$$L := \lim_{x \to 0} \frac{\sin x - x}{x \sin x}$$

l'Hospital:

$$L \overset{l'H \frac{0}{0}}{=} \lim_{x \to 0} \frac{\cos x - 1}{\sin x + x\cos x} \overset{l'H \frac{0}{0}}{=} \lim_{x \to 0} \frac{-\sin x}{\cos x + \cos x - x\sin x} \overset{\text{VoAL, spojitost při}}{=} \frac{0}{2} = 0$$

Taylor:

2/2

$$L = \lim_{x \to 0} \frac{x - \frac{x^3}{6} + o(x^4) - x}{x\left(x - \frac{x^3}{6} + o(x^4)\right)} = \lim_{x \to 0} \frac{-\frac{x^3}{6} + o(x^4)}{x^2 - \frac{x^4}{6} + o(x^5)} =$$

$$= \lim_{x \to 0} \frac{x^2}{x^2} \cdot \frac{-\frac{x}{6} + o(x^2)}{1 - \frac{x^2}{6} + o(x^3)} = \lim_{x \to 0} \frac{-\frac{x}{6} + o(x^2)}{1 - \frac{x^2}{6} + o(x^3)} \overset{\text{VoAL, spojitost}}{=} \frac{0}{1} = 0.$$

BOLTUAN M.

2) $L := \lim_{x\to 0} \frac{a^x - a^{\sin x}}{x^3}, \quad a > 0$

$$L \overset{l'H \text{ "}\frac{0}{0}\text{"}}{=} \lim_{x\to 0} \frac{a^x \ln a - a^{\sin x}\cos x \ln a}{3x^2} = \lim_{x\to 0} \frac{\ln a \left(a^x - a^{\sin x}\cos x\right)}{3x^2} =$$

$$\overset{l'H \frac{0}{0}}{=} \lim_{x\to 0} \frac{\ln a \left(a^x \ln a - \left(a^{\sin x}\cos x \ln a \cdot \cos x + a^{\sin x}(-\sin x)\right)\right)}{6x} =$$

$$= \lim_{x\to 0} \frac{\ln a}{6}\left(\frac{a^x \ln a - \ln a\, a^{\sin x}\cos^2 x + a^{\sin x}\sin x}{x}\right) \overset{l'H \frac{0}{0}}{=}$$

$$= \lim_{x\to 0} \frac{\ln a}{6}\left(a^x \ln a \cdot \ln a - \ln a\left(a^{\sin x}\cos x \ln a \cdot \cos^2 x + a^{\sin x} 2\cos x(-\sin x)\right) +\right.$$

$$\left. + \left(a^{\sin x}\cos x \ln a \sin x + a^{\sin x}(+\cos x)\right)\right) \overset{\text{VOAL + SPOJITOST}}{=}$$

$$= \frac{\ln a}{6}\left(\ln^2 a - \ln a(\ln a + 0) + (0+1)\right) = \frac{\ln a}{6}(1) = \underline{\underline{\frac{\ln a}{6}}}.$$

$f(x) = \frac{\cos x}{\cos 2x}$

a) $D_f = \mathbb{R} \setminus \{\frac{\pi}{4} + k\frac{\pi}{2}, k \in \mathbb{Z}\}$ — dhm fce je spojitá na D_f jelikož $\cos x$, $\cos 2x$ jsou spojité fce a $\frac{f(x)}{g(x)}$ je spojitá, pro $g(x) \neq 0$.

$\cos 2x \neq 0$

$x \neq \frac{\pi}{4} + k\frac{\pi}{2}, k \in \mathbb{Z}$

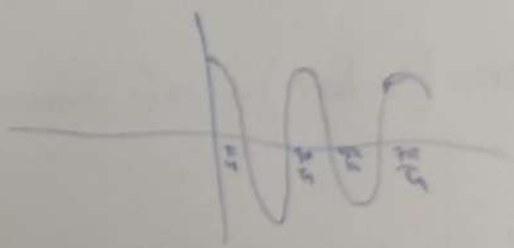
fce má periodu 2π (čitatel 2π, jmenovatel π)

fce je sudá (čitatel i jmenovatel sudé fce)

b) → Fce je periodická, stačí tedy zkoumat jen limity v $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$\lim_{x \to \frac{\pi}{4}+} \frac{\cos x}{\cos 2x} = \lim_{x \to \frac{\pi}{4}+} \cos x \cdot \lim_{x \to \frac{\pi}{4}+} \frac{1}{\cos 2x} = \frac{\sqrt{2}}{2} \cdot (-\infty) = -\infty$$

$$\lim_{x \to \frac{\pi}{4}-} \frac{\cos x}{\cos 2x} = +\infty$$



$\lim_{x \to \frac{3\pi}{4}+} f(x) = -\infty$ $\qquad$ $\lim_{x \to \frac{3\pi}{4}-} f(x) = +\infty$

$\lim_{x \to \frac{5\pi}{4}+} f(x) = +\infty$ $\qquad$ $\lim_{x \to \frac{5\pi}{4}-} f(x) = -\infty$ $\qquad$ c) $H_f = \mathbb{R}$

$\lim_{x \to \frac{7\pi}{4}+} f(x) = +\infty$ $\qquad$ $\lim_{x \to \frac{7\pi}{4}-} f(x) = -\infty$

6) Nulové body: $f(x) = 0 \iff \cos x = 0$

$x = \frac{\pi}{2} + k\pi$

fce je periodická → limity v $\pm\infty$ nejsou

3) $f'(x) = \frac{-\sin x \cos 2x - \cos x(-\sin 2x)\cdot 2}{(\cos 2x)^2} = \frac{2\cos x \sin 2x - \sin x \cos 2x}{(\cos 2x)^2} =$

$= \frac{4\cos^2 x \sin x - \sin x \cos^2 x + \sin^3 x}{(\cos 2x)^2} = \frac{\sin x\left(3\cos^2 x + \sin^2 x\right)}{(\cos 2x)^2} = \frac{\sin x\left(1+2\cos^2 x\right)}{(\cos 2x)^2}$

$f'(x) = 0 \iff \sin x\left(1+2\cos^2 x\right) = 0 \quad \cos^2 x \geq 0$ ~~arsinx~~

$\sin x = 0 \quad x = 0,\ x = \pi$ } zkoumáme pouze periodu pre $f(x)$ ~~dlw~~ ktorá je 2π

$\rightarrow f'(x) = 0 \quad x = 0 \quad x = \pi$

$f'(x) > 0 \Rightarrow f$ rostoucí na: $(0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{3\pi}{4}), (\frac{3\pi}{4}, \pi)$ ..

$f'(x) < 0 \Rightarrow f$ klesající na: $(\pi, \frac{5\pi}{4}), (\frac{5\pi}{4}, \frac{7\pi}{4}), (\frac{7\pi}{4}, 2\pi)$ ..

7) $f''(x) = \frac{\left(\cos x\left(1+2\cos^2 x\right) + \sin x(-4\cos x \sin x)\right)(\cos 2x)^{\cancel{2}} - \sin x\left(1+\cos^2 x\right)2\cos 2x\,(-\underbrace{\sin 2x}_{-2\sin x\cos x})2}{(\cos 2x)^{\cancel{4}\,3}} =$

$= \frac{\left(\cos x + 2\cos^3 x - 4\cos x \sin^2 x\right)\cos 2x + 8\sin^2 x \cos x\left(1+2\cos^2 x\right)}{(\cos 2x)^3} =$

$= \frac{\cos x}{(\cos 2x)^3}\cdot\left[\left(\underbrace{1+2\cos^2 x - 4\sin^2 x}_{3\cos 2x}\right)^{\cancel{4}}\cos 2x + 8\sin^2 x + 16\sin^2 x \cos^2 x\right] =$

$= \frac{\cos x}{(\cos 2x)^3}\cdot\left[3(\cos 2x)^2 + 4(\sin 2x)^2 + 8\sin^2 x\right] = \frac{\cos x}{(\cos 2x)^3}\cdot\underbrace{\left[3 + (\sin 2x)^2 + 8\sin^2 x\right]}_{=:A}$

$f''(x) = 0 \iff \cos x \cdot A = 0$ ale $A > 0$ vždy

$\Rightarrow \cos x = 0 \quad x = \frac{\pi}{2} \quad x = \frac{3\pi}{2}$ (na periodě 2π)

5) lokální extrémy: MIN: $x = 0 \quad f(0) = 1$

MAX: $x = \pi \quad f(\pi) = -1$

$f'' > 0: (0, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{3\pi}{4}), (\frac{5\pi}{4}, \frac{3\pi}{2}), (\frac{7\pi}{4}, 2\pi)$

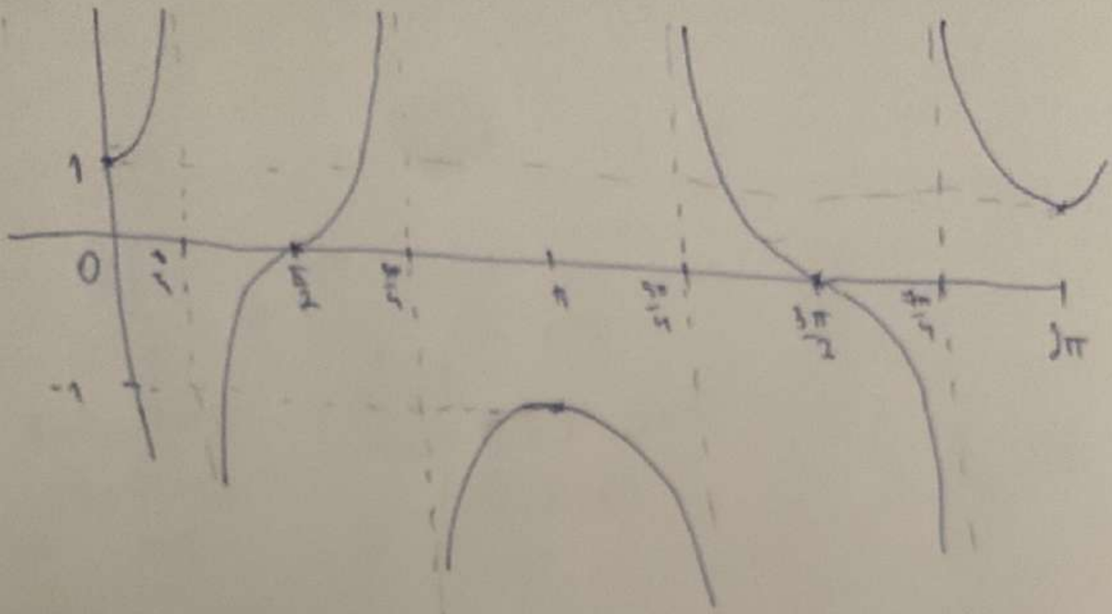
konvexní

$f'' < 0: (\frac{\pi}{4}, \frac{\pi}{2}), (\frac{3\pi}{4}, \frac{5\pi}{4}), (\frac{3\pi}{2}, \frac{7\pi}{4})$

konkávní

asymptoty nejsou

Nota:


1
0
-1
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{4}$
$\frac{3\pi}{2}$
$\frac{7\pi}{4}$
2π

$f(x) = \arccos \frac{2x}{x^2+1}$

1) $D_{\arccos} = [-1, 1]$

$$\frac{2x}{x^2+1} \geq -1 \quad \wedge \quad \frac{2x}{x^2+1} \leq 1$$

$$x^2+2x+1 \geq 0 \qquad 0 \leq x^2-2x+1$$

$$(x+1)^2 \geq 0 \qquad 0 \leq (x-1)^2$$

$$x \in \mathbb{R} \qquad x \in \mathbb{R}$$

$$\Bigg\} \; D_f = \mathbb{R}$$

- fce je spojitá na $D_f = \mathbb{R}$, protože arccos je spojitá fce.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{\pi}{2}$$

růstové řady x^2 jde k ∞ rychleji než x $\Rightarrow \arccos 0 = \frac{\pi}{2}$

2) fce není sudá, lichá ani periodická

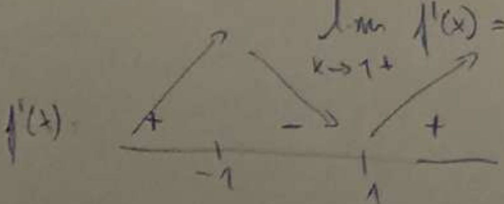
3)

$$f'(x) = \frac{-1}{\sqrt{1-\left(\frac{2x}{x^2+1}\right)^2}} \cdot \frac{2(x^2+1) - \overbrace{2x \cdot 2x}^{4x^2}}{(x^2-1)^2} = \frac{2x^2-2}{\sqrt{\frac{(x^2+1)^2-(2x)^2}{\cancel{(x^2+1)^2}} \cdot (x^2+1)^{\cancel{4}\,2}}} = \frac{2x^2-2}{\sqrt{\underbrace{(x^4-2x^2+1)}_{(x^2-1)^2}(x^2+1)^2}}$$

$$= \frac{2x^2-2}{(x^2+1)|x^2-1|} = \frac{2(\cancel{x^2-1})}{(x^2+1)\,\mathrm{sgn}(x^2-1)(\cancel{x^2-1})} = \frac{2}{(x^2+1)\,\mathrm{sgn}(x^2-1)}$$

$f'(x)$ neexistuje v $x = \pm 1$ protože:

$$\lim_{x\to 1^+} f'(x) = 1 \quad \lim_{x\to 1^-} f'(x) = -1 \quad \wedge \quad \lim_{x\to -1^+} f'(x) = -1 \quad \lim_{x\to -1^-} f'(x) = 1$$



$f'(x) > 0$ na $(-\infty, -1)$, $(1, \infty)$ ... rostoucí

$f'(x) < 0$ na $(-1, 1)$ ... klesající

4) $x = -1$ je lok. MAX $f(-1) = \pi$

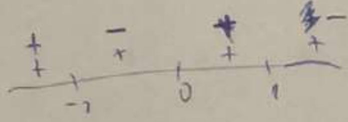
$x = 1$ —"— MIN $f(1) = 0$

5) $H_f = [0, \pi]$

6) $f(x) = 0 \iff \frac{2x}{x^2+1} = 1 \quad x = 1 \quad f(1) = 0$

7) $f''(x) = \left(\frac{2}{x^2+1} \operatorname{sgn}(x^2-1)\right)' = \frac{0 - 2\operatorname{sgn}(x^2-1) \cdot 2x}{(x^2+1)^2} = \frac{-4x \operatorname{sgn}(x^2-1)}{(x^2+1)^2}$

$f''(x) = 0$: $x = 0$, $x = \pm 1$

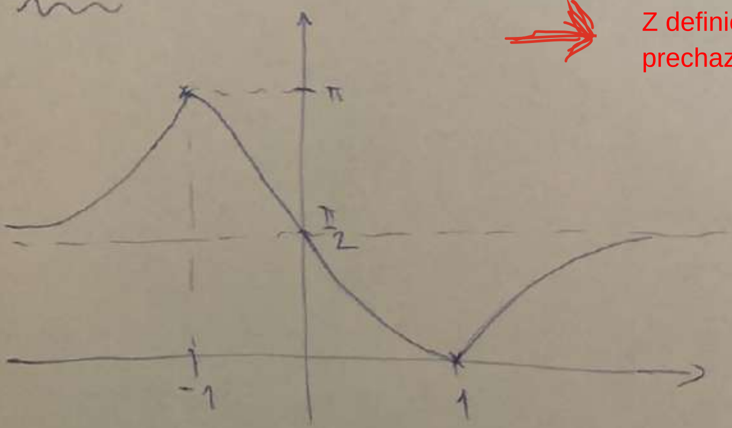


$f''(x) > 0$ na $(-\infty,-1)$, $(0,1)$ … konvexní

$f''(x) < 0$ na $(-1,0)$, $(1,\infty)$ … konkávní

$\Bigg\}$ $x = 0$ inflexní bod, $f(0) = \frac{\pi}{2}$

Nárt:



Z definice z vasi prednasky plyne, ze i -1, 1 jsou bodem inflexe - prechazi tam konvexnost v konkavnost respektive naopak.

Bod nabyti inflexe je definovan jako takovy, kde konvexnost prechazi v konkavnost nebo naopak. Specilně tedy v nem nemusi existovat druha a ani prvni derivace. Zde jsou tedy inflexni tedy i body -1 a 1

1,8/2

Hezke. Nemam co vytknout .