|_{-1,r'}\eta^{\frac{1-r}{r}} < \frac{2}{3}\frac{C_1}{2r}|\rho|^r.$$

Then it is clear (as $r > 1$) that setting ρ great enough there holds

$$\frac{1}{3}\frac{C_1}{2r}|\rho|^r - C\|\boldsymbol{b}\|_{-1,r'}|\rho| - C \geq 0,$$

and thus

$$\mathcal{P}([\boldsymbol{c}^N, \boldsymbol{d}^N]) \cdot [\boldsymbol{c}^N, \boldsymbol{d}^N] \geq 0 \tag{3.118}$$

for $|\boldsymbol{d}^N| = \rho$ and all $\boldsymbol{c}^N$. Moreover, for each $|\boldsymbol{d}^N| < \rho$ (3.118) holds for $\boldsymbol{c}^N$ great enough, say $|\boldsymbol{c}^N| \geq \sigma(|\boldsymbol{d}^N|)$, where σ is bounded.

Applying the Brouwer fixed point theorem 3.16 formulated below we thus obtain a solution $(\boldsymbol{v}^N, p^N)$ fulfilling

$$\|p^N\|_{1,2} + \|\boldsymbol{D}(\boldsymbol{v}^N)\|_r \leq C \tag{3.119}$$

C being a constant which doesn't depend on N.

We would like to make a remark that from Theorem 3.16 also follows the estimation

$$\|\boldsymbol{D}(\boldsymbol{v}^N)\|_r \leq c(\rho), \tag{3.120}$$

where $c(\rho)$ doesn't even depend on ε (note that ρ also doesn't depend on ε). This fact could be used in passing the limit $\varepsilon \to 0$.

We can then find a subsequence (which we denote same as the original sequence) such that

$$\boldsymbol{v}^N \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } \mathbf{W}^{1,r}(\Omega)^d \tag{3.121a}$$
$$\boldsymbol{v}^N \to \boldsymbol{v} \qquad \text{strongly in } \mathbf{L}^q(\Omega)^d \text{ for } q < \frac{dr}{d-r} \tag{3.121b}$$
$$p^N \rightharpoonup p \qquad \text{weakly in } \mathbf{W}^{1,2}(\Omega) \tag{3.121c}$$
$$p^N \to p \qquad \text{strongly in } \mathbf{L}^2(\Omega) \tag{3.121d}$$
$$\nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N) \rightharpoonup \boldsymbol{\chi} \qquad \text{weakly in } \mathbf{L}^{r'}(\Omega)^{d\times d}, \tag{3.121e}$$

which allows us to pass to the limit in (3.115)-(3.116). We obtain

$$\varepsilon(\nabla p, \nabla\xi) + \varepsilon(p, \xi) + (\operatorname{div}\boldsymbol{v}, p^N) = 0 \qquad \forall\xi \in \mathbf{W}^{1,2}(\Omega) \tag{3.122}$$
$$\begin{aligned}
&\left(v_i\frac{\partial\boldsymbol{v}}{\partial x_i}, \boldsymbol{\psi}\right) + \frac{1}{2}((\operatorname{div}\boldsymbol{v})\boldsymbol{v}, \boldsymbol{\psi}) + \\
&\qquad +(\boldsymbol{\chi}, \boldsymbol{D}(\boldsymbol{\psi})) - (p, \operatorname{div}\boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi}\rangle \qquad \forall\boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d.
\end{aligned} \tag{3.123}$$

In particular, testing $\xi := p$ in (3.122) and $\boldsymbol{\psi} := \boldsymbol{u} = \boldsymbol{v} - \boldsymbol{\Phi}^{\boldsymbol{\eta}}$ in (3.123) and summing the equations we obtain

$$\left(v_i \frac{\partial \boldsymbol{v}}{\partial x_i}, \boldsymbol{u}\right) + \frac{1}{2}\left((\operatorname{div} \boldsymbol{v})\boldsymbol{v}, \boldsymbol{u}\right) + \varepsilon \|p\|_{1,2}^2 + (\boldsymbol{\chi}, \boldsymbol{D}(\boldsymbol{u})) = \langle \boldsymbol{b}, \boldsymbol{u}\rangle. \tag{3.124}$$

All we need in order to complete the proof is to identify $\boldsymbol{\chi}$ as $\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))$. We do it as soon as we show that

$$\boldsymbol{D}(\boldsymbol{v}^N) \to \boldsymbol{D}(\boldsymbol{v}) \quad \text{and} \quad p^N \to p \quad \text{a. e. in } \Omega \tag{3.125}$$

at least for a subsequence. From Vitali's theorem (formulated in lemma 3.4) we conclude, in the same way as in the Stokes case,

$$\int_\Omega \nu(p^N, |\boldsymbol{D}(\boldsymbol{v}^N)|^2)\boldsymbol{D}(\boldsymbol{v}^N) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x \to \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x = \int_\Omega \boldsymbol{\chi} : \boldsymbol{D}(\boldsymbol{\psi})\,\mathrm{d}x.$$

In order to conclude (3.125) it is enough to show, at least for a subsequence, that

$$\lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r = 0.$$

Since we know (3.121d), (3.125) then follows.

Let us recall (3.106) and (3.107) with $p^{\varepsilon_n} = p^N$ and $\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) = \boldsymbol{D}(\boldsymbol{v}^N)$. We have

$$\begin{aligned}
C\|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 &\le \\
\le\; & \int_\Omega [\boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) - \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v}))] : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 = \\
=\; & \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u}^N)\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x - \\
& - \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 = \\
\stackrel{(3.117)}{=}\; & \langle \boldsymbol{b}, \boldsymbol{u}^N\rangle - I_C - \varepsilon\|p^N\|_2^2 - \varepsilon\|\nabla p^N\|_2^2 + \frac{\gamma_0^2}{2C_1}\|p^N - p\|_2^2 - \\
& - \int_\Omega \boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) : (\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v}))\,\mathrm{d}x - \int_\Omega \boldsymbol{S}(p^N, \boldsymbol{D}(\boldsymbol{v}^N)) : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x.
\end{aligned}$$

Using $\lim_{N\to\infty} \|\nabla p^N\|_2^2 \ge \liminf_{N\to\infty} \|\nabla p^N\|_2^2 \ge \|\nabla p\|_2^2$ we obtain

$$\varepsilon\|\nabla p\|_2^2 + C \cdot \lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \le \langle \boldsymbol{b}, \boldsymbol{u}\rangle - \varepsilon\|p\|_2^2 - \int_\Omega \chi : \boldsymbol{D}(\boldsymbol{u})\,\mathrm{d}x.$$

Then from (3.124) directly follows

$$C \cdot \lim_{N\to\infty} \|\boldsymbol{D}(\boldsymbol{v}^N) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \le 0 \tag{3.126}$$

which implies (3.125) and the proof is thus complete. □

3.6 Uniqueness of solutions

3.6.1 Uniqueness of solution to the generalized Stokes system

Theorem 3.14 (Uniqueness of solution to the system ($\mathbf{P}_\mathrm{S}$)) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with the Lipschitz boundary $\partial\Omega$, $d = 2$ or 3. Let the assumptions* **(1)** *and* **(2)** *be satisfied with $r \in (1, 2)$.*

Then the weak solution $(\boldsymbol{v}, p)$

$$\boldsymbol{v} \in \mathbf{W}^{1,r}_{\operatorname{div}}(\Omega)^d \quad \textit{and} \quad p \in \mathbf{L}^{r'}(\Omega),\ r' = \frac{r}{r-1} \tag{3.127}$$

to the problem (P_S) is unique.

PROOF. Let $(\boldsymbol{v}^1, p^1)$, $(\boldsymbol{v}^2, p^2)$ be two solutions to the same boundary condition. Remind that there holds (3.127) and each solution $(\boldsymbol{v}, p)$ has to fulfil

$$\big(\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{\psi})\big) - (p, \operatorname{div} \boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle, \quad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d, \tag{3.128}$$

where $\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})) = \nu(p, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v})$, and

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{a. e. in } \Omega. \tag{3.129}$$

Subtracting the equations (3.128) for these two solutions we obtain

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{\psi})\big) = (p^1 - p^2, \operatorname{div} \boldsymbol{\psi}) \quad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d. \tag{3.130}$$

Let us set $\boldsymbol{\psi} := \tilde{\boldsymbol{\psi}}$ where $\tilde{\boldsymbol{\psi}}$ solves:

$$\begin{aligned} \operatorname{div} \tilde{\boldsymbol{\psi}} &= p^1 - p^2 \quad \text{in } \Omega \\ \tilde{\boldsymbol{\psi}} &= \mathbf{0} \quad \text{on } \partial\Omega \\ \|\tilde{\boldsymbol{\psi}}\|_{1,2} &\le C_{\mathrm{div},2} \|p^1 - p^2\|_2. \end{aligned} \tag{3.131}$$

It then follows from (3.130) that

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) = \|p^1 - p^2\|_2^2. \tag{3.132}$$

Denoting further $p^s := p^1 + s(p^2 - p^1)$ and $\boldsymbol{D}^s := \boldsymbol{D}(\boldsymbol{v}^1) + s(\boldsymbol{D}(\boldsymbol{v}^2) - \boldsymbol{D}(\boldsymbol{v}^1))$ we can write

$$\begin{aligned} &(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2))) : \boldsymbol{D}(\tilde{\boldsymbol{\psi}}) = \\ &= \int_0^1 (p^1 - p^2) \frac{\partial \boldsymbol{S}(p^s, \boldsymbol{D}(\boldsymbol{v}^1))}{\partial p} : \boldsymbol{D}(\tilde{\boldsymbol{\psi}}) \,\mathrm{d}s + \int_0^1 \frac{\partial \boldsymbol{S}(p^2, \boldsymbol{D}^s)}{\partial \boldsymbol{D}} \cdot \Big(\boldsymbol{D}(\tilde{\boldsymbol{\psi}}) \otimes (\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2))\Big) \,\mathrm{d}s, \end{aligned}$$

and assumptions **(1)** and **(2)** then lead to

$$\begin{aligned} &\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) \le \\ &\le \int_\Omega \gamma_0 |p^1 - p^2| |\boldsymbol{D}(\tilde{\boldsymbol{\psi}})| \,\mathrm{d}x + \int_\Omega \int_0^1 C_2 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)| |\boldsymbol{D}(\tilde{\boldsymbol{\psi}})| \,\mathrm{d}s \,\mathrm{d}x \le \\ &\overset{\text{Hölder}}{\le} \gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 \left(\int_\Omega \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{r-2} |\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)|^2 \,\mathrm{d}s \,\mathrm{d}x\right)^{\frac{1}{2}} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2, \end{aligned}$$

using that $(1 + \omega^2)^{r-2} \le (1 + \omega^2)^{\frac{r-2}{2}}$ as $r < 2$,

$$\begin{aligned} &\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) \le \\ &\le \gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 \left(\int_\Omega \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)|^2 \,\mathrm{d}s \,\mathrm{d}x\right)^{\frac{1}{2}} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2. \end{aligned} \tag{3.133}$$

Setting $\boldsymbol{\psi} := \boldsymbol{v}^1 - \boldsymbol{v}^2$ in (3.130) we obtain (as $\operatorname{div} \boldsymbol{v}^1 = \operatorname{div} \boldsymbol{v}^2 = 0$ a. e. in Ω)

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)\big) = 0.$$

From Lemma 3.9 then follows that

$$\int_\Omega \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)|^2 \,\mathrm{d}s \,\mathrm{d}x \le \frac{\gamma_0^2}{C_1^2} \|p^1 - p^2\|_2^2 \tag{3.134}$$

which together with (3.133) leads to

$$\begin{aligned}
&\big(\boldsymbol{S}(p^1,\boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2,\boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) \le \\
&\le \gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 \frac{\gamma_0}{C_1} \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 = \\
&= \gamma_0 \frac{C_1 + C_2}{C_1} \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2.
\end{aligned}$$

Recalling $\|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 \le \|\tilde{\boldsymbol{\psi}}\|_{1,2}$, (3.131) and (3.132) we finally conclude that

$$\|p^1 - p^2\|_2^2 \le \gamma_0 \frac{C_1 + C_2}{C_1} C_{\mathrm{div},2} \|p^1 - p^2\|_2^2.$$

Since we assume $\gamma_0 < \frac{1}{C_{\mathrm{div},2}} \frac{C_1}{C_1 + C_2}$, this directly implies that

$$\|p^1 - p^2\|_2 = 0$$

i. e.

$$p^1 = p^2 \quad \text{a. e. in } \Omega. \tag{3.135}$$

Looking back to (3.134),

$$\int_\Omega \int_0^1 (1 + |\boldsymbol{D}^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)|^2 \,\mathrm{d}s\,\mathrm{d}x \le \frac{\gamma_0^2}{C_1^2} \|p^1 - p^2\|_2^2 = 0$$

we see that

$$\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2) = 0 \quad \text{a. e. in } \Omega$$

which implies, due to Korn's inequality and since $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ fulfil the same boundary conditions,

$$\boldsymbol{v}^1 = \boldsymbol{v}^2 \quad \text{a. e. in } \Omega. \tag{3.136}$$

□

3.6.2 Uniqueness of solution to the generalized Navier-Stokes system

Theorem 3.15 (Uniqueness of solution to the system (P)) *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with the Lipschitz boundary $\partial\Omega$, $d = 2$ or 3. Let the assumptions **(1)** and **(2)** be satisfied with r fulfilling*

$$\frac{3d}{d+2} < r < 2.$$

Let $(\boldsymbol{v}^1, p^1)$, $(\boldsymbol{v}^2, p^2)$ be two weak solutions to the problem (P),

$$\boldsymbol{v}^1, \boldsymbol{v}^2 \in \mathbf{W}^{1,r}_{\mathrm{div}}(\Omega)^d \quad \textit{and} \quad p^1, p^2 \in \mathbf{L}^{r'}(\Omega),\ r' = \frac{r}{r-1},$$

both fulfilling

$$\|\boldsymbol{v}^1\|_{1,r}, \|\boldsymbol{v}^2\|_{1,r} < \delta,$$

where δ is some small, positive constant, that will be specified.

Then $\boldsymbol{v}^1 = \boldsymbol{v}^2$ a. e. and $p^1 = p^2$ a. e.

PROOF. Each solution $(\boldsymbol{v}, p)$ has to fulfill

$$\left(v_i \frac{\partial \boldsymbol{v}}{\partial x_i}, \boldsymbol{\psi} \right) + \big(\boldsymbol{S}(p, \boldsymbol{D}(\boldsymbol{v})), \boldsymbol{D}(\boldsymbol{\psi})\big) - (p, \operatorname{div} \boldsymbol{\psi}) = \langle \boldsymbol{b}, \boldsymbol{\psi} \rangle, \tag{3.137}$$
$$\text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d,$$

and

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{a. e. in } \Omega.$$

Subtracting the equations (3.137) we obtain

$$\begin{aligned} &\left(v_i^1 \frac{\partial \boldsymbol{v}^1}{\partial x_i} - v_i^2 \frac{\partial \boldsymbol{v}^2}{\partial x_i}, \boldsymbol{\psi} \right) + \\ &+ \big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{\psi})\big) = (p^1 - p^2, \operatorname{div} \boldsymbol{\psi}), \\ &\qquad \text{for all } \boldsymbol{\psi} \in \mathbf{W}_0^{1,r}(\Omega)^d, \end{aligned} \tag{3.138}$$

where the first term can be rewritten as

$$\int_\Omega (v_i^1 - v_i^2) \frac{\partial v_j^1}{\partial x_i} \psi_j \,\mathrm{d}x + \int_\Omega v_i^2 \frac{\partial (v_j^1 - v_j^2)}{\partial x_i} \psi_j \,\mathrm{d}x. \tag{3.139}$$

Let us set $\boldsymbol{\psi} = \boldsymbol{v}^1 - \boldsymbol{v}^2$ and we conclude (as $\operatorname{div} \boldsymbol{v}^1 = \operatorname{div} \boldsymbol{v}^2 = 0$) that

$$\begin{aligned} &\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\big) = \\ &\qquad = \int_\Omega (v_i^1 - v_i^2) \frac{\partial v_j^1}{\partial x_i} (v_j^1 - v_j^2) \,\mathrm{d}x + \frac{1}{2} \int_\Omega v_i^2 \frac{\partial}{\partial x_i} (v_j^1 - v_j^2)^2 \,\mathrm{d}x. \end{aligned}$$

The last integral vanishes due to the Green theorem, and Hölder inequality then implies

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\big) \le \|\nabla \boldsymbol{v}^1\|_r \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'}^2.$$

Applying the imbedding theorem, using the fact that $2r' = 2\frac{r}{r-1} < \frac{rd}{d-r}$ as soon as $r > \frac{3r}{d+2}$, we see that

$$\|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'} < C_{\mathrm{IMB},2r'} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r},$$

and we can thus write

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\big) \le C_{\mathrm{IMB},2r'}^2 \|\nabla \boldsymbol{v}^1\|_r \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}^2. \tag{3.140}$$

The Lemma 3.9 shows that

$$\begin{aligned} Y &\le \frac{\gamma_0^2}{C_1^2} \|p^1 - p^2\|_2^2 + \frac{2}{C_1} \big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)\big) \\ &\overset{(3.140)}{\le} \frac{\gamma_0^2}{C_1^2} \|p^1 - p^2\|_2^2 + C_{\mathrm{IMB},2r'}^2 \frac{2}{C_1} \|\nabla \boldsymbol{v}^1\|_r \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}^2, \end{aligned}$$

which also means

$$Y^{1/2} \le \frac{\gamma_0}{C_1} \|p^1 - p^2\|_2 + C_{\mathrm{IMB},2r'} \left(\frac{2}{C_1} \|\nabla \boldsymbol{v}^1\|_r \right)^{\frac{1}{2}} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r}, \tag{3.141}$$

where we have defined Y as

$$\begin{aligned} Y &:= \int_\Omega \int_0^1 (1 + |D^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^1 - \boldsymbol{v}^2)|^2 \,\mathrm{d}s \,\mathrm{d}x, \\ D^s &:= \boldsymbol{D}(\boldsymbol{v}^2) + s(\boldsymbol{D}(\boldsymbol{v}^1) - \boldsymbol{D}(\boldsymbol{v}^2)). \end{aligned}$$

Let us set $\boldsymbol{\psi} := \tilde{\boldsymbol{\psi}}$ where $\tilde{\boldsymbol{\psi}}$ solves:

$$\begin{aligned} \operatorname{div} \tilde{\boldsymbol{\psi}} &= p^1 - p^2 \quad \text{in } \Omega \\ \tilde{\boldsymbol{\psi}} &= \mathbf{0} \quad \text{on } \partial\Omega \\ \|\tilde{\boldsymbol{\psi}}\|_{1,2} &\leq C_{\mathrm{div},2} \|p^1 - p^2\|_2. \end{aligned} \tag{3.142}$$

From assumptions **(1)** and **(2)** we deduce in the same way as in (3.133), that

$$\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) \leq \gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 Y^{1/2} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2, \tag{3.143}$$

which together with (3.141) gives

$$\begin{aligned} &\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) \leq \gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + \\ &+ C_2 \frac{\gamma_0}{C_1} \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 C_{\mathrm{IMB},2r'} \left(\frac{2}{C_1} \|\nabla \boldsymbol{v}^1\|_r \right)^{\frac{1}{2}} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2. \end{aligned} \tag{3.144}$$

Setting $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}}$ into equation (3.137) (and using (3.139)) we obtain

$$\begin{aligned} &\int_\Omega (v_i^1 - v_i^2) \frac{\partial v_j^1}{\partial x_i} \tilde{\psi}_j \,\mathrm{d}x + \int_\Omega v_i^2 \frac{\partial (v_j^1 - v_j^2)}{\partial x_i} \tilde{\psi}_j \,\mathrm{d}x \quad + \\ &\quad + \big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) = \|p^1 - p^2\|_2^2, \end{aligned}$$

where the convective term integrals can be fashioned using the Green theorem, writing

$$\begin{aligned} \int_\Omega v_i^2 \frac{\partial (v_j^1 - v_j^2)}{\partial x_i} \tilde{\psi}_j \,\mathrm{d}x &= - \int_\Omega v_i^2 (v_j^1 - v_j^2) \frac{\partial \tilde{\psi}_j}{\partial x_i} \,\mathrm{d}x, \\ \int_\Omega (v_i^1 - v_i^2) \frac{\partial v_j^1}{\partial x_i} \tilde{\psi}_j \,\mathrm{d}x &= - \int_\Omega (v_i^1 - v_i^2) v_j^1 \frac{\partial \tilde{\psi}_j}{\partial x_i} \,\mathrm{d}x, \end{aligned}$$

and with help of Hölder inequality and Korn's inequality ($\|\boldsymbol{u}\|_{1,2} \leq k_2 \|\boldsymbol{D}(\boldsymbol{u})\|_2$) and the imbedding theorem ($\|\boldsymbol{u}\|_q \leq C_{\mathrm{IMB},q} \|\boldsymbol{u}\|_{1,r}$) each of them can be estimated by

$$\begin{aligned} \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'} \|\nabla \tilde{\boldsymbol{\psi}}\|_2 &\overset{\text{Korn}}{\leq} k_2 \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 \\ &\overset{\text{Imbedding}}{\leq} k_2 C_{\mathrm{IMB},2r'} \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 \end{aligned}$$

such that

$$\begin{aligned} \|p^1 - p^2\|_2^2 \leq &\big(\boldsymbol{S}(p^1, \boldsymbol{D}(\boldsymbol{v}^1)) - \boldsymbol{S}(p^2, \boldsymbol{D}(\boldsymbol{v}^2)), \boldsymbol{D}(\tilde{\boldsymbol{\psi}})\big) + \\ &+ 2k_2 C_{\mathrm{IMB},2r'} \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \|\boldsymbol{D}(\boldsymbol{\psi})\|_2. \end{aligned} \tag{3.145}$$

From (3.145) and (3.144) we can see that

$$\begin{aligned} \|p^1 - p^2\|_2^2 \leq \ &\gamma_0 \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + C_2 \frac{\gamma_0}{C_1} \|p^1 - p^2\|_2 \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + \\ &+ C_2 C_{\mathrm{IMB},2r'} \left(\frac{2}{C_1} \|\nabla \boldsymbol{v}^1\|_r \right)^{\frac{1}{2}} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \|\boldsymbol{D}(\tilde{\boldsymbol{\psi}})\|_2 + \\ &+ 2k_2 C_{\mathrm{IMB},2r'} \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{2r'} \|\boldsymbol{D}(\boldsymbol{\psi})\|_2, \end{aligned}$$

and (3.142) then gives us

$$\begin{aligned} &\|p^1 - p^2\|_2^2 \leq C_{\mathrm{div},2} \gamma_0 \|p^1 - p^2\|_2^2 + C_{\mathrm{div},2} C_2 \frac{\gamma_0}{C_1} \|p^1 - p^2\|_2^2 + \\ &+ C_{\mathrm{div},2} C_{\mathrm{IMB},2r'} \left[C_2 \left(\frac{2}{C_1} \|\nabla \boldsymbol{v}^1\|_r \right)^{\frac{1}{2}} + 2k_2 \|\boldsymbol{v}^1 + \boldsymbol{v}^2\|_{2r} \right] \|\boldsymbol{v}^1 - \boldsymbol{v}^2\|_{1,r} \|p^1 - p^2\|_2, \end{aligned} \tag{3.146}$$

from which we directly observe

$$A_1\|p^1-p^2\|_2 \le A_2\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r}$$

as we have defined

$$A_1 := \frac{1}{C_{\mathrm{div},2}} - \gamma_0\frac{C_1+C_2}{C_1} \tag{3.147}$$

$$A_2 := C_{\mathrm{IMB},2r'}\left[C_2\left(\frac{2}{C_1}\|\nabla\boldsymbol{v}^1\|_r\right)^{\frac{1}{2}} + 2k_2\|\boldsymbol{v}^1+\boldsymbol{v}^2\|_{2r}\right]. \tag{3.148}$$

As soon as the assumption **(1)** is met and A_1 is thus positive, we conclude

$$\|p^1-p^2\|_2 \le \frac{A_2}{A_1}\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r}. \tag{3.149}$$

Introducing this last inequality into (3.141) it follows

$$Y^{1/2} \le \left[\frac{\gamma_0}{C_1}\frac{A_2}{A_1} + C^2_{\mathrm{IMB},2r'}\left(\frac{2}{C_1}\|\nabla\boldsymbol{v}^1\|_r\right)^{\frac{1}{2}}\right]\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r}. \tag{3.150}$$

Recall then (3.155) from the proof of Lemma 3.17 (we just need to put $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ instead of $\boldsymbol{v}^{\varepsilon_n}$ and $\boldsymbol{v}$), observe

$$\|\boldsymbol{D}(\boldsymbol{v}^1)-\boldsymbol{D}(\boldsymbol{v}^2)\|_r^2 \le \|1+|\boldsymbol{D}(\boldsymbol{v}^1)|+|\boldsymbol{D}(\boldsymbol{v}^2)|\|_r^{2-r},$$

and applying the Korn's inequality $\|\omega\|_{1,r} \le k_r\|\boldsymbol{D}(\omega)\|_r$,

$$\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r}^2 \le k_r^2(|\Omega|^{1/r}+\|\boldsymbol{v}^1\|_r+\|\boldsymbol{v}^2\|_r)^{2-r}\,Y,$$

we finally conclude from (3.150), that

$$\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r} \le A_3\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r} \tag{3.151}$$

where A_3 is defined as

$$A_3 := k_r(|\Omega|^{1/r}+\|\boldsymbol{v}^1\|_r+\|\boldsymbol{v}^2\|_r)^{2-r}\left[\frac{\gamma_0}{C_1}\frac{A_2}{A_1} + C^2_{\mathrm{IMB},2r'}\left(\frac{2}{C_1}\|\nabla\boldsymbol{v}^1\|_r\right)^{\frac{1}{2}}\right].$$

When $\|\boldsymbol{v}^1\|_{1,r}, \|\boldsymbol{v}^2\|_{1,r} < \delta$, we easily see that

$$\begin{aligned} A_2 &\le C_{\mathrm{IMB},2r'}\left[C_2\left(\frac{2\delta}{C_1}\right)^{\frac{1}{2}} + 4k_2C_{\mathrm{IMB},2r}\,\delta\right] \\ &= C_{\mathrm{IMB},2r'}\left[C_2\left(\frac{2}{C_1}\right)^{\frac{1}{2}} + 4k_2C_{\mathrm{IMB},2r}\,\delta^{\frac{1}{2}}\right]\delta^{\frac{1}{2}} \\ A_3 &\le k_r(|\Omega|^{1/r}+2\delta)^{2-r}\left[\frac{\gamma_0}{C_1A_1}A_2 + C_{\mathrm{IMB},2r'}\left(\frac{2\delta}{C_1}\right)^{\frac{1}{2}}\right] \end{aligned}$$

so it is clear that setting δ small enough, we achieve $A_3 < 1$. With that (3.151) gives $\|\boldsymbol{v}^1-\boldsymbol{v}^2\|_{1,r} = 0$, (3.149) then gives $\|p^1-p^2\|_2 = 0$ and the proof is complete. □

3.7 Auxiliary lemmas

Lemma 3.16 (Fixed point) *Let $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let K be an open set, star-shaped with respect to $\mathbf{0}$, i. e. there exists a continuous function $\kappa : \partial B_1(\mathbf{0}) \to (0, +\infty)$, such that*

$$\boldsymbol{x}\kappa\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \in \overline{K}, \quad \forall \boldsymbol{x} \in B_1(\mathbf{0}), \boldsymbol{x} \neq \mathbf{0}$$
$$\boldsymbol{x}\kappa(\boldsymbol{x}) \in \partial K, \quad \forall \boldsymbol{x} \in \partial B_1(\mathbf{0}),$$

($B_1(\mathbf{0})$ being a closed ball in $\mathbb{R}^n$). Let $\mathcal{P}$ fulfil

$$\big(\mathcal{P}(\boldsymbol{x}), \boldsymbol{x}\big) \geq 0, \quad \forall \boldsymbol{x} \in \partial K. \tag{3.152}$$

Then there exists $\boldsymbol{x}_0 \in \overline{K}$ such that $\mathcal{P}(\boldsymbol{x}_0) = \mathbf{0}$.

<u>Proof</u>. We use the Brouwer theorem formulated in Lemma 3.6, see Evans [18] or Lions [17]. Assume that such an $\boldsymbol{x}_0$ doesn't exist, i. e.

$$\mathcal{P}(\boldsymbol{x}) \neq 0, \quad \forall \boldsymbol{x} \in \overline{K}.$$

Define a mapping $\boldsymbol{a} : B_1(\mathbf{0}) \to \overline{K}$:

$$\boldsymbol{a}(\boldsymbol{x}) := \begin{cases} \boldsymbol{x}\kappa\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), & \boldsymbol{x} \neq \mathbf{0} \\ \mathbf{0}, & \boldsymbol{x} = \mathbf{0}, \end{cases} \tag{3.153}$$

we see that $\boldsymbol{a}$ is continuous and

$$\boldsymbol{a}(\boldsymbol{x}) \in \partial K \quad \forall \boldsymbol{x} \in \partial B_1(\mathbf{0}). \tag{3.154}$$

Define a continuous mapping $\mathcal{M} : B_1(\mathbf{0}) \to B_1(\mathbf{0})$ (in fact, it is on $\partial B_1(\mathbf{0})$):

$$\mathcal{M}(\boldsymbol{x}) := \frac{-\mathcal{P}(\boldsymbol{a}(\boldsymbol{x}))}{|\mathcal{P}(\boldsymbol{a}(\boldsymbol{x}))|}.$$

It then follows from the Brouwer theorem that there exists some $\boldsymbol{c} \in B_1(\mathbf{0})$ such that

$$\mathcal{M}(\boldsymbol{c}) = \boldsymbol{c},$$

(moreover $\boldsymbol{c} \in \partial B_1(\mathbf{0})$). We multiply this equality by $\boldsymbol{a}(\boldsymbol{c})$ and using $\boldsymbol{a}(\boldsymbol{c}) \in \partial K$ (3.154) we obtain:

$$0 \overset{(3.152)}{\geq} -\frac{1}{|\mathcal{P}(\boldsymbol{a}(\boldsymbol{c}))|}\big(\mathcal{P}(\boldsymbol{a}(\boldsymbol{c})), \boldsymbol{a}(\boldsymbol{c})\big) = \big(\boldsymbol{c}, \boldsymbol{a}(\boldsymbol{c})\big) \overset{(3.153)}{=} \kappa\left(\frac{\boldsymbol{c}}{|\boldsymbol{c}|}\right)|\boldsymbol{c}|^2 > 0,$$

which gives the contradiction. □

Lemma 3.17 *Assuming $\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})\|_r \leq K$ and $\|\boldsymbol{D}(\boldsymbol{v})\|_r \leq K$, there holds*

$$\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq C(K)Y^n. \tag{3.107}$$

<u>Proof</u>. Let us recall the definitions

$$D^s := \boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})),$$
$$Y^n := \int_\Omega \int_0^1 (1 + |D^s|^2)^{\frac{r-2}{2}} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^2 \,\mathrm{d}s\,\mathrm{d}x.$$

Then

$$\begin{aligned}
\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^r &= \int_\Omega |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^r \,\mathrm{d}x = \\
&= \int_\Omega \left[(1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|)^{r-2} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^2\right]^{\frac{r}{2}} \\
&\qquad (1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|)^{(2-r)\frac{r}{2}} \,\mathrm{d}x \leq \\
&\overset{\text{Hölder}}{\leq} \left(\int_\Omega (1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|)^{r-2} |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})|^2 \,\mathrm{d}x\right)^{\frac{r}{2}} \\
&\qquad \left(\int_\Omega (1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|)^{(2-r)\frac{r}{2}\frac{2}{2-r}} \,\mathrm{d}x\right)^{\frac{2-r}{2}}.
\end{aligned}$$

For all $s \in \langle 0, 1\rangle$ clearly $(1 + |\boldsymbol{D}(\boldsymbol{v}) + s(\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v}))|^2) \leq (1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|)^2$ and thus

$$\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^r \leq (Y^n)^{\frac{r}{2}} \|1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|\|_r^{\frac{2-r}{2}r},$$

i. e.

$$\|\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n}) - \boldsymbol{D}(\boldsymbol{v})\|_r^2 \leq Y^n \|1 + |\boldsymbol{D}(\boldsymbol{v}^{\varepsilon_n})| + |\boldsymbol{D}(\boldsymbol{v})|\|_r^{2-r}. \tag{3.155}$$

From the assumptions then (3.107) follows. □

4 Numerical results

4.1 Numerical method

For numerical simulations we use the software package `featflow`, the finite element method package developed initially to solve the Navier-Stokes equations and modified in order to solve also the Navier-Stokes-like system with the pressure- and the shear- dependent viscosities. Information about the basic numerical methods used in the package, about the efficiency and the mathematical background, as well as the software itself, is available on the Internet on `www.featflow.de`. We will give just a brief survey, which mostly retells what can be found in `featflow` manual [23] and in the book by S. Turek [22].

4.1.1 The finite elements used

For solving the problem (P) the finite element approach is used. Let $\mathbf{T}_h$ be a regular decomposition of the domain Ω into quadrilaterals (see figure 2). An example of decom-

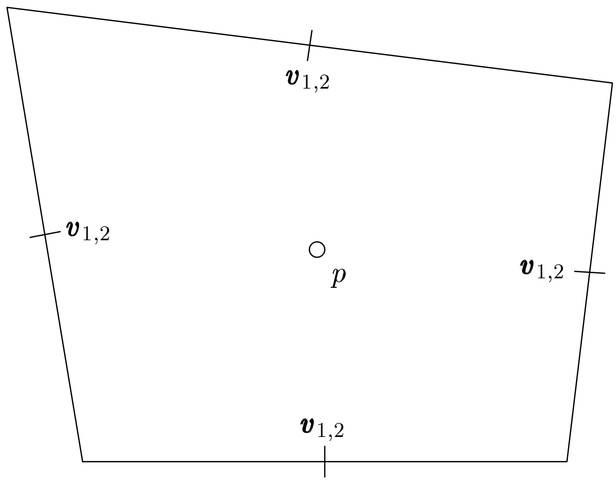


Figure 2: Quadrilateral element geometry

position of the domain can be seen in figure 3. The velocity and the pressure field is then approximated as follows. The $\tilde{Q}1/Q0$ Stokes element (see Turek [22]) uses "rotated bilinear" shape functions for velocity and piecewise constants for the pressure. For each $T \in \mathbf{T}_h$ independently we set

$$\tilde{Q}_1(T) := \{q \in \operatorname{span}\langle \chi^2 - \eta^2, \chi, \eta, 1\rangle\},$$

with respect to the coordinate system (χ, η) spanned by the directions connecting the midpoints of sides of T. We define the corresponding finite element spaces $\boldsymbol{H}_h$ and $\boldsymbol{L}_h$:

$$\begin{aligned}
\boldsymbol{T}_h &\quad \text{be a regular decomposition of the domain } \Omega \text{ into quadrilaterals,} \\
\boldsymbol{L}_h &= \{q_h \in \mathbf{L}^2(\Omega); q_h|_T = \text{const.}, \forall T \in \mathbf{T}_h\}, \\
\boldsymbol{S}_h &= \{v_h \in \mathbf{L}^2(\Omega); v_h|_T \in \tilde{Q}_1(T), \forall T \in \mathbf{T}_h, \\
&\quad v_h \text{ continuous w. r. t. the values in midpoints of edges, and} \\
&\quad v_h \text{ be zero in the midpoints on } \partial\Omega\}, \\
\boldsymbol{H}_h &= \boldsymbol{S}_h \times \boldsymbol{S}_h.
\end{aligned}$$

Since the space $\boldsymbol{H}_h$ is nonconforming, i. e. $\boldsymbol{H}_h \not\subset \mathbf{W}_0^{1,2}(\Omega)$, we have to work with elementwise defined bilinear forms and corresponding energy norms

$$
\begin{aligned}
a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &:= \sum_{T \in \mathbf{T}_h} \int_T \nabla \boldsymbol{u}_h \cdot \nabla \boldsymbol{v}_h \,\mathrm{d}x, \\
\|\boldsymbol{v}_h\|_h &:= (a_h(\boldsymbol{v}_h, \boldsymbol{v}_h))^{1/2}, \\
b_h(q_h, \boldsymbol{v}_h) &:= -\sum_{T \in \mathbf{T}_h} q_h|_T \int_T \operatorname{div} \boldsymbol{v}_h \,\mathrm{d}x.
\end{aligned}
$$

There exists $j_h : \mathbf{L}_0^2(\Omega) \to \boldsymbol{L}_h$ the operator of piecewise constant interpolation, modified to preserve the zero-mean value property, which satisfies for $q \in \mathbf{L}_0^2(\Omega) \cap \mathbf{W}^{1,2}(\Omega)$

$$
\|q - j_h q\|_2 \leq ch\|q\|_{1,2}.
$$

Let $i_h : \mathbf{W}_0^{1,2}(\Omega) \to \boldsymbol{H}_h$ be the global interpolation operator, which is determined by

$$
(i_h \boldsymbol{v})(m_i) = \boldsymbol{v}(m_i), \qquad \text{in all midpoints.}
$$

Then, as it can be found in [22], there holds the optimal error estimate

$$
\|\boldsymbol{v} - i_h \boldsymbol{v}\|_2 + h\|\boldsymbol{v} - i_h \boldsymbol{v}\|_h \leq ch^2 \|\boldsymbol{v}\|_{2,2}, \quad \forall \boldsymbol{v} \in \mathbf{W}_0^{1,2}(\Omega) \cap \mathbf{W}^{2,2}(\Omega),
$$

and, moreover, under the additional assumption that the meshes $\mathbf{T}_h$ are sufficiently uniform, there holds also the stability estimate

$$
\beta \|p_h\|_2 \leq \max_{\boldsymbol{v}_h \in \boldsymbol{H}_h} \frac{b_h(p_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_h}, \qquad \forall p_h \in \boldsymbol{L}_h, \ \int_\Omega p_h \,\mathrm{d}x = 0.
$$

where β is even independent of the mesh aspect ratio.

A convergence analysis for this element pair is refered to [24]. One of the features of this element choice is that it admits the simple upwind strategies which lead to matrices with better properties, these methods are included in `featflow` and we use it without providing any further description. For details see [22].

4.1.2 Discrete formulation of the problem

The discrete formulation for the classical Navier-Stokes problem customary reads:

$$
\begin{aligned}
a_h(\boldsymbol{v}_h, \boldsymbol{\psi}_h) + b_h(p_h, \boldsymbol{\psi}_h) + \eta_h(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{\psi}_h) + b_h(q_h, \boldsymbol{v}_h) &= (\boldsymbol{b}, \boldsymbol{\psi}_h), \\
&\forall \{\boldsymbol{\psi}_h, q_h\} \in \boldsymbol{H}_h \times \boldsymbol{L}_h.
\end{aligned}
$$

where we use the following bilinear forms:

$$
\begin{aligned}
a_h(\boldsymbol{v}_h, \boldsymbol{\psi}_h) &:= \nu \sum_{T \in \mathbf{T}_h} \int_T \nabla \boldsymbol{v}_h \nabla \boldsymbol{\psi}_h \,\mathrm{d}x, \\
b_h(p_h, \boldsymbol{\psi}) &:= \sum_{T \in \mathbf{T}_h} p_h|_T Q_T(\operatorname{div} \boldsymbol{\psi}_h), \\
Q_T(\operatorname{div} \boldsymbol{\psi}_h) &:= \sum_{\Gamma \subset \partial T} |\Gamma| \boldsymbol{\psi}_h(m_\Gamma) \cdot \boldsymbol{n}_\Gamma,
\end{aligned}
$$

($\boldsymbol{n}_\Gamma$ being an outer normal unit vector to the boundary of an element T, m_Γ being a midpoint of the edge Γ) and where the trilinear form η_h is some (upwind, in our case) discretisation of the trilinear form representing the convective term

$$
\eta_h(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{\psi}_h) \ \approx \ \eta(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\psi}) = \int_\Omega u_i \frac{\partial v_j}{\partial x_i} \psi_j \,\mathrm{d}x.
$$

As we solve the generalised Navier-Stokes problem (P), we define a form

$$\mu_h(\boldsymbol{u}_h, p_h, \boldsymbol{v}_h, \boldsymbol{\psi}_h) := \sum_{T\in\mathbf{T}_h} \int_T \nu(p_h, |\boldsymbol{D}(\boldsymbol{u}_h)|^2) D_{ij}(\boldsymbol{v}_h) D_{ij}(\boldsymbol{\psi}_h)\,\mathrm{d}x,$$

which is linear only with respect to $\boldsymbol{v}_h$ and $\boldsymbol{\psi}_h$. The definition of the discrete weak solution to (P) thus reads:

Definition 4.1 *The pair $(\boldsymbol{v}_h, p_h) \in \boldsymbol{H}_h \times \boldsymbol{L}_h$ is a discrete weak solution if for all $(\boldsymbol{\psi}_h, q_h) \in \boldsymbol{H}_h \times \boldsymbol{L}_h$*

$$\begin{aligned} \mu_h(\boldsymbol{v}_h, p_h, \boldsymbol{v}_h, \boldsymbol{\psi}_h) + \eta_h(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{\psi}_h) + b_h(p_h, \boldsymbol{\psi}_h) &= (\boldsymbol{b}, \boldsymbol{\psi}_h), \\ b_h(q_h, \boldsymbol{v}_h) &= \mathbf{0}. \end{aligned}$$

As we look for solution in the form

$$\boldsymbol{v}_h = \sum_{i=1}^N V_h^i \boldsymbol{\sigma}_h^i, \quad p_h = \sum_{i=1}^N P_h^i \pi_h^i,$$

($\{\boldsymbol{\sigma}_h^i\}_{i=1}^N$, $\{\pi_h^i\}_{i=1}^N$ being the bases in $\mathbf{H}_h$, $\mathbf{L}_h$), and as we denote the matrix corresponding to the two nonlinear terms by $\boldsymbol{N}_h(\boldsymbol{V}_h, \boldsymbol{P}_h)$, we get the formula

$$\begin{aligned} \boldsymbol{N}_h(\boldsymbol{V}_h, \boldsymbol{P}_h)\,\boldsymbol{V}_h + \boldsymbol{B}_h \boldsymbol{P}_h &= \boldsymbol{b}_h, \\ \boldsymbol{B}_h^T \boldsymbol{V}_h &= \mathbf{0}. \end{aligned}$$

$\boldsymbol{B}_h$ is the gradient matrix and $-\boldsymbol{B}_h^T$ the transposed divergence matrix.

To solve this system of (nonlinear) algebraic equations, the adaptive fixed point defect correction method is used. The basic iteration looks like:

- Having the previous iterations $\boldsymbol{V}_h^{n-1}$, $\boldsymbol{P}_h^{n-1}$
- calculate the nonlinear residual (defect)

$$\begin{pmatrix} d_{\boldsymbol{v}}^{n-1} \\ d_p^{n-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{N}_h(\boldsymbol{V}_h^{n-1}, \boldsymbol{P}_h^{n-1})\,\boldsymbol{V}_h^{n-1} + \boldsymbol{B}_h \boldsymbol{P}_h^{n-1} - \boldsymbol{b}_h \\ \boldsymbol{B}_h^T \boldsymbol{V}_h^{n-1} \end{pmatrix},$$

- and solve the Oseen-like subproblem

$$\begin{bmatrix} \tilde{\boldsymbol{N}}_h(\boldsymbol{V}_h^{n-1}, \boldsymbol{P}_h^{n-1}) & \boldsymbol{B}_h \\ \boldsymbol{B}_h^T & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{V}_h \\ \boldsymbol{P}_h \end{pmatrix} = \begin{pmatrix} d_{\boldsymbol{v}}^{n-1} \\ d_p^{n-1} \end{pmatrix}.$$

- Choose an appropriate ω^{n-1} and obtain $\boldsymbol{V}_h^n, \boldsymbol{P}_h^n$.

$$\begin{pmatrix} \boldsymbol{V}_h^n \\ \boldsymbol{P}_h^n \end{pmatrix} = \begin{pmatrix} \boldsymbol{V}_h^{n-1} \\ \boldsymbol{P}_h^{n-1} \end{pmatrix} - \omega^{n-1} \begin{pmatrix} \boldsymbol{V}_h \\ \boldsymbol{P}_h \end{pmatrix},$$

  ω^{n-1} is a step length parameter, which is adaptively computed.

Within the fixed point method we choose $\tilde{\boldsymbol{N}}_h(\boldsymbol{V}_h^{n-1}, \boldsymbol{P}_h^{n-1}) = \boldsymbol{N}_h(\boldsymbol{V}_h^{n-1}, \boldsymbol{P}_h^{n-1})$, for more details see [22].

Linear problems resulting in each step are solved by an efficient multi-grid method, where Vanka-like block-Gauß-Seidel scheme is used both as a smoother and a solver. For all details, documentation and further analysis we refer to `www.featflow.de`, the `featflow` manual [23] and the book [22]. An example of the mesh refinement is given in figure 3.

Since we use the formulation including the deformation tensor $\boldsymbol{D}(\boldsymbol{v})$ this approach in itself is unstable due to its failure to satisfy a discrete Korn's inequality. In the *power-law*

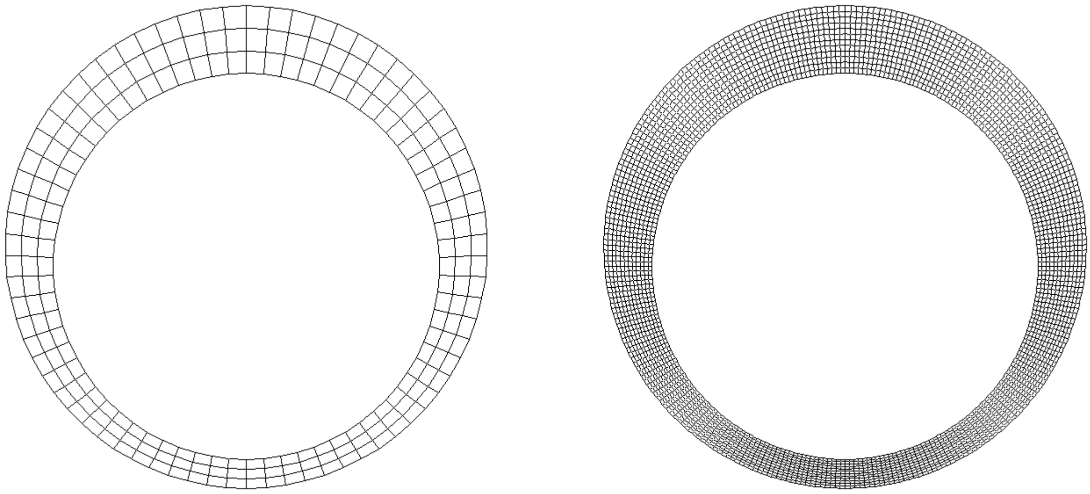


Figure 3: An example of the coarse and the fine mesh.

version of `featflow` the stabilisation technique is thus included. For details we refer to the paper [25].

We note, that in the `featflow` version available on the Internet, there is only shear-dependent problem solver included. Fortunately, the method used in the solver enables us to add the pressure dependence of viscosity by just a small modification. In fact, we did change nothing concerning the method used, we only prescribed the pressure dependence in the computation of the proper viscosity values (that are computed in each nonlinear step in the course of the nonlinear matrix set-up) and we do one more modification concerning the mean value of the pressure. Solving the linear subproblem (basically the same as it would be in the classical Navier-Stokes case) there is no constraint on the pressure level – the pressure field can be shifted arbitrarily, while it always satisfy the linear equations. We thus introduce the constraint of the pressure mean value simply by projecting the pressure field to the prescribed mean value after each nonlinear step.

4.2 Non-dimensional form of generalized Navier-Stokes equations

In the classical Navier-Stokes equations it is custamary to characterize the flow problem by the non-dimensional Reynolds number, defined as

$$\mathrm{Re} = \frac{UV}{\nu},$$

where U and V are characteristic length and characteristic velocity, respectivelly, and ν is the viscosity (which is constant in Navier-Stokes case.) This is a consequence of the fact, that if we introduce these characteristic quantities into equations, writing thus the equations in the terms of non-dimensional velocity, pressure and length, the only term by that the equations for different problems would differ, is exactly the Reynolds number. Therefore, the flow problems specified by similar geometries and by the same Reynolds numbers result in the same behavior.

As soon as we consider that viscosity depends non-trivially on the pressure and/or on the velocity gradient, this is not the case in general; at least not without some amendment. Since the viscosity is not constant, converting the equations into the non-dimensional form we, in addition to the classical Navier-Stokes case, have to "adapt" the form, by that the viscosity depends on this non-dimensional quantities.

Let U be the characteristic length (in our journal bearing problem let it be the radius of the outer circle, i. e. of the bearing), and let V be the characteristic velocity (in our case let it be the velocity prescribed on the inner circle, i. e. on the journal wall). Then the non-dimensional quantities $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}}$, $\hat{p}$, and $\hat{\boldsymbol{b}}$ we define by the following formulas:

$$\begin{aligned} \boldsymbol{v} &= V\hat{\boldsymbol{v}}, & p &= V^2\rho\hat{p}, \\ \boldsymbol{x} &= U\hat{\boldsymbol{x}}, & \boldsymbol{b} &= \tfrac{V^2}{U}\hat{\boldsymbol{b}}, \end{aligned} \tag{4.156}$$

Introducing this transformation into the generalized Navier-Stokes equation (P):

$$v_i\frac{\partial \boldsymbol{v}}{\partial x_i} + \nabla\left(\frac{p}{\rho}\right) - \operatorname{div}\left[\nu\left(\frac{p}{\rho}, |\boldsymbol{D}(\boldsymbol{v})|^2\right)\boldsymbol{D}(\boldsymbol{v})\right] = \boldsymbol{b}$$

we obtain the non-dimensional form (the continuity equation doesn't change)

$$\hat{v}_i\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{x}_i} + \nabla\hat{p} - \frac{1}{UV}\operatorname{div}\left[\nu\left(V^2\hat{p}, \frac{V^2}{U^2}|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2\right)\boldsymbol{D}(\hat{\boldsymbol{v}})\right] = \hat{\boldsymbol{b}}. \tag{4.157}$$

In order to compare the results of our pressure- and shear- dependent model with different parameters set and also with the classical Navier-Stokes model, we redefine our viscosity formula writing $\nu(p, \boldsymbol{D}) \equiv \nu_0\hat{\nu}(p, \boldsymbol{D})$ such that we can define a Reynolds number

$$\mathrm{Re}^* = \frac{UV}{\nu_0} \tag{4.158}$$

again. The notation "Raynolds number" for this definition we could justify, as soon as we specify ν_0.

We should realise that the classical Navier-Stokes model and our generalized Navier-Stokes model are not in contradiction. We can consider the classical model with constant viscosity as a good approximation of the generalized one, in the case when the pressure and the shear rate are not too great. Following this idea, it seems to be reasonable to define the Reynolds number using the viscosity value, which appears for the small pressures and the small shear rates.

Therefore, we define

$$\nu_0 := \nu(0, 0), \tag{4.159}$$

and

$$\hat{\nu}(\hat{p}, |\boldsymbol{D}(\hat{\boldsymbol{v}})|^2) := \frac{1}{\nu_0}\,\nu\left(\frac{p}{\rho}, |\boldsymbol{D}(\boldsymbol{v})|^2\right) = \frac{1}{\nu_0}\,\nu\left(V^2\hat{p}, \frac{V^2}{U^2}|\boldsymbol{D}(\hat{\boldsymbol{v}})|\right). \tag{4.160}$$

According to (4.157), (4.158), (4.159) and (4.160), we shall write the non-dimensional generalized Navier-Stokes equations (P) in the form

$$\hat{v}_i\frac{\partial \hat{\boldsymbol{v}}}{\partial \hat{x}_i} + \nabla\hat{p} - \frac{1}{\mathrm{Re}^*}\operatorname{div}\left[\hat{\nu}(\hat{p}, |\boldsymbol{D}(\hat{\boldsymbol{v}})|^2)\boldsymbol{D}(\hat{\boldsymbol{v}})\right] = \hat{\boldsymbol{b}}, \tag{4.161}$$

where, in addition, we see that $\hat{\nu}$ fulfills

$$\hat{\nu}(0, 0) = 1.$$

We emphasize that, in this case, the Reynolds number defined in (4.158) does not have exactly the same role as in the classical Navier-Stokes model. In the classical case, if we change the parameters of the real-case problem such that the Reynolds number remains unchanged, for instance, if we at once make the velocity twice as big and make all the distances twice as short, the non-dimensional equations and the resulting flow behavior does not change at all, since the Reynolds number does not change. On the contrary, in the generalized model discussed here, this is true only for small pressure and shear rate range, as long as the term $\hat{\nu}$ doesn't have significant impact. Outside of this range we must heed (4.160) and observe that the viscosity dependence both on the pressure and the shear should be adapted in order to get the same resulting flow.

4.2.1 The non-dimensional force

One of the most observed quantities in this work is the force acting on the journal or on the bearing as well. We thus recall that the force $\boldsymbol{f}$ acting on a small face equipped by the normal unit vector $\boldsymbol{n}$ and with the area S, in the space point $\boldsymbol{x}$, is described by

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{n}) = S\,[\boldsymbol{T}(\boldsymbol{x})]\boldsymbol{n},$$

where in our context the Cauchy stress tensor has the form given by (2.8)

$$\boldsymbol{T} = -p\boldsymbol{I} + \rho\nu\left(\frac{p}{\rho}, |\boldsymbol{D}(\boldsymbol{v})|^2\right)\boldsymbol{D}(\boldsymbol{v}).$$

By the transformation (4.156) we obtain the non-dimensional force $\hat{\boldsymbol{f}}$, such that there holds

$$\boldsymbol{f} = \rho V^2 U^2 \hat{\boldsymbol{f}},$$

where we compute $\hat{\boldsymbol{f}}$ from the non-dimensional quantities

$$\begin{aligned}\hat{\boldsymbol{f}} &= \hat{S}\hat{\boldsymbol{T}}\hat{\boldsymbol{n}},\\ \hat{\boldsymbol{T}} &= -\hat{p}\boldsymbol{I} + \frac{1}{\mathrm{Re}^*}\hat{\nu}(\hat{p}, |\boldsymbol{D}(\hat{\boldsymbol{v}})|^2)\boldsymbol{D}(\hat{\boldsymbol{v}}).\end{aligned}$$

(This transformation works in three dimensions, thus we have to remember that both the dimensional and the non-dimensional force relates to the (dimensional or non-dimensional) unit of length of the journal bearing.)

In this work we observe the force acting on the journal (the inner circle) as an integral over the inner circle boundary

$$\hat{\boldsymbol{f}} = \int_{\Gamma_I} \hat{S}\hat{\boldsymbol{T}}\hat{\boldsymbol{n}}\,\mathrm{d}s,$$

approximated by simply summing over the boundary elements.

We do not provide any error estimate of the force observed by such a way. On the other hand, we compute also the force acting on the outer circle, which might be the same from the physical reasons. The difference of these two forces we take as a rough estimate of possible error. In the results provided here this is seldom more then 1% of the presented value.

4.3 Studied form of viscosity

As we have stated in section 2.3.1, the dependence of the viscosity on the pressure is most often considered to be exponential. On the other hand, we have established the existence due to the assumptions **(1)** and **(2)** (on page 18), i. e., among others, due to the assumption that the partial derivative of $\nu(p, |\boldsymbol{D}|^2)$ with respect to pressure is bounded. Therefore, any exponential model of the form

$$\nu(p, |\boldsymbol{D}|) = \nu_{\boldsymbol{D}}(|\boldsymbol{D}|^2)\exp(\alpha p)$$

or similar doesn't met our assumptions and, up to our knowledge, the existence of a solution for such a case is not clear.

However, in some bounded range of pressures we can get close to the exponential pressure-dependence without abondoning the required constraint **(2)**. We can find for instance in Málek, et al. [5], in Franta, et al. [1] or in Hron et al. [4] following viscosity forms, which satisfy the conditions **(1)** and **(2)**:

$$\nu_i(p, |\boldsymbol{D}|^2) = (A + \gamma_i(p) + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}, \quad i = 1, 2, 3,$$

where $A \in (0,\infty)$, $r \in (1,2)$ and γ_i takes one of the following forms for $\alpha, q > 0$:

$$\gamma_1 = (1+\alpha^2 p^2)^{-\frac{q}{2}},$$
$$\gamma_2 = (1+\exp(\alpha p))^{-q},$$
$$\gamma_3 = \begin{cases} \exp(-\alpha q p) & \text{if } p > 0, \\ 1 & \text{if } p \le 0, \end{cases}$$

and the parameters A, α, q, r satisfy

$$\alpha q(2-r) \le \frac{r-1}{2}\left(\frac{A}{2}\right)^{\frac{2-r}{2}} \frac{1}{C_{\mathrm{div},2}}.$$

($C_{\mathrm{div},2}$ is presented in section 3.5.) When $r = 2$, the above forms for the viscosity reduce to the classical Navier-Stokes model, while when $q = 0$ or $\alpha = 0$ it reduces to a subclass of the general Stokesian fluid (for $|\boldsymbol{D}|^2 >> A$ similar to the power-law model $\nu = |\boldsymbol{D}|^{r-2}$).

In this work, we are going to study the model, also introduced and numerically studied in Hron et al. [4], of the following form:

$$\nu(p, |\boldsymbol{D}|^2) = \tilde{\nu}_0\big(A + (\beta + \exp(\alpha p))^{-q} + |\boldsymbol{D}|^2\big)^{\frac{r-2}{2}}. \tag{4.162}$$

In the following lemma we show the condition under which the form (4.162) meets the assumptions **(1)** and **(2)**.

Lemma 4.2 (Parameters for model (4.162)**)** *Assume A, α, β, q be positive and $r \in (1,2)$. Let $A, \beta \le 1$ and the condition*

$$\frac{\alpha q}{\beta^q} \le \frac{1}{C_{\mathrm{div},2}} \frac{r-1}{2-r}\left(\frac{A\beta^q}{2}\right)^{\frac{2-r}{2}} \tag{4.163}$$

be satisfied. Then the viscosity of the form

$$\nu(p, |\boldsymbol{D}|^2) = \big(A + (\beta + \exp(\alpha p))^{-q} + |\boldsymbol{D}|^2\big)^{\frac{r-2}{2}} \tag{4.164}$$

*meet the assumptions **(1)** and **(2)**.*

PROOF. Let us set $\sigma := \frac{2-r}{2} > 0$ and denote

$$Q := A + (\beta + \exp(\alpha p))^{-q} + |\boldsymbol{D}|^2,$$

such that we shall write

$$\nu(p, |\boldsymbol{D}|^2) = Q^{-\sigma}. \tag{4.165}$$

As $A \le 1$ we can easily observe

$$Q \ge A + |\boldsymbol{D}|^2 \ge A(1 + |\boldsymbol{D}|^2), \tag{4.166}$$

and as $\beta^{-q} \ge 1$ also

$$Q \le A + \beta^{-q} + |\boldsymbol{D}|^2 \le (A + \beta^{-q})(1 + |\boldsymbol{D}|^2) \le 2\beta^{-q}(1 + |\boldsymbol{D}|^2). \tag{4.167}$$

There also holds

$$\begin{aligned} Q &= \frac{1 + (\beta + \exp(\alpha p)^q (A + |\boldsymbol{D}|^2)}{(\beta + \exp(\alpha p))^q} \\ &\ge (\beta + \exp(\alpha p))^{-q}(1 + (\beta + \exp(\alpha p)^q |\boldsymbol{D}|^2). \end{aligned} \tag{4.168}$$

We can differentiate $\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}$ as in assumption **(1)** and we obtain

$$\frac{\partial[\nu(p,|\boldsymbol{D}|^2)D_{ij}]}{\partial D_{kl}} = \left(\delta_{ij,kl} - 2\sigma D_{ij}D_{kl}Q^{-1}\right)Q^{-\sigma}, \quad i,j,k,l = 1,\ldots,d,$$

where $\delta_{ij,kl} = 1$ if $i = k$,$j = l$, and $\delta_{ij,kl} = 0$ otherwise.

We thus conclude

$$\frac{\partial[\nu(p,|\boldsymbol{D}|^2)D_{ij}]}{\partial D_{kl}} \leq \left(1 - \frac{2\sigma D_{ij}D_{kl}}{Q}\right)Q^{-\sigma} \overset{(4.166)}{\leq} A^{-\sigma}(1+|\boldsymbol{D}|^2)^{-\sigma},$$

and using

$$\frac{2\sigma D_{ij}^2}{Q} \overset{(4.166)}{\leq} \frac{2\sigma|\boldsymbol{D}|^2}{A+|\boldsymbol{D}|^2} \leq 2\sigma$$

we obtain

$$\frac{\partial[\nu(p,|\boldsymbol{D}|^2)D_{ij}]}{\partial D_{ij}} \geq \left(1 - \frac{2\sigma D_{ij}D_{ij}}{Q}\right)Q^{-\sigma} \overset{(4.167)}{\geq} (1-2\sigma)\left(\frac{\beta^q}{2}\right)^{\sigma}(1+|\boldsymbol{D}|^2)^{-\sigma}.$$

We can therefore set $C_1 := (1-2\sigma)\left(\frac{\beta^q}{2}\right)^{\sigma} = (r-1)\left(\frac{\beta^q}{2}\right)^{\frac{2-r}{2}}$ and $C_2 := A^{\frac{r-2}{2}}$ in **(1)**. In order to fulfil **(2)** we are looking for γ_0 such that

$$\left|\frac{\partial\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}}{\partial p}\right| \leq \gamma_0(1+|\boldsymbol{D}|^2)^{-\frac{\sigma}{2}}.$$

Differentiating (4.165) with respect to p we observe (using $\sigma - \frac{1}{2} < 0$)

$$\begin{aligned}
\frac{\partial\nu(p,|\boldsymbol{D}|^2)\boldsymbol{D}}{\partial p} &= -\sigma Q^{-\sigma-1}(-q)(\beta+\exp(\alpha p))^{-q-1}\alpha\exp(\alpha p)|\boldsymbol{D}| \leq \\
&\overset{(4.168)}{\leq} \alpha q\sigma(\beta+\exp(\alpha p))^{-q}\frac{|\boldsymbol{D}|}{(\beta+\exp(\alpha p))^{-q(\sigma+1)}[1+(\beta+\exp(\alpha p))^q|\boldsymbol{D}|^2]^{\sigma+1}} = \\
&= \alpha q\sigma(\beta+\exp(\alpha p))^{q(\sigma-\frac{1}{2})}\frac{[(\beta+\exp(\alpha p))^q|\boldsymbol{D}|^2]^{\frac{1}{2}}}{[1+(\beta+\exp(\alpha p))^q|\boldsymbol{D}|^2]^{\sigma+1}} \leq \\
&\leq \alpha q\beta^{-q}\sigma(1+|\boldsymbol{D}|^2)^{-\sigma-\frac{1}{2}}.
\end{aligned}$$

Since $-\sigma - \frac{1}{2} < -\frac{\sigma}{2}$ we can set $\gamma_0 = \sigma\alpha q\beta^{-q} = \frac{2-r}{2}\alpha q\beta^{-q}$.

Finally, we need to meet the condition $\gamma_0 < \frac{1}{C_{\text{div},2}}\frac{C_1}{C_1+C_2}$ but it is enough to show

$$\gamma_0 \leq \frac{1}{C_{\text{div},2}}\frac{C_1}{2C_2} \tag{4.169}$$

which is stronger (as $C_1 < C_2$). We substitute C_1, C_2 and γ_0 which we have set above and we obtain

$$\frac{2-r}{2}\frac{\alpha q}{\beta^q} \leq \frac{1}{C_{\text{div},2}}(r-1)\left(\frac{\beta^q}{2}\right)^{\frac{2-r}{2}}\frac{1}{2}A^{\frac{2-r}{2}}.$$

The last inequality is equivalent to the ones we have set in (4.163). ☐

Remark: If we consider (4.162)

$$\nu(p,|\boldsymbol{D}|^2) = \tilde{\nu}_0\left(A+(\beta+\exp(\alpha p))^{-q}+|\boldsymbol{D}|^2\right)^{\frac{r-2}{2}} \tag{4.170}$$

instead of (4.164), we easily come to the condition

$$\tilde{\nu}_0\frac{\alpha q}{\beta^q} \leq \frac{1}{C_{\text{div},2}}\frac{r-1}{2-r}\left(\frac{A\beta^q}{2}\right)^{\frac{2-r}{2}} \tag{4.171}$$

instead of (4.163).

4.4 The investigated range of parameters

Looking at (4.162), one can see an amount of parameters that we have to set. In a real-life case, of course, we fit them to come into agreement with experiment. Here we do not so but we discuss the influence of some selected parameters on the behaviour of the resulting flow. Nevertheless, not all parameters will be discussed here in such a way; some of them will be set by a numerical reasons for example.

In this work, we focus on to study the flows with various Reynolds numbers Re* (defined in (4.158)) and with various geometry parameters (namely the eccentricity). Several results to the chosen form of viscosity within the geometry of eccentric annular rings are presented in the paper of Hron et al. [4], where also the different values of parameters A and β are discussed.

Let us point out the properties of the model (4.162) in a short survey.

- First of all, we immediately see that

$$\begin{aligned} \nu(p,\cdot) \quad & \text{is decreasing function of } |\boldsymbol{D}|^2 \text{ for arbitrary } p \in \mathbb{R}, \\ \nu(\cdot,|\boldsymbol{D}|^2) \quad & \text{is increasing function of } p \text{ for arbitrary } |\boldsymbol{D}|^2, \end{aligned}$$

  as soon as $r < 2$.

- Moreover, setting $|\boldsymbol{D}|$ great enough we notice that the remaining terms are bounded by $A + \beta^{-q}$ such that the shear dependence become dominant. We see that, asymptotycally, (4.162) behaves like the power-law model for big shear rates:

$$\nu(p,|\boldsymbol{D}|^2) \sim |\boldsymbol{D}|^{r-2}, \quad \text{as } |\boldsymbol{D}| \to \infty,\ p \text{ arbitrary.}$$

- If we set the pressure large, such that $\exp(\alpha p) >> \beta$ and, in the same moment, we assume the A and $|\boldsymbol{D}|$ be fixed, we recognize that our viscosity is indeed bounded; the pressure term vanishes and

$$\nu(p,|\boldsymbol{D}|) \sim (A + |\boldsymbol{D}|^2)^{\frac{r-2}{2}}, \quad \text{as } p \to \infty,\ A, |\boldsymbol{D}| \text{ fixed.}$$

  Setting $|\boldsymbol{D}| = 0$ the viscosity draws near to its supremum,

$$\nu(p,|\boldsymbol{D}|) < A^{\frac{r-2}{2}}.$$

- Finally, we consider the pressure large enough, such that we can neglect β, but still not too large, such that there still holds $A + |\boldsymbol{D}|^2 << \exp(-q\alpha p)$. In this situation we obtain

$$\nu(p,|\boldsymbol{D}|) \sim \exp(\alpha p)^{q\frac{2-r}{2}}, \quad \text{for } A, |\boldsymbol{D}| << 1 \text{ and } p \text{ being in some feasible range.} \tag{4.172}$$

Let us remind what is presented in section 2.3.1, that the most used model for describing the pressure dependence of viscosity in practise is the simple exponential law $\nu(p) = \exp(\alpha p)$. For this reason and the last finding (4.172) we set in our simulations

$$q := \frac{2}{2-r}, \tag{4.173}$$

such that the exponencial law is approximately fulfilled at least in some range of pressures and when the shear is small. From the same reasons we put A and β small:

$$A := \beta := 10^{-5}.$$

Our theoretical results give us the existence of a solution to (P) only for $r \in (\frac{3}{2}, 2)$ (in two dimensions). However, I provided the simulations with r within the range $r \in (1, 2)$, and I didn't notice any significant change in behaviour near the value $r = \frac{3}{2}$.

There is an amount of various parameters that could be documented here, among others the influence of several α, r, p_0, R_J should be investigated. However, the scope of this work is limited, thus we do not study all these parameters. In the following section, we fix the journal radius, the mean value of the pressure and both the parameters r and α. We focus on the differences between the Navier-Stokes model and our generalized model (P) and mainly on the influence of the eccentricity and of the Reynolds number on the resulting flow.

4.5 Numerical results

4.5.1 The eccentricity and the Reynolds number influence – classical Navier-Stokes model

As the first journal bearing simulations we show the results of the classical Navier-Stokes model applied to the journal bearing geometry. The main aim of this section is to show the influence of the varying eccentricity of the journal and the behaviour of the Navier-Stokes model with various Reynolds numbers.

We set in all simulations the velocity prescribed on the inner circle to be 1, i. e. to be equal to the characteristic velocity in the real problem. Similarly, we set 1 the radius of the outer circle. The radius of the inner circle we set to be 0.8, which gives us the possible range of absolute eccentricity $\varepsilon \in (0,1) \approx e \in (0,0.2)$.

The resulting pressure p distributions for the Reynolds numbers 1, 100 and 1000 and for the eccentricities 0.3 and 0.8 are shown in figure 4. In figure 5 there is shown the distribution of $|\boldsymbol{D}(\hat{\boldsymbol{v}})|$ and figure 6 shows the streamlines of the resulting flow.

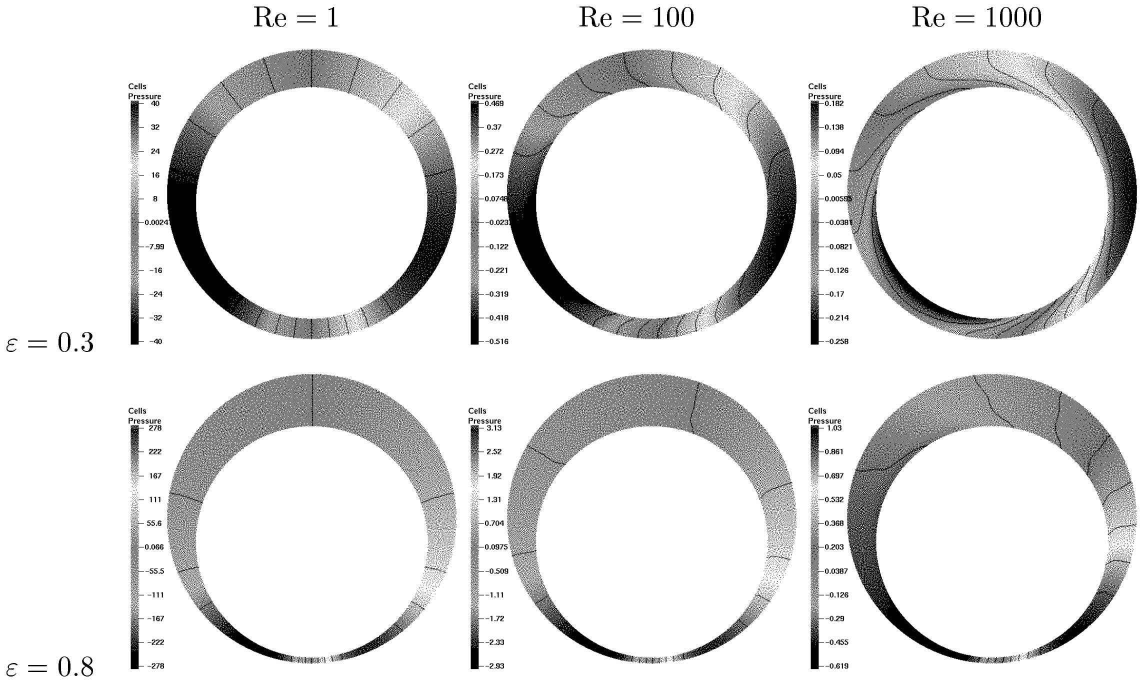
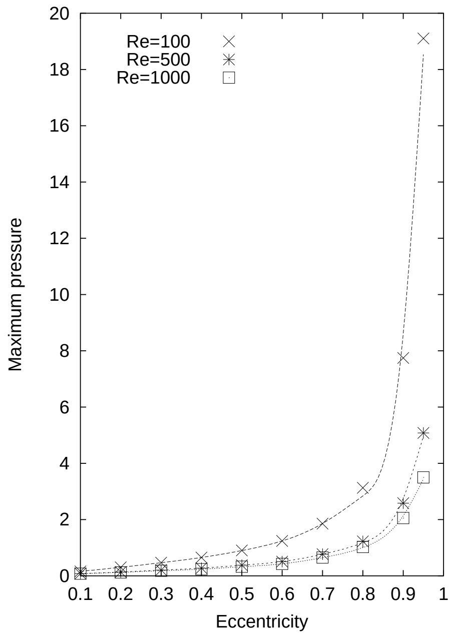


Figure 4: The pressure $\hat{p}$ distribution for the Navier-Stokes model

Since in the Navier-Stokes problem the pressure is given up to the constant, we prescribe the meanvalue of the pressure to be zero in order to see the values better. We see that the resulting pressure range is getting shorter just as the Reynolds number increases. We show the maximum pressure values for several Reynolds numbers and several eccentricities in table and graph 1. We do not present the minimum pressure values, we just note that they show similar behaviour (but negative).

The most signigicant quantity, observed from the (steady-state) journal bearing computation, is the force acting on the rotating journal by the fluid. For the manner how we compute this force see section 4.2.1. In table and graph 2 we present the (non-dimensional) magnitude of caused force, while its direction is shown in table and graph 3. The direction is described by an angle/π[rad] measured clockwise from the direction from centre to the left on the figures; i. e. the value 0 means 9 o'clock, 0.5 means 12 o'clock, -0.5 means 6 o'clock. We see that the magnitude increases by increasing the eccentricity, and decreases by increasing the Reynolds number. The differences in the direction of the force grow up with the Reynolds number.



	Re = 1	100	500	1000
$\varepsilon = 0.1$	12	0.16	0.08	0.08
0.2	25	0.30	0.13	0.12
0.3	40	0.5	0.21	0.18
0.4	56	0.7	0.27	0.24
0.5	78	0.9	0.38	0.33
0.6	109	1.2	0.51	0.43
0.7	162	1.9	0.77	0.65
0.8	278	3.1	1.2	1.0
0.9	712	7.8	2.6	2.1
0.95	1831	19.1	5.1	3.5

Table 1: Maximum pressure $\hat{p}$ values for the Navier-Stokes model

Looking at the pressure distribution in figure 4 already, we see that the pressure reaches its maximum somewhere on the left-hand side (upstream the contraction) where the lubricant arrives to the narrow gap, while on the right-hand side (downstream), where the fluid leaves the contraction, the minimum values occure. In the case of Stokes flow (behaving as a limit Re $\to 0$) the pressure distribution is exactly symmetric with respect to the vertical axis and the reacting force is parallel to the horizontal axis. A diversion from this configuration is caused by the influence of the convective term.

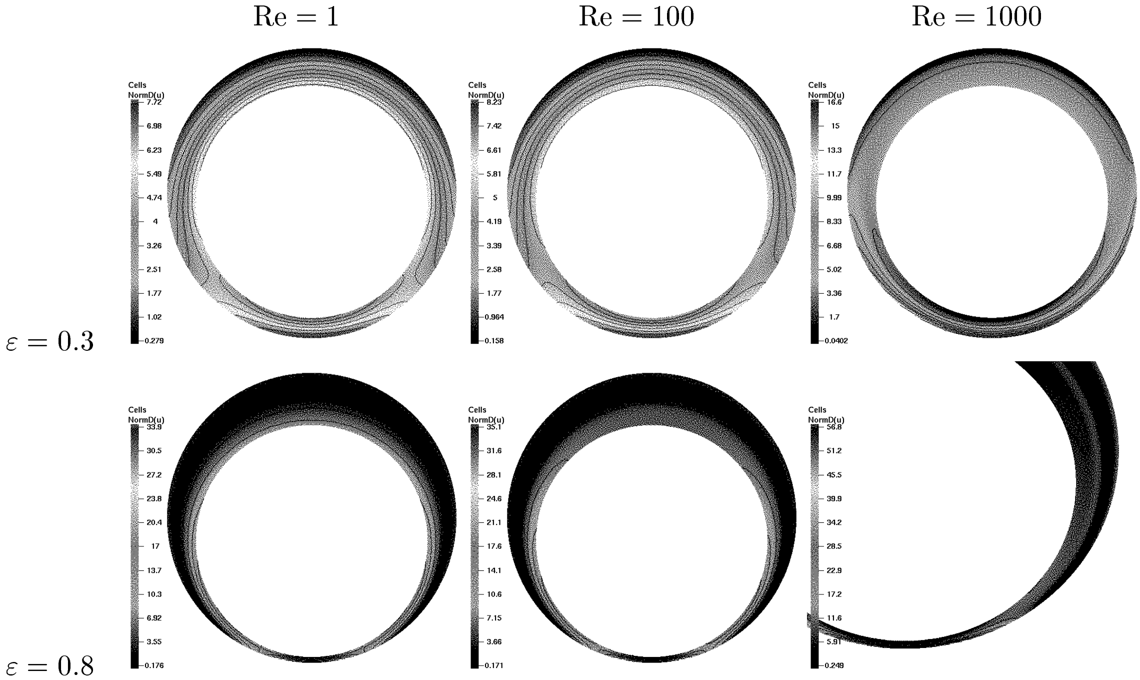


Figure 5: $|\boldsymbol{D}(\hat{\boldsymbol{v}})|$ distribution for the Navier-Stokes model

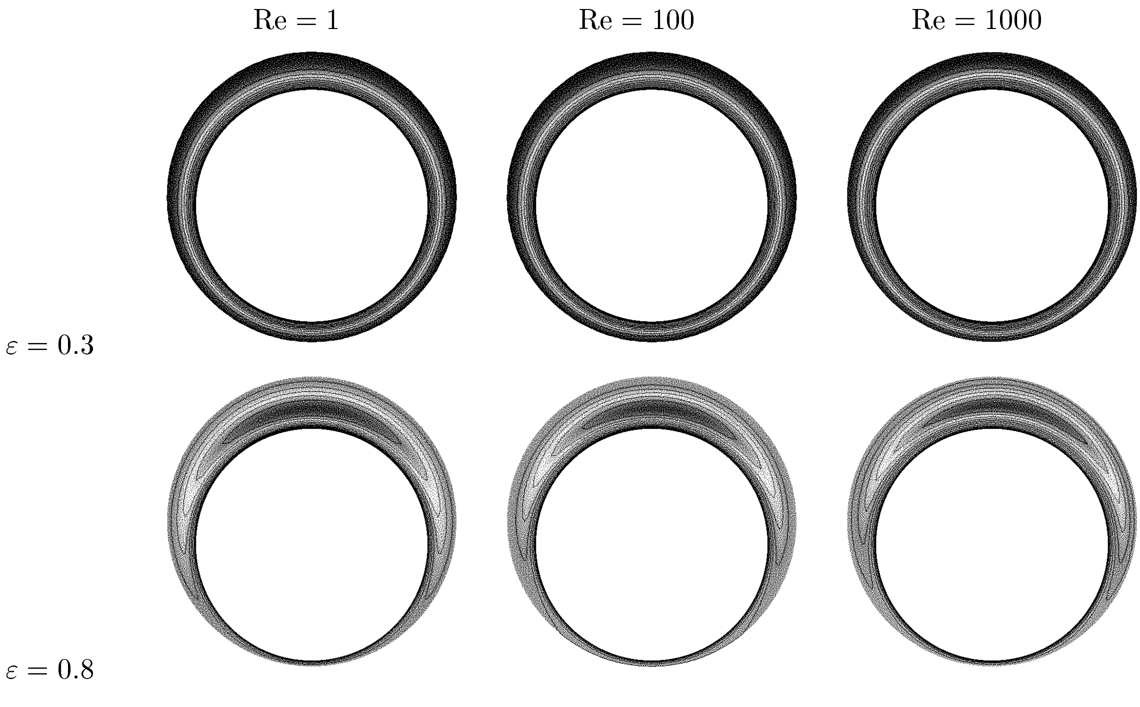
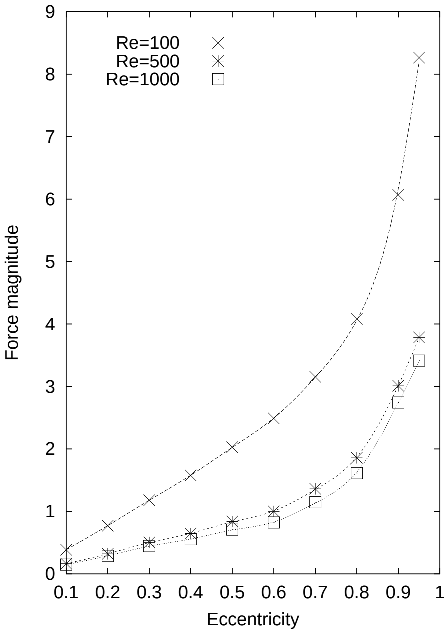
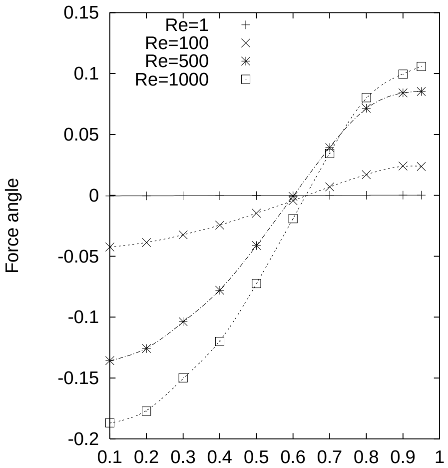


Figure 6: The stream-lines for the Navier-Stokes model



	Re = 1	100	500	1000
$\varepsilon = 0.1$	34	0.38	0.16	0.15
0.2	67	0.77	0.32	0.29
0.3	101	1.18	0.50	0.45
0.4	136	1.58	0.65	0.55
0.5	172	2.03	0.84	0.71
0.6	213	2.49	1.00	0.82
0.7	265	3.15	1.36	1.15
0.8	339	4.08	1.86	1.61
0.9	493	6.07	3.01	2.74
0.95	694	8.27	3.79	3.41

Table 2: Force magnitude for the Navier-Stokes model



	Re = 1	100	500	1000
$\varepsilon = 0.1$	-0.00047	-0.042	-0.136	-0.19
0.2	-0.00043	-0.039	-0.126	-0.18
0.3	-0.00037	-0.032	-0.104	-0.15
0.4	-0.00029	-0.025	-0.078	-0.12
0.5	-0.00019	-0.015	-0.041	-0.07
0.6	-0.00009	-0.004	-0.001	-0.02
0.7	0.00002	0.007	0.039	0.03
0.8	0.00012	0.017	0.072	0.08
0.9	0.00019	0.024	0.084	0.10
0.95	0.00020	0.024	0.085	0.11

Table 3: Force direction for the Navier-Stokes model

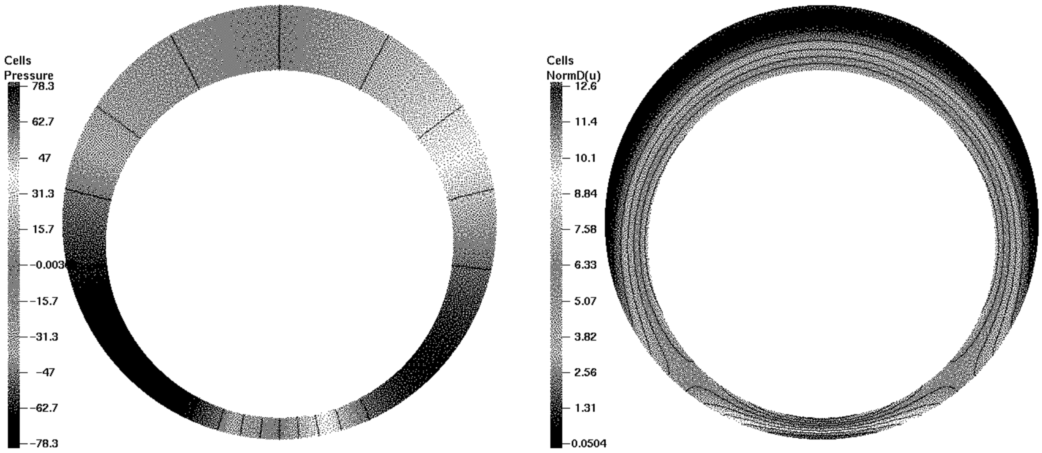
4.5.2 The eccentricity influence for the problem (P), $\mathrm{Re}^* = 1$.

In this section we would like to show the differences that occure when we introduce the generalized model, with the viscosity of the form (4.162)

$$\nu(p, |\boldsymbol{D}|^2) = \tilde{\nu}_0\big(A + (\beta + \exp(\alpha p))^{-q} + |\boldsymbol{D}|^2\big)^{\frac{r-2}{2}}. \qquad (4.162)$$

We set $r = 1.5$, as the lower border of the range within we proved the existence, and in order to keep (4.173) we set $q = 4$. For this moment, we set the Reynolds number (defined in (4.158)) $\mathrm{Re}^* = 1$, where the (non-dimensional) pressure results to the greatest range in the Navier-Stokes case.

In order to set the pressure-dependence parameter α, let us look into the paper of Gwynllyw et al. [10], where a different viscosity model is introduced together with the material constants following the experimental data. Although the model presented in [10] is not at all the same as the one presented here, the pressure-dependence of the viscosity is essentially determined (considering $|\boldsymbol{D}| = 0$) by the term $\exp(\alpha p)$, where $\alpha = 1.12 \times 10^{-8}/\mathrm{Pa}$. Therefore, in order to roughly approximate the real situation, we assume $\alpha := 10^{-8}$ herein. We emphasize that we do not aspire to present a real-life model at all. All what is provided here follows the strictly illustrative purpose.



values of the pressure $\hat{p}$ values of $|\boldsymbol{D}(\hat{v})|$

Figure 7: Some Navier-Stokes results for $\varepsilon = 0.5$

Let us look to the previous section simulations. We can see, observing for instance the case $\varepsilon = 0.5$ in figure 7, that on the majority part of the flow domain $|\boldsymbol{D}|^2 \sim 30$. On the other side, the (non-dimensional) pressure results somewhere in the range $\hat{p} \sim -100 \cdots +100$. Since we would like to have a positive pressure values (realising that in the generalized Navier-Stokes case this becomes important, in contrast to the Navier-Stokes case) we set the meanvalue $\hat{p}_0 = 100$ such that we can expect $\hat{p} \sim 0 \ldots 200$ from the Navier-Stokes case. If we set $\rho = U = V = 1$ in the non-dimensional transformation (4.160), we would obtain $(\beta + \exp(\alpha\hat{p}))^{-q} \sim$ from $1 - 8 \times 10^{-6}$ to 1. Note that setting the Reynolds number (in the Navier-Stokes case in previous section) higher than one we obtain even smaller range of the pressure field. Naturally, we wouldn't obtain any visible dependence of the viscosity on the pressure setting the parameters in this manner.

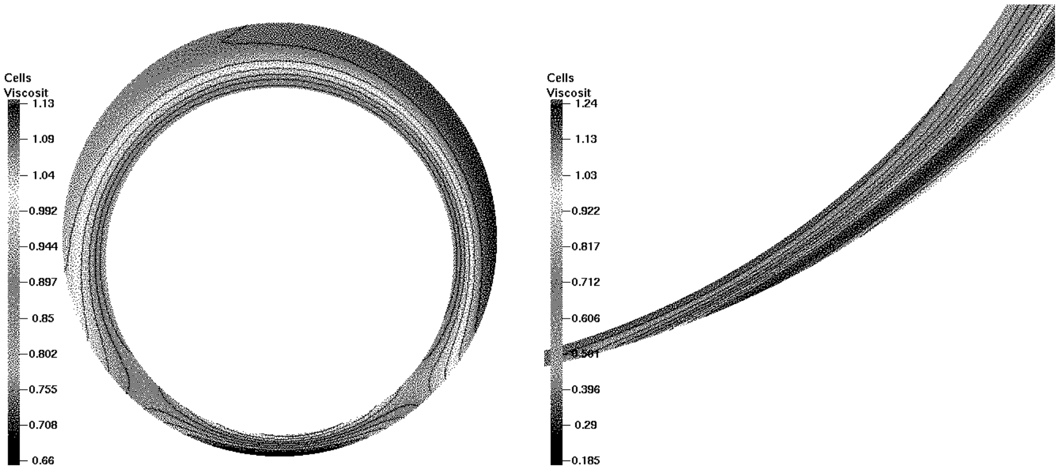
Hereafter we assume $\rho = 1$ for simplicity. We would like to recall the non-dimensional transformation (4.160),

$$\hat{\nu}(\hat{p}, |\boldsymbol{D}(\hat{\boldsymbol{v}})|^2) = \frac{1}{\nu_0}\nu(V^2\hat{p}, \frac{V^2}{U^2}|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2),$$

which we shall employ in order to balance the pressure- and the shear- dependence in (4.162) better. ("Better" means in order to demonstrate the abilities of the model, not in order to approach the reality.) Note that all the time we keep the Reynolds number $\mathrm{Re}^* = 1$ in what follows.

First, we set the characteristic velocity V in such a way that $(\beta + \exp(\alpha V^2 \hat{p}))^{-q} \sim 0.5$ for $p \sim 200$. We thus set $V = 300$. We notice that for $p \sim 100$, that is for the prescribed mean value of the pressure, we obtain $(\beta + \exp(\alpha p))^{-q} \sim 0.7$. Therefore, as a second step, we set the characteristic length U such that for $|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2 \sim 30$ we obtain $|\boldsymbol{D}(\boldsymbol{v})|^2 = \frac{V^2}{U^2}|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2 \sim 0.7$. We thus set $U = 2000$. As we have promised to keep $\mathrm{Re}^* = 1$, we must set $\tilde{\nu}_0$ such that $\nu_0 = \nu(0,0) := UV = 6 \times 10^5$. The last choice is, of course, a very unrealistic one. We see that the claim to get $(\beta + \exp(\alpha V^2 \hat{p}))^{-q} \sim 0.5$ same as the demand to keep $\mathrm{Re}^* = 1$ together with the wish to balance the shear- and the pressure-dependence of the viscosity at once is vindicable at most as a numerical experiment.

In figure 8 we show the viscosity field for the case $\varepsilon = 0.5$. We see that the viscosity is somewhere greater then one, somewhere smaller. In table 4 we present the comparison of



$\varepsilon = 0.5$ $\varepsilon = 0.95$

Figure 8: The viscosity field for the problem (P), $\mathrm{Re}^* = 1$.

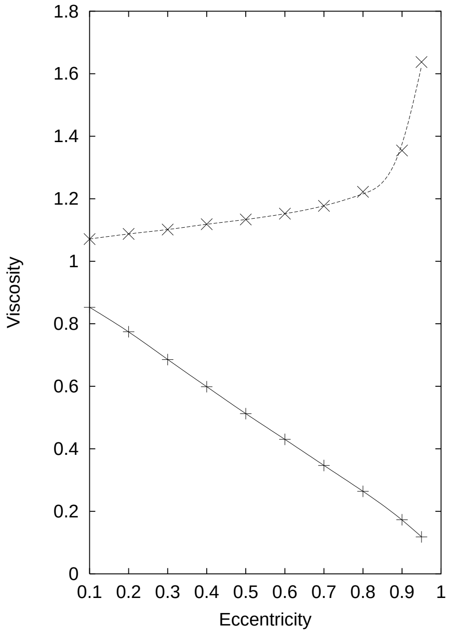
the following quantities for the Navier-Stokes and for the generalized Navier-Stokes case: we show the minimum and maximum values of the pressure (we shifted the pressure to the meanvalue 100 in the N.-S. case), of the shear $|\boldsymbol{D}(\hat{\boldsymbol{v}})|$ and of the viscosity, then we show the (non-dimensional) force magnitude and its direction.

	$\hat{p}_{\min}$	$\hat{p}_{\max}$	$\lvert\boldsymbol{D}(\hat{\boldsymbol{v}})\rvert_{\min}$	$\lvert\boldsymbol{D}(\hat{\boldsymbol{v}})\rvert_{\max}$	$\hat{\nu}_{\min}$	$\hat{\nu}_{\max}$	force mag.	force dir.
N.-S.	22	178	0.05	12.6	1	1	172	0.4998
(P)	41	159	0.002	612	0.51	1.13	137	0.5011

Table 4: A comparison between N.-S. and (P) model, $\mathrm{Re}^* = 1$, $\varepsilon = 0.5$.

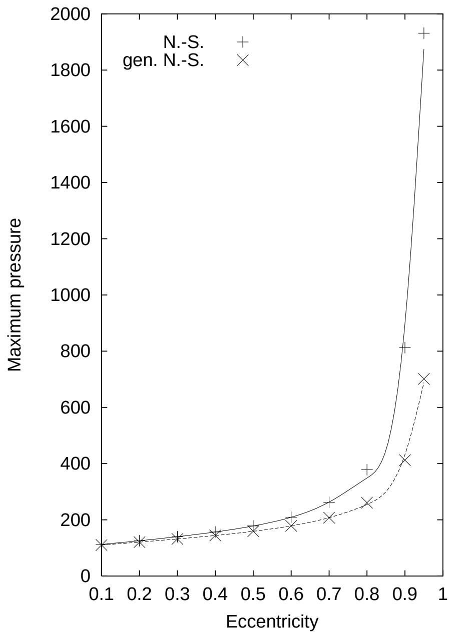
In table and graph 5 we show the minimum and maximum viscosities for several eccentricities for $\mathrm{Re}^* = 1$. In tables 6, 7 and 8 we present the maximum pressure and $|\boldsymbol{D}(\hat{\boldsymbol{v}})|$ values, the force magnitude and its direction, compared with the N.-S. case for several eccentricities.

We clearly see that the chosen model is indeed able both of the pressure thickening and the shear thinning.



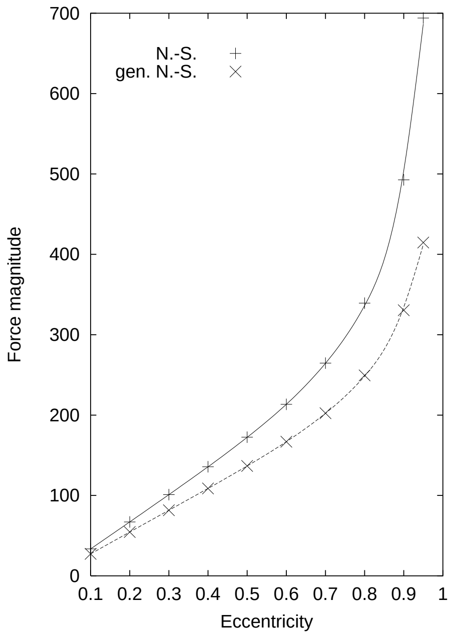
	$\hat{\nu}_{\min}$	$\hat{\nu}_{\max}$
$\varepsilon = 0.1$	0.85	1.07
0.2	0.77	1.09
0.3	0.69	1.10
0.4	0.60	1.12
0.5	0.51	1.13
0.6	0.43	1.15
0.7	0.35	1.18
0.8	0.26	1.22
0.9	0.17	1.35
0.95	0.12	1.64

Table 5: The minimum and maximum viscosities for (P), $\mathrm{Re}^* = 1$.



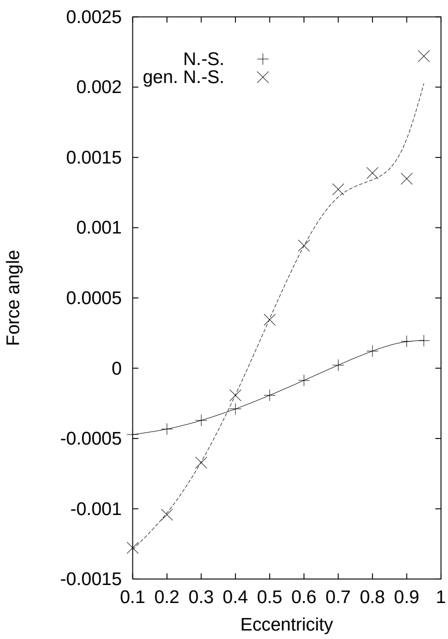
	N.-S.	gen. N.-S.
$\varepsilon = 0.1$	113	110
0.2	126	121
0.3	140	132
0.4	157	144
0.5	178	159
0.6	209	179
0.7	262	208
0.8	378	261
0.9	813	412
0.95	1931	701

Table 6: Maximum pressure $\hat{p}$ values, N.-S. and (P) model, $\mathrm{Re}^* = 1$.



	N.-S.	gen. N.-S.
$\varepsilon = 0.1$	34	27
0.2	67	55
0.3	101	82
0.4	136	109
0.5	172	137
0.6	213	167
0.7	265	202
0.8	339	249
0.9	493	330
0.95	694	415

Table 7: Force magnitude, comparison between N.-S. and (P) model, $\mathrm{Re}^* = 1$.



	N.-S.	gen. N.-S.
$\varepsilon = 0.1$	-0.00047	-0.0013
0.2	-0.00043	-0.0010
0.3	-0.00037	-0.0007
0.4	-0.00029	-0.0002
0.5	-0.00019	0.0003
0.6	-0.00009	0.0009
0.7	0.00002	0.0013
0.8	0.00012	0.0014
0.9	0.00019	0.0014
0.95	0.00020	0.0022

Table 8: Force direction, comparison between N.-S. and (P) model, $\mathrm{Re}^* = 1$.

4.5.3 Three examples how to get $\mathrm{Re}^* = 100$ in the problem (P).

In the previous section we presented the differences occuring when the generalized model (P) is introduced instead of the classical Navier-Stokes model. Here we would like to emphasize, and to show by the numerical results, how could the flow differ for model (P) with the same Reynolds number Re^* but for different characteristic velocity V and characteristic length U.

In the previous case $\mathrm{Re}^* = 1$ we have set (in order to show both the pressure thickening and the shear thinning capability of the model) the characteristic velocity $V = 300$ and the characteristic length $U = 2000$. The viscosity used in the Reynolds number definition, i. e. the viscosity value for both the pressure and the shear rate being zero, has been set $\nu_0 = 6 \times 10^5$ in order to preserve $\mathrm{Re}^* = 1$.

Herein, we provide the simulations with $\mathrm{Re}^* = 100$ and we present three different ways, how the Reynolds number can be changed.

Example I The most "simple" possibility is to change (decrease) the viscosity ν_0. In this case the (non-dimensional) viscosity formulation doesn't change, the viscous term is, essentially, purely weakened with respect to the convective term. Recalling the sample $\mathrm{Re}^* = 1$ we shall set $V = 300$, $U = 2000$ and $\nu_0 = 6 \times 10^3$.

Example II Another way is to increase the characteristic length U. In that case we should look to the viscosity formulation (4.160) and we see that the influence of the shear $|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2$ shall be weakened with respect to the pressure dependence (that will not change). In this example we set $V = 300$, $U = 2 \times 10^5$ and $\nu_0 = 6 \times 10^5$.

Example III The last instance changes Re^* by changing the characteristic velocity V (which is the same as to increase the velocity prescribed on the boundary). We can guess, looking to (4.160) that this will intensify both the shear and the pressure impact on the viscosity. We set $V = 3 \times 10^4$, $U = 2000$ and $\nu_0 = 6 \times 10^5$.

As we should keep the same mean value of the pressure, $p_0 = V^2 \hat{p}_0$ ($\rho = 1$), which is in the first two cases equal to $300^2 \times 100 = 9 \times 10^6 = 9\mathrm{MPa}$, we should re-set the non-dimensional mean value and prescribe it to $\hat{p}_0 = \frac{9 \times 10^6}{(3 \times 10^4)^2} = 0.01$.

In figure 9 we show the resulting viscosity fields to these three examples for the eccentricities $\varepsilon = 0.3,\ 0.8$ while on figures 10 and 11 we present all the pressure $\hat{p}$ and $|\boldsymbol{D}(\hat{\boldsymbol{v}})|$ fields, stream-lines and also the viscosity field for the eccentricity $\varepsilon = 0.5$.

In tables and graphs 9, 10, 11 and 12 we present the maximum and minimum values of viscosity $\hat{\nu}$, maximum pressure, the magnitude and the direction of the force acting on the journal.

We see (compare with the case $\mathrm{Re}^* = 1$ on figure 8) that in case I, where the Reynolds number was increased from 1 to 100 by simply decreasing the viscosity ν_0, the resulting character of viscosity distribution does not change much. Although, since the resulting pressure range is much shorter, the pressure impact on the viscosity also reduces.

In example II, on the contrary, the shear influence on viscosity nearly disappeares; the viscosity field qualitatively reproduces the pressure distribution in the domain.

The parameters that have been set in III bring forth a litle bit problematic behaviour for higher eccentricities; the shear thinning effect is in this case so enhanced, that the viscosity $\frac{1}{\mathrm{Re}^*}\hat{\nu}$ near the small gap goes down close to 1×10^{-7} ($\approx$ Re $\sim 10^7$) for the case of $\varepsilon = 0.8$. The convergence of the performed numerical method then becomes slow and for $\varepsilon > 0.8$ the computation fails. We guess that, due to such extreme shear thinning and consequently due to such small viscosities, the steady-state flow becomes unstable.

Both the shear-thinning and the pressure-thickening qualities of the model give rise to some problems occuring in the numerical simulations. For instance, the exponential of the pressure causes numerical problems for higher pressures, at least until the approximative solution converges enough. We thus simply cut the exponential for the pressures too large and we relax this limitation once the solution converges enough.

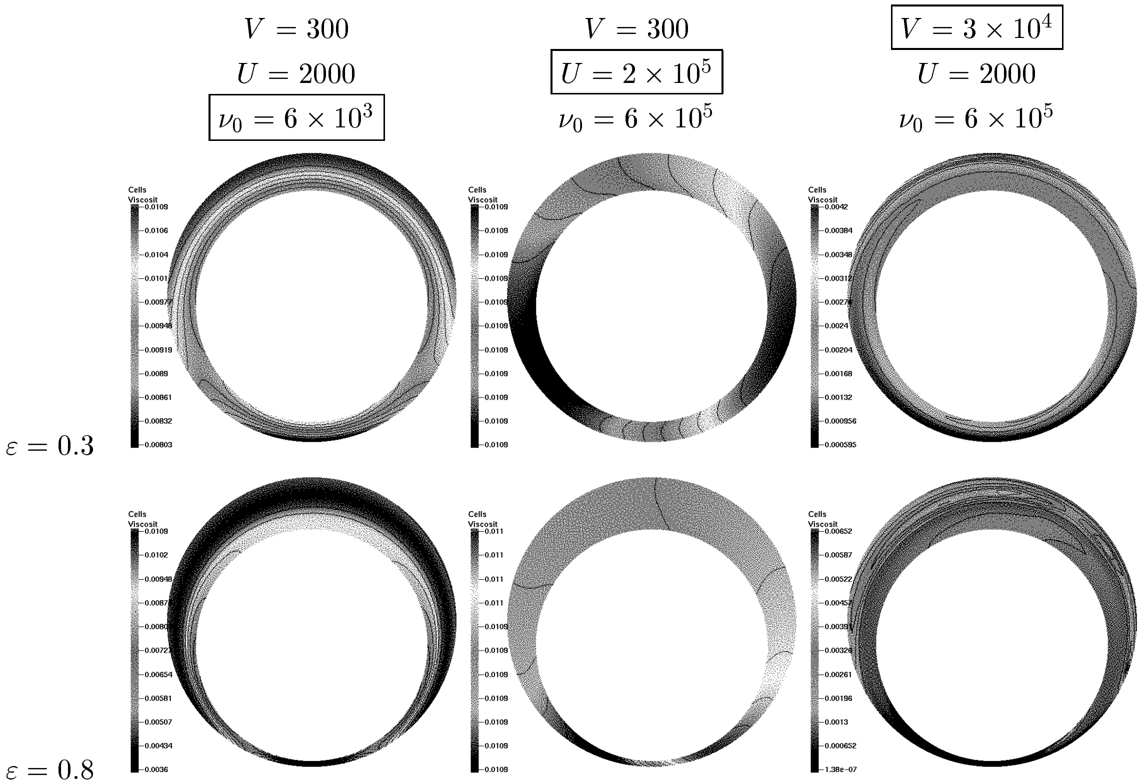
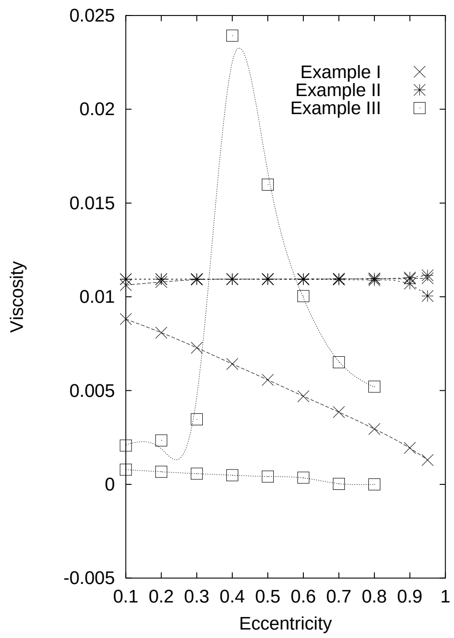


Figure 9: The viscosity field for the three examples of (P), $\mathrm{Re}^* = 100$.



	Example I		Example II		Example III	
$\varepsilon = 0.1$	0.88	1.063	1.094	1.094	0.078	0.21
0.2	0.81	1.079	1.094	1.095	0.067	0.23
0.3	0.73	1.092	1.094	1.095	0.057	0.35
0.4	0.64	1.094	1.093	1.095	0.049	2.39
0.5	0.56	1.095	1.093	1.095	0.041	1.16
0.6	0.47	1.095	1.093	1.096	0.036	1.00
0.7	0.39	1.095	1.092	1.096	0.0023	0.65
0.8	0.30	1.096	1.088	1.098	0.0001	0.52
0.9	0.19	1.097	1.070	1.102	-	-
0.95	0.13	1.099	1.005	1.115	-	-

table shows $\hat{\nu}$, graph shows $\hat{\nu}/\mathrm{Re}^* = 0.01\hat{\nu}$.

Table 9: Maximum and minimum viscosity $\hat{\nu}$, three examples of $\mathrm{Re}^* = 100$ for (P).

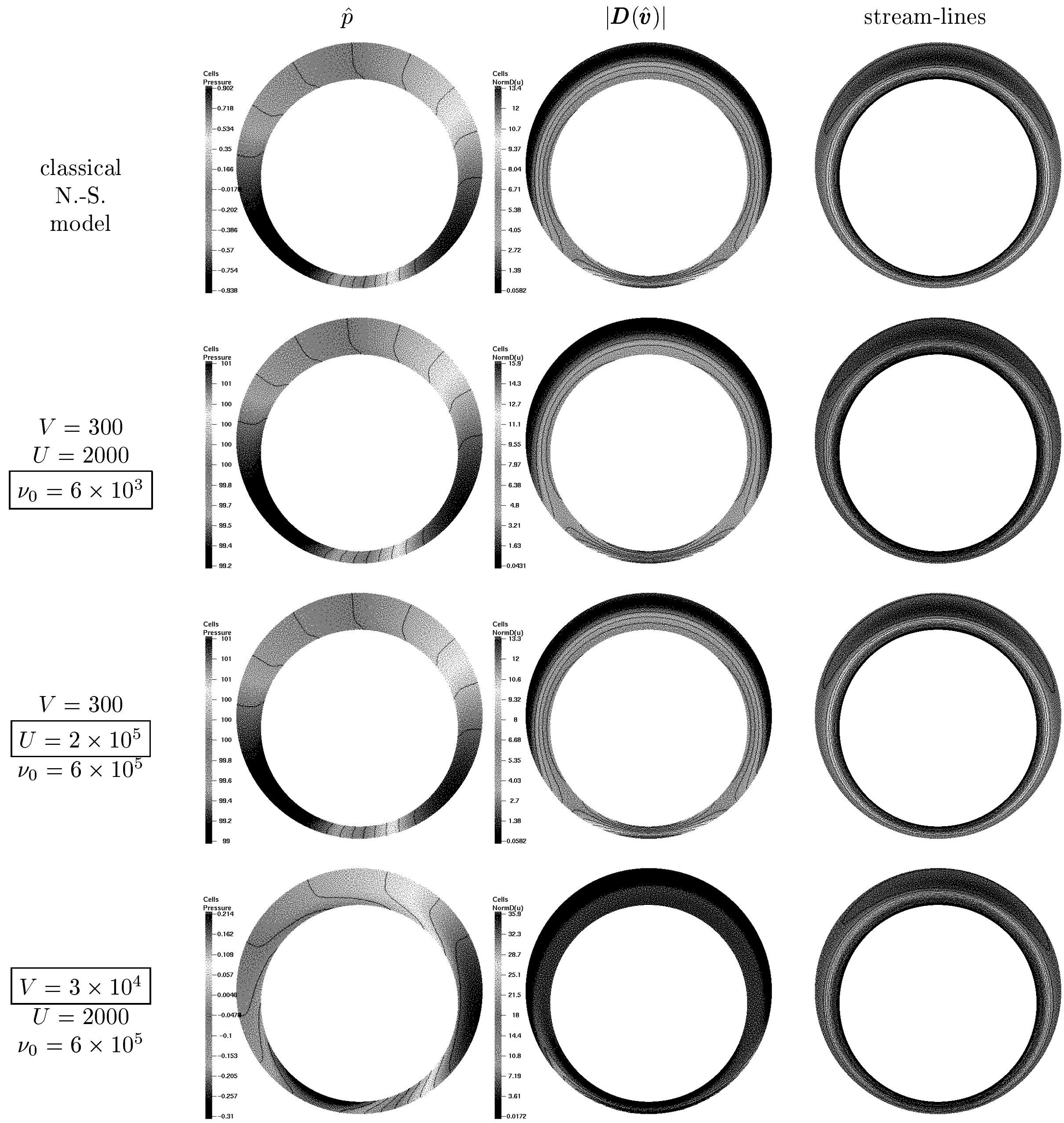


Figure 10: A comparison of three examples for problem (P), $\mathrm{Re}^* = 100$, $\varepsilon = 0.5$.

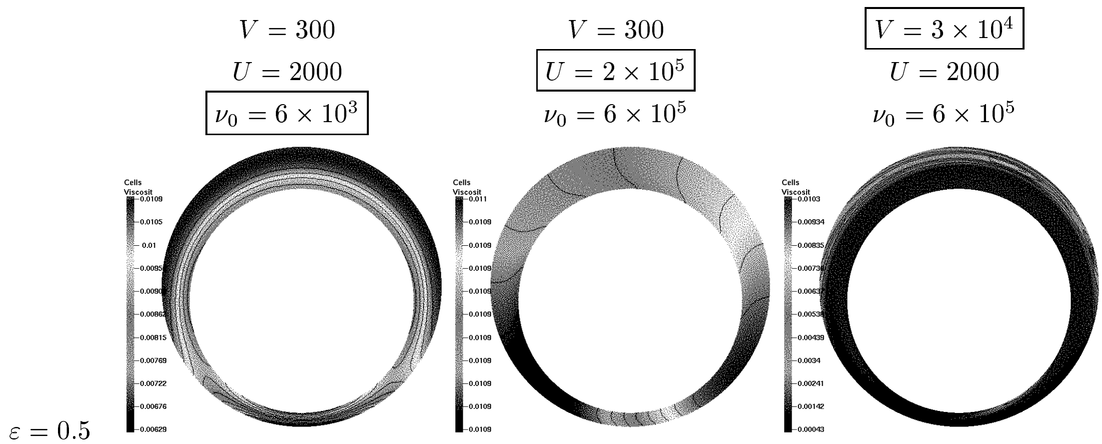
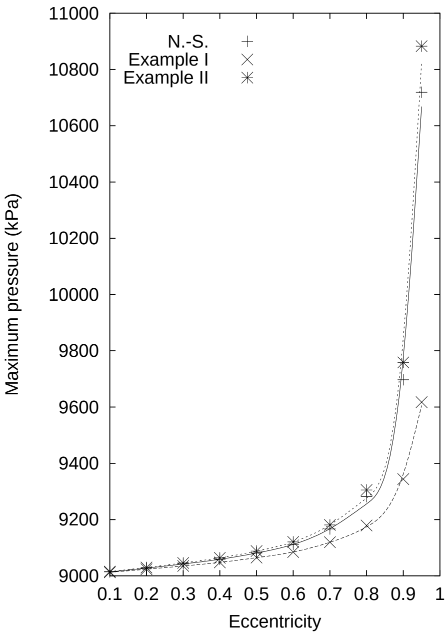


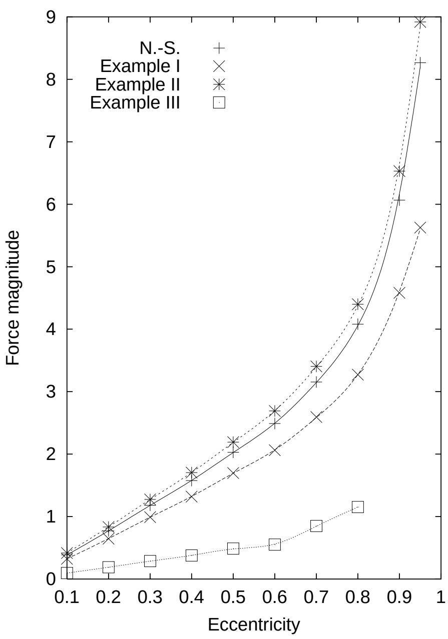
Figure 11: The viscosity field for the three examples of (P), $\mathrm{Re}^* = 100$, $\varepsilon = 0.5$.



	N.-S.	I	II	III
$\varepsilon = 0.1$	9.01	9.01	9.02	60
0.2	9.03	9.02	9.03	84
0.3	9.04	9.04	9.05	112
0.4	9.04	9.05	9.06	141
0.5	9.08	9.06	9.09	193
0.6	9.11	9.08	9.12	248
0.7	9.17	9.12	9.18	371
0.8	9.28	9.18	9.31	487
0.9	9.70	9.34	9.76	-
0.95	10.72	9.62	10.88	-

We show the pressure p/kPa in the table.
Example III is not in the graph.

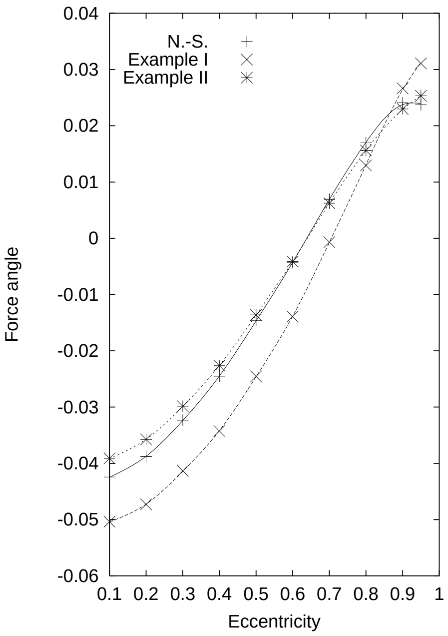
Table 10: Maximum pressure, three examples of $\text{Re}^* = 100$ for problem (P).



	N.-S.	I	II	III
$\varepsilon = 0.1$	0.4	0.3	0.4	0.1
0.2	0.8	0.6	0.8	0.2
0.3	1.2	1.0	1.3	0.3
0.4	1.6	1.3	1.7	0.4
0.5	2.0	1.7	2.2	0.5
0.6	2.5	2.1	2.7	0.6
0.7	3.2	2.6	3.4	0.8
0.8	4.1	3.3	4.4	1.2
0.9	6.1	4.6	6.5	-
0.95	8.3	5.6	8.9	-

We show the values of $\hat{\boldsymbol{f}}$.
Note that $\boldsymbol{f} = \rho U^2 V^2 \hat{\boldsymbol{f}}$.

Table 11: Force magnitude, three examples of $\text{Re}^* = 100$ for problem (P).



	N.-S.	I	II	III
$\varepsilon = 0.1$	-0.042	-0.050	-0.039	-0.35
0.2	-0.038	-0.047	-0.036	-0.33
0.3	-0.032	-0.041	-0.030	-0.23
0.4	-0.025	-0.034	-0.023	-0.14
0.5	-0.015	-0.025	-0.014	-0.00
0.6	-0.004	-0.014	-0.004	0.15
0.7	0.007	-0.001	0.006	-0.01
0.8	0.017	0.013	0.016	-0.16
0.9	0.024	0.027	0.023	-
0.95	0.024	0.031	0.025	-

Table 12: Force direction, three examples of $\mathrm{Re}^* = 100$ for problem (P).

For more perceptible pressure dependence of viscosity there occure peaks of great viscosity that cause a slow-down of convergence of the method. In the next section we find that this effect comes from numerical reasons as it vanishes by refining the mesh. The shear-thinnig effect of the fluid model, for once, could cause problems near the boundary where the areas with large velocity gradient occure. We could help us by refining the mesh in such areas. In the geometry presented in this work this has been partly ensured automatically since the large velocity gradient occures in the region of the small gap between the eccentrical circles where the mesh is also most fine. In the next section we introduce also the modified coarse mesh with smaller elements near the boundary.

4.5.4 Dependence of the quantities on the applied mesh

As we have no error estimate about how much the finite dimensional approximation solution differ from the exact "continuous" solution which we are looking for, it is customary to verify that the approximative solution within the provided mesh refinement is good enough by showing the mesh independence of the solution. It means that we should show that the solution observed on the mesh we have chosen is close enough to the solution observed on, say, twice finer or coarser mesh.

For the examples presented in this work we use the multigrid level 3, the greatest one we are able to provide for all the described eccentricies. As our coarse mesh counts from 180 elements for small eccentricities up to 600 for the case $\varepsilon = 0.95$, the multigrid-level 3 then leads to the count from 2880 up to 9600 elements. (It might be that such a choice of the coarse mesh is unnecessarily fine, on the other hand, this choice leads to very small range of aspect ratios of quadrilaterals.) Although we haven't been able to provide the multigrid-level 4 for all eccentricities and for that reason we present here all the computational results on level 3 only, for the eccentricity $\varepsilon = 0.5$ we present also the level 4 results. Moreover, we introduce a modification of the coarse mesh with smaller elements near the boundary. This modificated coarse mesh can be seen in figure 12 while the coarse and the level-3-fine mesh used in the whole work are shown in figure 3.

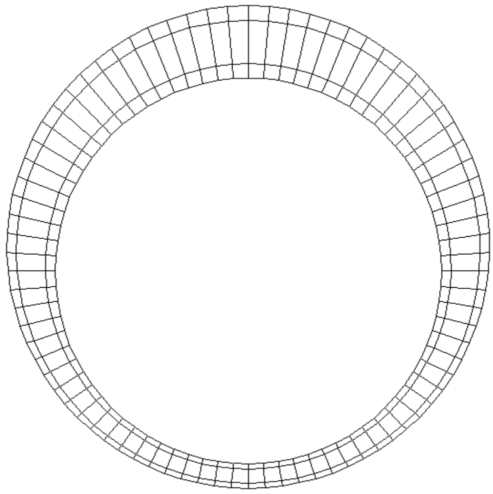


Figure 12: The modified coarse mesh.

In table 13 we systematically compare the values resulting from multigrid levels 2, 3, 4 and from two types of coarse mesh, for Navier-Stokes equations and for the three examples from section 4.5.3 for $\mathrm{Re}^* = 100$ and $\varepsilon = 0.5$. We note that the maximum and minimum values are quite sensitive to the choice of mesh since they could be determined by "sharp" peaks occuring near the boundary where high gradients occure. Unfortunately, the force acting on the journal, which is an important outcome of our simulations, is given just by the values of pressure and of shear-rate on the boundary. The resulting values of acting force are thus also quite responsive to the mesh refinement.

In figure 13 and 14 we show the viscosity and pressure fields resulting in examples I, II and III for $\varepsilon = 0.5$ for multigrid level 4 on the standard mesh and on the modified mesh (from figure 12). We remind that the viscosity and pressure fields on level 3 on the standard mesh are shown in figures 10 and 11. We do not show the stream lines nor the velocity field since no visible differences show up.

As far as the Navier-Stokes case and the examples I and II are concerned, we can take the resulting flow behaviour as sufficient for illustrative purposes. The worst findings concerns the example III where the acting force differs between level 3 and 4 by more then 30%. We also see (figures 13 and 14) that for example III both the viscosity and the pressure fields differ qualitatively. Unfortunately, we were not able to provide all the simulations on level 4 or higher.

	mesh	mg.	$\hat{p}_{\min}$	$\hat{p}_{\max}$	$\|\boldsymbol{D}(\hat{\boldsymbol{v}})\|_{\min}$	$\|\boldsymbol{D}(\hat{\boldsymbol{v}})\|_{\max}$	$\hat{\nu}_{\min}$	$\hat{\nu}_{\max}$	$\|\hat{\boldsymbol{f}}\|$	$\hat{\boldsymbol{f}}$ dir.
		2	98.84	101.15	0.016	307	1	1	2.67	-0.010
	1	3	99.02	100.95	0.014	398	1	1	2.14	-0.014
N.-S.		4	99.11	100.85	0.0006	485	1	1	1.89	-0.017
		2	98.95	100.99	0.011	781	1	1	2.29	-0.019
	2	3	99.06	100.88	0.0008	1237	1	1	1.99	-0.020
		4	99.12	100.82	0.0001	1785	1	1	1.84	-0.020
		2	99.02	100.96	0.02503	360	0.58	1.0944	2.33	-0.018
	1	3	99.20	100.76	0.00164	399	0.57	1.0947	1.81	-0.023
I		4	99.29	100.66	0.00111	528	0.53	1.0946	1.54	-0.028
		2	99.19	100.74	0.00579	615	0.51	1.0945	1.81	-0.031
	2	3	99.27	100.66	0.00060	985	0.46	1.0946	1.58	-0.033
		4	99.31	100.62	0.00003	1492	0.41	1.0946	1.46	-0.033
		2	98.76	101.23	0.01477	316	1.0929	1.0954	2.84	-0.010
	1	3	98.95	101.02	0.01467	406	1.0930	1.0952	2.30	-0.013
II		4	99.03	100.92	0.00070	490	1.0931	1.0951	2.05	-0.015
		2	98.88	101.06	0.00569	816	1.0925	1.0952	2.45	-0.018
	2	3	98.98	100.96	0.00072	1280	1.0924	1.0951	2.16	-0.018
		4	99.04	100.90	0.00008	1822	1.0920	1.0951	2.00	-0.018
		2	-0.52	0.41	0.19321	2186	0.0089	0.369	1.01	-0.13
	1	3	-0.34	0.25	0.00193	1959	0.0388	1.231	0.57	-0.15
III		4	-0.24	0.16	0.00002	813	0.0483	2.890	0.35	-0.17
		2	-0.43	0.23	0.96876	3005	0.0209	0.260	0.68	-0.23
	2	3	-0.28	0.14	0.00055	1426	0.0420	1.456	0.40	-0.26
		4	-0.21	0.10	0.0000001	1072	0.0451	1.654	0.28	-0.26

Table 13: The mesh dependence for N.-S. and the three examples of (P), $\varepsilon = 0.5$, mesh=2 denotes the modified mesh from figure 12.

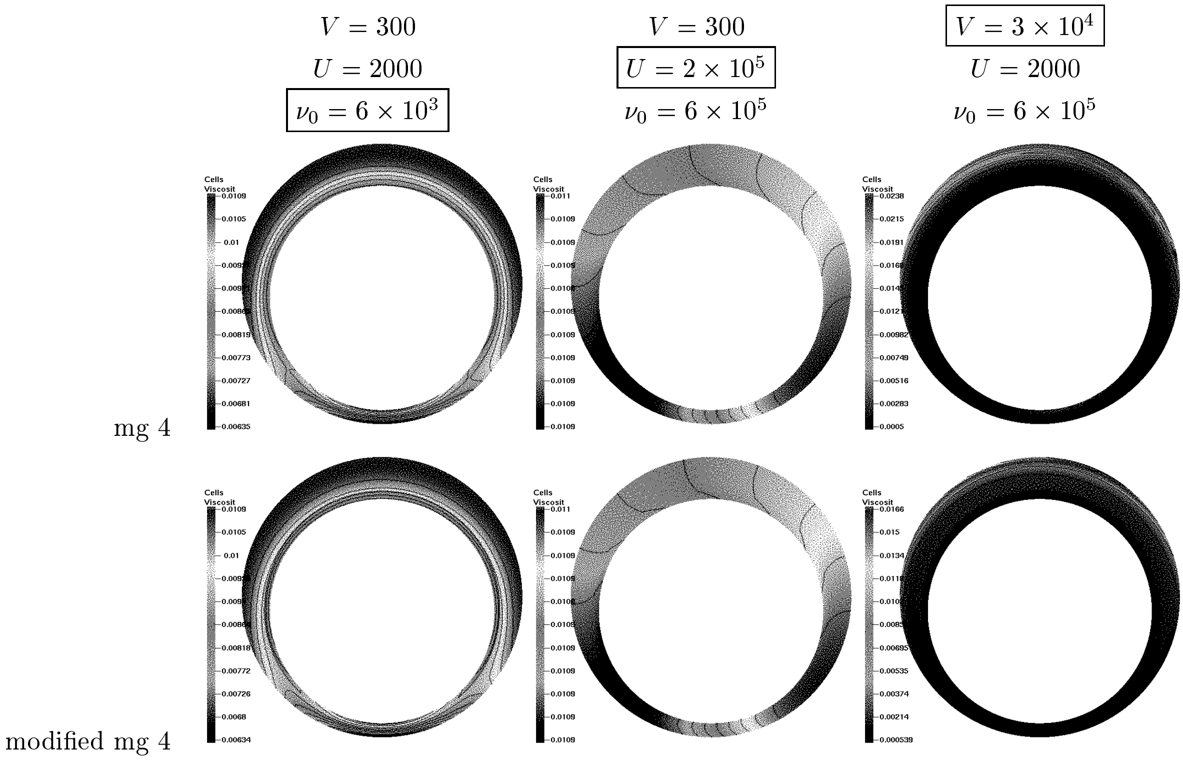


Figure 13: The mesh dependence of viscosity field for the three examples of (P), $\varepsilon = 0.5$.

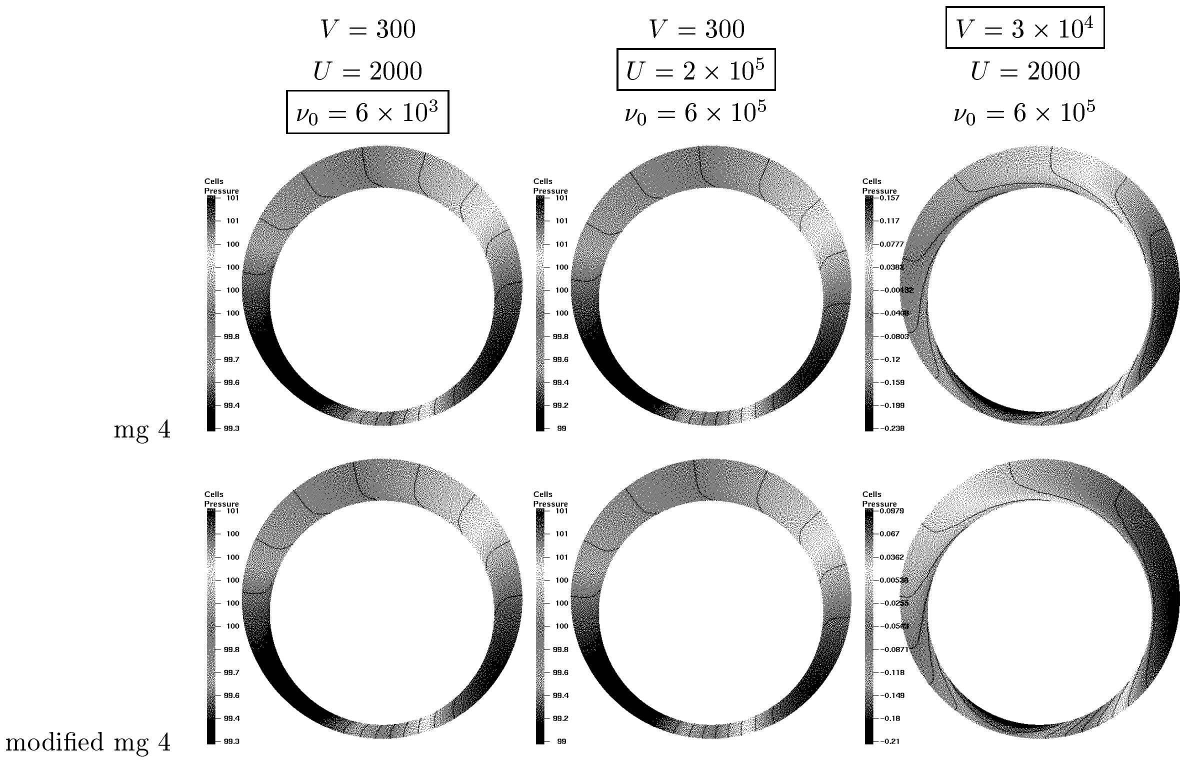


Figure 14: The mesh dependence of pressure field for the three examples of (P), $\varepsilon = 0.5$.

5 Conclusion

The main result of this work is the proof of existence of a steady-state solution to one of the generalizations of the Navier-Stokes problem with non-homogeneous Dirichlet boundary condition. In Theorem 3.13 we assume that there is no inflow or outflow through the boundary (such that the boundary consists of solid walls, for example) but there is the tangential velocity prescribed, without any further "smallness" condition. In the context of journal bearing, we thus establish the existence of a steady solution to the flow of the lubricant with the pressure- and the shear- dependent viscosity between the eccentric rotating cylinders (circles) where the speed of journal rotation is assumed to be arbitrary.

The proof strongly uses the assumption of no inflow and outflow and the result in Kaplický, Málek, Stará [2], which assume the two-dimensional flow. Nevertheless, I believe that for the special case of journal bearing geometry this result could be generalized to three dimensions. This might be worth further study.

We also present the existence result to the Stokes-like system, which is provided under more general conditions, in comparison with the Navier-Stokes-like problem. Among others, we do not need the constraint of two-dimensions. Further, the uniqueness of solution is stated, although for the Navier-Stokes-like system (P) for small data only.

The pressure level (which is considered up to the constant in the Navier-Stokes case but which becomes into the new importance in the pressure dependent viscosity case) is fixed by the mathematically natural condition on the mean value over the domain. This is not the best formulation from the physical point of view, since the pressure level should be better fixed in some small subdomain for example. This is also the question for further analysis.

In the second part of the thesis numerical simulations of the journal bearing problem are presented for one chosen sample form of the viscosity which fulfills the conditions of our existence result. The eccentricity influence is systematically studied in order to compare the behaviour of the Navier-Stokes and the generalized Navier-Stokes fluid in some selected examples. The main aim is not to give any engineering prediction or quantitative results but to show the extended capabilities of the generalized model same as the need to determine and set the additional parameters occuring in the model.

Among others, we show both the pressure-thickening and the shear-thinning capability of the chosen model (see section 4.5.2). However, its relevance should be furthermore investigated in future. In practise, the exponential laws for the pressure dependence of the viscosity are mostly considered. This can not be directly applied in the context of our theoretical results, since we assume the derivative of the viscosity with respect to the pressure bounded. Still, in some limited range of the pressure and in some limited range of the shear, the exponential behaviour could be approximated by the viscosity form (4.162),

$$\nu(p, |\boldsymbol{D}|^2) = \tilde{\nu}_0 \big(A + (\beta + \exp(\alpha p))^{-q} + |\boldsymbol{D}|^2\big)^{\frac{r-2}{2}},$$

introduced in our numerical experiments. How satisfactory is the reality approached by this model, that is a question for further searching.

The numerical method used in this work, i. e. the finite element approach processed by the `featflow` software package using the fixed point iteration technique as a non-linear solver and the multigrid linear solver (see section 4.1), seems to be general enough to solve the nonlinear viscosity model we have chosen. Anyhow, the fixed point convergence rate, especially at the geometry with higher eccentricities, starts to signify that it is not the most efficient method. The Newton-method might be examined as another possible way.

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