Full name: ____

Final test – sample with solutions
Variant A

In every problem, justify all steps properly.

1. (4 points) Find the limit

$$\lim_{n\to\infty} \frac{(n+3)^3-(n-1)^3}{(2n+5)^{\frac{3}{2}}\sqrt{2n+3}}.$$

Answer: 3

2. (4 points) Find the derivative of the function

$$f(x) = \ln\left(\frac{3x}{x-5}\right)$$

and find the domain of definition of the function $f(x)$ and of its derivative $f'(x)$.

Answer: $f'(x) = \frac{5}{5x-x^2}$, $D_f = (-\infty, 0) \cup (5, +\infty)$, $D_{f'} = (-\infty, 0) \cup (0, 5) \cup (5, +\infty)$

3. (12 points) A parabola is given as a graph of the function

$$f(x) = 2x^2 - 3x - 2.$$

Find all points $x_0 \in \mathbb{R}$ at which the tangent line to the parabola has its slope equal to 1. At every such point, find the equation of the tangent. Draw the parabola including its itercepts with the axes and vertex. Draw the previously found tangent line into the same picture, including its intercepts with the axes and its contact point with the parabola.

Answer: tangent $y = x - 4$, contact point $[1, -3]$, roots $-1/2, 2$, vertex $[3/4, -25/8]$

For graph, click here.

4. (20 points) Examine the course of the function

$$f(x) = \frac{8(3-x)}{(x-4)^2}$$

i.e. find its domain of definition D_f, limits at endpoints of D_f, intercepts with axes, derivative of f, intervals of monotony, global and local extrema, intervals of convexity and concavity, asymptotes. Draw the graph.

Answer: $D_f = \mathbb{R} - \{4\}$, $P_x = [3,0], P = [0,3/2]$, $\lim_{x\to\pm\infty} f(x) = 0, \lim_{x\to 4} f(x) = -\infty$, therefore asymptote at $\pm' infty$ is $y = 0$, and a vertical asymptote $x = 4$, $f'(x) = \frac{8(x-2)}{(x-4)^3}$, f increases in $(-\infty, 2)$ and in $(4, +\infty)$, decreases in $(2,4)$, globmax $[2,2]$, $f''(x) = \frac{-16(x-1)}{(x-4)^4}$, f convex in $(-\infty, 1)$, concave in $(1,4)$ and in $(4, +\infty)$.

For graph, click here.

5. (20 points) Find the extremes of the function $f(x,y) = x^2 + y^2 - 4y + 3x$ on the set

$$M = \{[x,y] \in \mathbb{R}^2;\ x^2 + y^2 \leq 100;\ x + 2y + 10 \geq 0\}.$$

For candidates at the curved part of the boundary of M (except for vertices), find the value of λ. Draw the set M including all candidates for an extreme.

Answer – all candidates and values:
$f(-10,0) = 70, f(6,-8) = 150$ MAX, $f(-3/2,2) = -6,25$ MIN, $f(-4,-3) = 25, f(-6,8) = 50, \lambda = 3/4$

For picture of M, click here.

Full name: ____

Final test – sample
Variant B

In every problem, justify all steps properly.

1. (4 points) Find the limit

$$\lim_{x\to 3} \frac{\ln(x^2+x-11)}{x^2-7x+12}.$$

Answer: -7

2. (4 points) Find the derivative of the function

$$f(x) = \frac{(x+2)^2}{\sqrt{x+1}}$$

and find the domain of definition of the function $f(x)$ and of its derivative $f'(x)$.

Answer: $f'(x) = \frac{(x+2)(3x+2)}{2(x+1)^{\frac{3}{2}}}$, $D_f = D_{f'} = (-1,\infty)$

3. (12 points) A hyperbola is given as a graph of the function

$$f(x) = \frac{x+1}{2x-6}.$$

Find all points $x_0 \in \mathbb{R}$ at which the tangent line to the parabola has its slope equal to -2. At every such point, find the equation of the tangent. Draw the hyperbola including its itercepts with the axes, center and asymptotes. Draw the previously found tangent line(s) into the same picture, including its intercepts with the axes and its contact point(s) with the hyperbola.

Answer: contact points: $[2,-3/2], [4,5/2]$; tangents $y = -2x + \frac{5}{2}, y = -2x + \frac{21}{2}$; center $[3,1/2]$, $P_y = [0,-1/6], P_x = [-1,0]$

For graph, click here.

4. (20 points) Examine the course of the function

$$f(x) = \frac{x^2 + 2x - 15}{x - 4}$$

i.e. find its domain of definition D_f, limits at endpoints of D_f, intercepts with axes, derivative of f, intervals of monotony, global and local extrema, intervals of convexity and concavity, asymptotes. Draw the graph.

Answer: $D_f = \mathbb{R} - \{4\}$, $P_{x1} = [-5, 0]$, $P_{x2} = [3, 0]$, $P = [0, 3/2]$, $\lim_{x \to \pm\infty} f(x) = \pm\infty$, $\lim_{x \to 4\pm} f(x) = \pm\infty$, $f'(x) = \frac{x^2-8x+7}{(x-4)^2}$, f increases in $(-\infty, 1)$ and in $(7, +\infty)$, decreases in $(1, 4)$ and in $(4, 7)$, locmax $[1, 4]$, locmin $[7, 16]$, $f''(x) = \frac{18}{(x-4)^3}$, f concave in $(-\infty, 4)$, convex in $(4, +\infty)$, asymptote at $\pm'infty$ is $y = x + 6$, and a vertical asymptote $x = 4$.

For graph, click here.

5. (20 points) Find the extremes of the function $f(x, y) = x^3 - 3x^2 + x + xy - y$ on the set

$$M = \{[x, y] \in \mathbb{R}^2 : 2x - 4 \leq y \leq -x^2 + 3x + 2\}.$$

For candidates at the curved part of the boundary of M (except for vertices), find the value of λ. Draw the set M including all candidates for an extreme.

Answer – all candidates and values:
$f(3; 2) = 7$ MAX, $f(-2; -8) = 2$, $f(1; 2) = -1$, $f(-1; -6) = 7$ MAX, $f(5/3; -2/3) = -67/27$ MIN, $f(0; 2) = -2, \lambda = -1$

For picture of M, click here.