Recall: F1

- $y_I = y_{i_1} \cdots y_{i_q}$, $I = (i_1, \dots, i_q)$, I non-decreasing, where $y_i := \sigma(X_i)$ & $(X_i)_{i=1}^n$ basis of $\mathfrak{g}$ $\Rightarrow$ $\{y_I \mid I \text{ non-decr.}\}$ is basis of $\mathcal{U}(\mathfrak{g})$ (Poincaré-Birkhoff-Witt).

- A a $\mathbb{k}$-algebra: 1) $A_p \subseteq A_{p+1}$ $\mathbb{k}$-modules (i.e., vect. spaces over $\mathbb{k}$)

  &

  2) $A_p \cdot A_q \subseteq A_{p+q}$, $p \in \mathbb{N}_0, q \in \mathbb{N}_0$

"Many" notions, classical, in study of algebras

$\Rightarrow$ $(A_p)_p$ is filtration of A

- $T(\mathfrak{g}) = \mathbb{k} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$ tensor algegebra $(= \otimes^p \mathfrak{g})$

  $T_p(\mathfrak{g}) = \mathbb{k} \oplus \mathfrak{g} \oplus \dots \oplus \underbrace{\mathfrak{g} \otimes \dots \otimes \mathfrak{g}}_{p\text{-times}}$

  $(T_p(\mathfrak{g}))_p$ is filtration [2) by definition of $\cdot$ as $\otimes$]

- $S(\mathfrak{g}) = \mathbb{k} \oplus \mathfrak{g} \oplus \odot^2 \mathfrak{g} \oplus \dots$ polynomial algebra

  ($\odot^2$: symm. power)

  $S_p(\mathfrak{g}) = \mathbb{k} \oplus \mathfrak{g} \oplus \dots \oplus \underbrace{\mathfrak{g} \odot \dots \odot \mathfrak{g}}_{p\text{-times}}$

  $(S_p(\mathfrak{g}))_p$ is a filtration [2) by properties of mult. of pol.]

$\Rightarrow$ Associated grading, assoc. graded algebra

Filtrat. $(A_p)_p$ $\rightsquigarrow$ $G^p := A_p / A_{p-1}$, $A_{-1} := \{0\}$

quotient of vect. spaces

$$\left(\begin{array}{l} G^0 = A_0/\{0\} \cong A_0 \\ G^1 = A_1/A_0,\ G_2 = A_2/A_1, \text{ etc.} \end{array} \right)$$

$\Rightarrow G := \bigoplus_{p \in \mathbb{N}_0} G^p$ assoc. graded is vect. space

$\Rightarrow [g'] \cdot [g''] := [g'g'']$, $g', g'' \in A$ ($\Rightarrow [a]+[b] = [a+b]$) $a, b \in A$ F2

- Is def correct?

$[g_k + u_{k-1}][g_\ell + u_{\ell-1}] = [(g_k+u_{k-1})(g_\ell+u_{\ell-1})] =$

$g_k \in A_k$, $u_{k-1} \in A_{k-1}$, $g_\ell \in A_\ell$, $u_{\ell-1} \in A_{\ell-1}$

A_{k-1} (may be from A_{k-2}, A_{k-3} etc)

$= [g_k g_\ell + \underbrace{g_k u_{\ell-1}}_{A_{k+\ell-1}} + \underbrace{u_{k-1} g_\ell}_{A_{k-1+\ell}} + \underbrace{u_{k-1} u_{\ell-1}}_{A_{k+\ell-2}}] =$

$= [g_k g_\ell]$. It is correct.

by quotiening out

- Similarly it is corr. defined on G.
- Similarly: $r \cdot [a] := [ra]$, $r \in \Bbbk$, $a \in A$, is corr. def
...

$\Rightarrow$ Apply this to $\mathcal{U}(g)$:

First $S^p(g) := S_p(g)/S_{p-1}(g)$

Thus just polynomials of homogeneity p.

Second $T^p(g) := T_p(g)/T_{p-1}(g) =$

$= (\Bbbk \oplus g \oplus \otimes^2 g \oplus \cdots \oplus \otimes^p g) / (\Bbbk \oplus \cdots \oplus \otimes^{p-1} g)$

$\cong \otimes^p g$. Thus homogeneus tensors; e.g.:

this $\underbrace{x_1 \otimes \cdots \otimes x_1}_{p\text{-times}} + \underbrace{x_n \otimes \cdots \otimes x_n}_{p\text{-times}}$. Summands $x_1 \otimes \cdots \otimes x_1$, $x_n \otimes \cdots \otimes x_n$ so called 'decomposable'

Third $\mathcal{U}(g)$: $\mathcal{U}_p(g) = T_p(g)/J$ = more

[we don't know whether $J \subseteq T_p(g)$] formally $\mathcal{U}_p(g) := \pi(T_p(g))$

$\pi : T(g) \to T(g)/J$

or even more formally $\mathcal{U}_p(g) = \pi_{|T_p(g)}(T_p(g)) =$

$= \mathrm{Im}(\pi|_{T_p(\mathfrak{g})})$. Convention: filtrat: lower indices / grading: upper case F3

Remark: • We guess: $\mathcal{U}_0(\mathfrak{g}) = \pi(\mathbb{k}) = \mathbb{k}$ since "J is not contained in $\mathbb{k}$"

• $\mathcal{U}_1(\mathfrak{g}) = \pi(\mathbb{k} \oplus \mathfrak{g}) = \overset{\text{by guess}}{\mathbb{k}} + \pi(\mathfrak{g}) \cong \mathbb{k} + \pi i(\mathfrak{g}) = \mathbb{k} + \sigma(\mathfrak{g}) \underset{\text{by PBW } (\sigma \text{ is mono})}{\cong} \mathbb{k} + \mathfrak{g}$

which even equals to $\mathbb{k} \oplus \mathfrak{g}$ since $\mathbb{k} \cap \mathfrak{g} = \{0\}$ (this is assumed actually).

The guess is a task for a student (tutorials/exerc.).

$G := \bigoplus \mathcal{U}^p(\mathfrak{g})$. We have $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}_p(\mathfrak{g})$, we have $\mathcal{U}^p(\mathfrak{g})$ (i.e., the associated graded denoted by G).

(thm.) ▷▷ Though $\mathcal{U}(\mathfrak{g})$ need not be commutative, G is commut.

Proof: $[X_1^{i_1} \cdots X_n^{i_n}] \cdot [X_1^{j_1} \cdots X_n^{j_n}] = $ ($\sum i_k = p$, $\sum j_e = q$)

$$= [X_1^{i_1} \cdots X_n^{i_n - 1} X_n \cdot X_1 X_1^{j_1 - 1} \cdots X_n^{j_n}]$$

$$= [X_1^{i_1} \cdots X_n^{i_n - 1} (X_n \cdot X_1) X_1^{j_1 - 1} \cdots X_n^{j_n}]$$

$$= [X_1^{i_1} \cdots X_n^{i_n - 1} (X_1 X_n + [X_n, X_1]) X_1^{j_1 - 1} \cdots X_n^{j_n}]$$

$$= [X_1^{i_1} \cdots X_n^{i_n - 1} X_1 X_n X_1^{j_1 - 1} \cdots X_n^{j_n}] + [\underbrace{X_1^{i_1} \cdots X_n^{i_n - 1}}_{p-1} \underbrace{[X_n, X_1]}_{1} \underbrace{X_1^{j_1 - 1} \cdots X_n^{j_n}}_{q-1}]$$ degree

F4

Thus the 2^{nd} term has degree $p+q-1 \Rightarrow$ it vanishes since A_{p+q-1} is mod out in G^{p+q} $(= A_{p+q}/A_{p+q-1})$.

In this way, x_1 is transfered to the left;

the rest $x_1^{j_1-1}$ is transfered in a similar way.

Similarly we transfer $x_2^{j_2}$, $x_3^{j_3}$, etc. as well, finally getting $[x_1^{j_1} \dots x_n^{j_n} x_1^{i_1} \dots x_n^{i_n}]$ ~~which~~ which is by def. of mult in G: $[x_1^{j_1} \dots x_n^{j_n}] \cdot [x_1^{i_1} \dots x_n^{i_n}]$.

Thus commutativity on these simple (decomposable) elements is proved. Of course G^p does not contain only decomposable elements (of degree p: $x_1^{i_1} \dots x_n^{i_n}$, $\sum_k i_k = p$), but also their finite sums. Thus one shall consider them as well, but this case reduces to the previous one, by the def. of addition in assoc. graded $[a]+[b]=[a+b]$ since, e.g., $\square$

$[(a_1+a_2)(b_1+b_2)] = [a_1 b_1] + [a_1 b_2] + [a_2 b_1] + [a_2 b_2]$.

▷ Now, we would like to compare $\mathcal{U}(g)$ and $S(g)$.

We have $g \xrightarrow{\sigma} \mathcal{U}(g)$, $\sigma = \pi \circ i$ (from „the beginning" of semester).

Set: $\varphi(x) := [\sigma(x)]$ ← quotient class in associated graded G, $x \in g$

$[\sigma(x)] = [x] \in \mathcal{U}_1(g) = G_1 \subseteq G$

by PBW; just notation!

[by def. of filtrat. it is $\pi(\mathbb{k} \oplus g)$, i.e., $\subseteq \mathbb{k} \oplus g$. By guess even $= \mathbb{k} \oplus g$

F5

So we have arrows $\overset{\varphi}{\nearrow}$ & $\searrow$ in:

$$
\begin{tikzcd}
 & G \arrow[r, Rightarrow] & \textcircled{?} \\
\mathfrak{g} \arrow[ur, "\varphi"] \arrow[dr, "\text{canonical}"'] \arrow[r, phantom, "\circlearrowleft"] & & \\
 & S(\mathfrak{g}) \arrow[uu, "\exists!\,\varphi' \text{ homom of algs}"'] &
\end{tikzcd}
$$

? ⟸ Question

$S(\mathfrak{g}) = \mathbb{k} \oplus \mathfrak{g} \oplus \odot^2 \mathfrak{g} \oplus \cdots$

canonical, i.e., $x_i \mapsto z_i$

$\exists\, \varphi'$? Set $\varphi'(z_1^{i_1} \cdots z_n^{i_n}) := \varphi(x_1)^{i_1} \cdots \varphi(x_n)^{i_n}$ *)

!? Also uniqueness is easy: φ' being homom, it must preserve multiplication in $S(\mathfrak{g})$

<u>Remark:</u> One can prove universality of $S(\mathfrak{g})$ among commutative unital assoc. algebras. Since φ can be proved to be 'compatible' (suitably compatible), the !-part for φ' could be deduced and need not be said that homom. must preserve (multiplication).

<u>Theorem:</u> $\varphi' : S(\mathfrak{g}) \longrightarrow G$ is an isomorphism of assoc. algebras.

<u>Proof</u>: 1) <u>epimorphism</u> (easy): $\sigma(\mathfrak{g})$ generates $\mathcal{U}(\mathfrak{g})$ [we mentioned that this is by def. (not by PBW)] $\Rightarrow \varphi(\mathfrak{g})$ generates. [Ordo analysis: "

class in G $\longrightarrow [x_I]$, $(x_1, \dots, x_n)$ basis of $\mathfrak{g}$

$x_I = x_{i_1} \cdots x_{i_q}$, $I = (i_1, \dots, i_q)$ %.

---

*) Better: $\varphi'(z_I) := \varphi(x_{i_1})\varphi(x_{i_2}) \cdots \varphi(x_{i_q})$ for $I = (i_1, \dots, i_q)$ (but equivalent by commutativity)

$[X_I] = \varphi'(Z_I)$? $\quad \varphi'(z_{i_1} \cdots z_{i_q}) = \varphi(z_{i_1}) \cdots \varphi(z_{i_q})$

$= [X_{i_1}] \cdots [X_{i_q}] =$

$= [X_{i_1} \cdots X_{i_q}] = [X_I]$ ✓

(Notice: It is not to convenient to use $z_1^{i_1} \cdots z_n^{i_n}$ since $(i_1 \cdots i_n)$ is a different multiindex!)

2) monom. (enhanced a bit to the lecture):

Consider

$$\varphi'\left(\sum c_I Z_I\right) = 0$$

I non decreasing in all elem. of the sum (convention of multiind. as used in last lecture)

$$\varphi'\Big(\sum_{|I|=p} c_I z_I + \underbrace{\sum_{|I|<p} c_I z_I}_{\text{this is mod-out in } G}\Big) = 0$$

Thus $\varphi'\left(\sum_{|I|=p} c_I z_I\right) = 0 \overset{1)}{\Rightarrow} \underset{\text{class in } G}{\left[\sum_{|I|=p} c_I X_I\right]} = 0 \overset{2)}{\Rightarrow}$

$\sum_{|I|=p} c_I \underset{\text{class in } G}{[X_I]} = 0$

$U(g) \ni \sum_{|I|=p} c_I X_I \in A_{p-1}$ $\overset{3)}{\Rightarrow}$ $\overset{4)}{\Rightarrow}$ $\sum_{|I|=p} c_I X_I = 0$ in $U(g)$, $(p-1<p)$

by PBW $\Rightarrow c_I = 0$ since X_I basis. $\square$

Remark: It is convenient to use $Z_I = z_{i_1} \cdots z_{i_q}$, than $Z_I = z_1^{i_1} \cdots z_m^{i_m}$ what we do sometimes, however. I recommend to use the first exclusively!

Remark: If $z_4^2 z_1^3 z_2^5$ in $P^{\le 10}[z_1, z_2, z_3, z_4]$

$\updownarrow$

(different use of the term multiindex)

$$z_1^3 z_2^5 z_4^2 \rightsquigarrow z_1 z_1 z_1 z_2 z_2 z_2 z_2 z_2 z_4 z_4$$

$$\rightsquigarrow I = (1, 1, 1, 2, 2, 2, 2, 2, 4, 4) \text{ our convention.}$$

In PDE's: $\dfrac{\partial^{10}}{\partial z_1^3 \, \partial z_2^5 \, \partial z_4^2} \rightsquigarrow I' = (3, 5, 0, 2)$

is the multiindex & $\partial_{I'}$ and

$z^{I'} = z_4^2 z_1^3 z_2^5$ is the convention. Do not use it in this lecture, since it may lead to mistake. We do not do PDEs (though we mentioned Laplace etc.)

This was solved at the end of the lecture since there were misunderstanding.

▷ It is just a recommendation, also for me.