! Lemma: $\forall p \in \mathbb{N}_0\ \exists!\ f_p : \mathfrak{g} \otimes P_p \longrightarrow P_{p+1}$ that is $\mathbb{K}$-linear E1
and satisfies $(\subseteq P)$

(A_p) $f_p(x_i \otimes z_I) = z_i z_I \quad \forall i \leq I \quad \forall z_I \in P_p$

(B_p) $f_p(x_i \otimes z_I) - z_i z_I \in P_{p-1} \quad \forall i\ \forall I\ \forall z_I \in P_p$

(C_p) $f_p(x_i \otimes f_{p-1}(x_j \otimes z_J)) = f_p(x_j \otimes f_{p-1}(x_i \otimes z_J))$
$+ f_p([x_i, x_j] \otimes z_J) \quad \forall z_J \in P_{p-1} \subseteq P_p$.

(Here, also f_{p-1} can be written) Moreover, $f_p|_{\mathfrak{g} \otimes P_{p-1}} = f_{p-1}$.

Proof: I) $p = 0$ ($P_{-1} = \{0\}$, $f_{-1} = 0$ we suppose it).
$f_0(x_i \otimes 1) := z_i$ (A_0); $f_0(x_i \otimes 1) - z_i = 0 \in P_0 (= \mathbb{K}1)$
(B_0); (C_0) $\quad f_0(x_i \otimes 0) = 0$, $f_0(x_j \otimes 0) = 0$
$f_0([x_i, x_j] \otimes 0) = f_0(0) = 0$
"moreover" part is trivial $f_0|_{\mathfrak{g} \otimes P_{-1}} = 0$ & $f_{-1} = 0$.

II) Suppose $\exists!\ f_{p-1}$ s.t. $A_{p-1}, B_{p-1}, C_{p-1}$ and "moreover"-part is satisfied (this is a choice of our approach)
$f_p(x_i \otimes z_I) = ?$ $\quad I$ nondecreasing is sufficient
since polynomials are commutative. (This is assumed to hold for all p)
a) $i \leq I \xRightarrow{\text{forced by } A_p} \boxed{f_p(x_i \otimes z_I) := z_i z_I}$
b) If not, write $I = (j, J)$ ($j \leq J$ automatic)
and now $j < i$.
$f_p(x_i \otimes z_I) = f_p(x_i \otimes z_j z_J) \overset{\text{forced by a) since } j \leq J \text{ and by "moreover"}}{=} f_p(x_i \otimes f_{p-1}(x_j \otimes z_J)) = f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) +$
forced by C_p
$+ f_p([x_i, x_j] \otimes z_J) =$
f_{p-1} by "moreover" $(\|J\| (= \sum j_k) = p-1)$

$= f_p(x_j \otimes (z_i z_J + w)) + f_{p-1}([x_i, x_j] \otimes z_J) =$ E2

$\Uparrow$
P_{p-1} ← forced by B_{p-1} (thus also induction is used)

$= f_p(x_j \otimes z_i z_J) + f_{p-1}(x_j \otimes w) + f_{p-1}([x_i, x_j] \otimes z_J) \overset{(*)}{=}$

Since $j \le \{i\} \cup J$ (we add i into J), we have by A_p (which is true by a)), that: $(*) = z_j z_i z_J$ + something determined + something determined. Thus f_p is determined by a) & b). A_p and B_p hold since we defined f_p in this way. We have to verify C_p. (Notice that we already proved the uniqueness of f_p)

(1) $j \le J$ & $j < i$: C_p satisfied by def. of f_p in item b)

Let us analyse the logical possibilities:

$$\begin{array}{ccc} i \le J & \xrightarrow{\neg} & i \not\le J \\ \lor & & \land \quad (5) \\ j \le J & & j \not\le J \end{array}$$

($\mathbb{N}_0$ is well ordered) $\to \Updownarrow$

$$(i \le J \land i \le j)^{(4)} \lor (i \le J \land i > j)^{(3)}$$
$$\lor^{(2)} \ (j \le J \land i \le j) \lor (j \le J \land i > j)^{(1)}$$

(2) $i \le j \le J$ C_p will follow by $\boxed{\text{antisymmetry}}$ (It does not follow by b) due to the different inequalities.)

(3) follow by b).

(4) $i \leq j \wedge i \leq J$ will follow by antisymmetry (again b) cannot be used).

(5) will be checked at the end and explained.

· The $\boxed{\text{antisymmetry}}$ ($i \leq j$) for the cases when

'?' $f_p(x_i, f_{p-1}(x_j \otimes z_J)) = f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) +$

$+ f_{p-1}([x_i, x_j] \otimes z_J)$

But by a), we know $\underline{f_p(x_j \otimes f_{p-1}(x_i \otimes z_J))} = f_p(x_i, f_{p-1}(x_j \otimes z_J))$

$+ f_{p-1}([x_j, x_i] \otimes z_J) = \underline{f_p(x_i, f_{p-1}(x_j \otimes z_J)) - f_{p-1}([x_j, x_i]}$ by antisymmetry of $[\,,]$

$\underline{\otimes z_J)}$. Consequently the '?' holds.

this was done for (2) and (4).

The item (5): Further subdividing of multiindices is used: $J = (k, K)$, $k \leq K$ (automatic), $k < i$ and $k < j$. This is what (5) expresses.

· We omit f_p, we write $f(\tilde{x}, f(\tilde{\tilde{x}} \otimes z)) = \tilde{x}\tilde{\tilde{x}}z$ as.

· Again, we shall subtract $x_i(x_j z_J)$ and $x_j(x_i \cdot z_J)$ (for checking C_p). First, compute: $\underline{x_i(x_j \cdot z_J)} = x_i(x_j x_k z_K) =$

$\underset{\text{by } C_{p-2}}{=} x_i(x_k(x_j z_K) + [x_j, x_k] z_K) =$

(paying by commutator)

$\underset{\text{distrib. in assoc. alg: it is a ring!}}{=} x_i(x_k(x_j z_K)) + x_i([x_j, x_k] z_K)$. We move twice $\underline{i}$ more to the right because in the term to be subtracted i is also more to the right. Of course, you can do it when you subtract, i.e., later. I.e. we get

$$X_k(X_i\cdot(X_j\cdot Z_K)) + [X_i, X_k](X_j\cdot Z_K) + [X_j, X_k](X_i\cdot Z_K)$$
$$+ [X_i, [X_j, X_k]]\, Z_K. \quad \text{F}$$

Now we subtract (we do it without assuming the result):

thus it is less uniform than lecture

$$X_i\cdot(X_j\cdot Z_J) - X_j\cdot(X_i Z_J) = X_k(X_i\cdot(X_j\cdot Z_K)) + \cancel{[X_i, X_k](X_j\cdot Z_K)}$$
$$+ \cancel{[X_j, X_k](X_i\cdot Z_K)} + [X_i, [X_j, X_k]]\, Z_K$$

← just interchanging i & j from the fmla F

$$- X_k(X_j\cdot(X_i\cdot Z_K)) - \cancel{[X_j, X_k](X_i\cdot Z_K)}$$
$$- \cancel{[X_i, X_k](X_j\cdot Z_K)} - [X_j, [X_i, X_k]]\, Z_K =$$
$$= X_k(X_i\cdot(X_j\cdot Z_K)) - X_k(X_j(X_i\cdot Z_K)) + [X_i, [X_j, X_k]]\, Z_K$$
$$- [X_j, [X_i, X_k]]\, Z_K \overset{(B_{p-2})}{=} X_k([X_i, X_j]\, Z_K) +$$
$$+ [X_i, [X_j, X_k]]\, Z_K - [X_j, [X_i, X_k]]\, Z_K$$

Now, we return k back / order it for Jacobi

$$= [X_i, X_j]\, X_k\, Z_K + [X_k, [X_i, X_j]]\, Z_K$$
$$+ [X_i, [X_j, X_k]]\, Z_K + [X_j, [X_k, X_i]]\, Z_K$$
$$= [X_i, X_j]\, Z_J, \text{ that is } C_p. \qquad \square$$

Remark: 1. $\sigma(x_i) = y_i$ recall

2. $\tau(x_i)(z_I) := f_p(x_i, z_I)$

$\tau : a_y \longrightarrow \mathrm{End}(P) =: A$

$\tau([x_i,x_j])z_I = f_p([x_i,x_j]\otimes z_I) \overset{C_p}{=}$

$= f_p(x_i\otimes f_{p-1}(x_j\otimes z_I)) - f_p(x_j\otimes f_{p-1}(x_i\otimes z_I))$

$= \tau(x_i)(\tau(x_j)z_I) - \tau(x_j)(\tau(x_i)z_I) =$

$= [\tau(x_i),\tau(x_j)]z_I$, Thus τ is a compatible morphism. By universality of $\mathcal{U}(g)$ $\exists!\,\tau'$ homom of ass. alg. with 1, s.t.

$$\begin{tikzcd} & \mathrm{End}(P) \\ g \arrow[ur, "\tau"] \arrow[r, "\sigma"'] & \mathcal{U}(g) \arrow[u, "\exists!\,\tau'"'] \end{tikzcd}$$

Theorem (Poincaré-Birkhoff-Witt): $\{y_I \mid I$ nondecreasing (incl. $\emptyset$)$\}$ is a basis (over k) of $\mathcal{U}(g)$

Proof: 1. We know that $\{y_I \mid \dots\}$ generates $\mathcal{U}(g)$

2. Lin. indep.: $\sum c_I y_I = 0$ / τ' from above

remark $\sum c_I \tau'(y_I) = 0$ / evaluation 1

polynomial

$\sum c_I \tau'(\sigma(x_{i_1})\dots\sigma(x_{i_q}))(1) = 0$

$\sum c_I z_I = 0 \Rightarrow c_I = 0\ \forall I$, since $\{z_I \mid I$ nondecr.$\}$ is a basis of P. $\square$

Remark: 1. (z_I), I nondecr. is basis. Why. Induction: $P[z_1,\dots,z_n] = (P(z_1,\dots,z_{n-1}))[z_n]$ and we know it in one variable, i.e., $\mathbb{P}[z_1]$. Why $\{1, z_1, z_1^2, \dots\}$ E6

basis of $P[z_1]$? Generates: by def. of $P[z_1]$ ✓

Lin. indep. (e.g.) $\sum_{i=0}^{n} c_i z_1^i = 0$ everywhere

$\xrightarrow{\text{division alg}}$ divisible by $(z_1 - q)$ $\forall q$. Thus

$\sum^{n} c_i z_1^i = \prod_{q \in \mathbb{R}} (z_1 - q)$ ↯ $n \neq \infty$.
order

<u>Remark</u>: 1. σ injective ($\Leftarrow$ PBW)

2. We (often) omit writing σ.