Primitivní funkce II

Nalezněte následující primitivní funkce na maximálních možných intervalech. Určete i tyto intervaly.

1. $\displaystyle\int \frac{x^3+1}{x^3-5x^2+6x}\,\mathrm{d}x$

2. $\displaystyle\int \frac{1}{(x^3+1)^2}\,\mathrm{d}x$

   Vhodnou substitucí převeďte integrály na integrály z racionálních funkcí a ty se pokuste vyřešit.

3. $\displaystyle\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\,\mathrm{d}x$

4. $\displaystyle\int x\sqrt{x^2-2x+2}\,\mathrm{d}x$

5. $\displaystyle\int \frac{x+\sqrt{1+x+x^2}}{1+x+\sqrt{1+x+x^2}}\,\mathrm{d}x$

6. $\displaystyle\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,\mathrm{d}x$

   Nalezněte následující primitivní funkce

7. $\displaystyle\int \frac{\sin^2 x}{1+\sin^2 x}\,\mathrm{d}x$

8. $\displaystyle\int \frac{1}{2\sin x-\cos x+5}\,\mathrm{d}x$

9. $\displaystyle\int \frac{\sin x\cos x}{1+\sin^3 x}\,\mathrm{d}x$

10. $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x$

11. $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x$

12. $\displaystyle\int \sqrt{a^2+x^2}\,\mathrm{d}x$

Důležité substituce: převod na racionální funkce

Jsou-li P, Q polynomy $\mathbb{R} \to \mathbb{C}$, pak $R := \frac{P}{Q}$ nazveme racionální funkce jedné reálné proměnné, platí $R(x) = \frac{P(x)}{Q(x)}$.

Obecněji, jsou-li P, Q polynomy dvou reálných proměnných, tj. $P, Q : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, kde $P(x,y) = \sum_{0 \le i+j \le n} a_{ij} x^i y^j$ a $Q(x,y) = \sum_{0 \le i+j \le m} b_{ij} x^i y^j$, pak $R := \frac{P}{Q}$ nazveme racionální funkce dvou reálných proměnných, platí $R(x,y) = \frac{P(x,y)}{Q(x,y)}$.

$$\boxed{\textbf{(I)} \qquad \int \mathbf{R}(\mathbf{e}^{\alpha \mathbf{x}}) \mathbf{dx}}$$

Substituce: $y = e^{\alpha x}, \quad x \in \mathbb{R}$

Tvar derivace: $dx = \frac{1}{\alpha y} dy$

Výsledek: $\int R(y) \frac{1}{\alpha y} dy$

$$\boxed{\textbf{(II)} \qquad \int \frac{\mathbf{R}(\ln \mathbf{x})}{\mathbf{x}} \mathbf{dx}}$$

Substituce: $y = \ln x, \quad x > 0$

Tvar derivace: $\frac{dx}{x} = dy$

Výsledek: $\int R(y) dy$

$$\boxed{\textbf{(III)} \qquad \int \mathbf{R}\left(\mathbf{x}, \left(\frac{\mathbf{ax+b}}{\mathbf{cx+d}}\right)^{\frac{1}{s}}\right) \mathbf{dx}}$$

Substituce: $t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}}$

Podmínky: $ad - bc \neq 0$; $\qquad s = 2k \implies \frac{ax+b}{cx+d} > 0$, $\qquad s = 2k-1 \implies x \neq -\frac{d}{c}$

Inverze: $x = \frac{-dt^s + b}{ct^s - a}$

Tvar derivace: $dx = (ad - bc)s \frac{t^{s-1}}{(ct^s - a)^2} dt$

Výsledek: $(ad - bc)s \int \frac{\hat{R}(t^s, t) t^{s-1}}{(ct^s - a)^2} dt$

$$\boxed{\textbf{(IV)} \qquad \int \mathbf{R}\left(\mathbf{x}, \sqrt{\mathbf{ax^2+bx+c}}\right) \mathbf{dx}}$$

`Eulerovy substituce`

Čtyři netriviální případy (někdy i dva najednou).

(1) $ax^2 + bx + c = a(x - x_1)(x - x_2),\ x_1 < x_2,\ x_1, x_2 \in \mathbb{R}$

Substituce: $t = \left(\frac{x - x_1}{x - x_2}\right)^{\frac{1}{2}}$ vede k **(III)**

(2) $a > 0$

Substituce: $\sqrt{ax^2 + bx + c} = \sqrt{a}x + t \implies x = (t^2 - c)/(b - 2\sqrt{a}t)$

(3) $c > 0$

Substituce: $\sqrt{ax^2 + bx + c} = \sqrt{c} + tx \implies x = (2\sqrt{c}t - b)/(a - t^2)$

(4) $a \le 0$ a $ax^2 + bx + c$ nemá v $\mathbb{R}$ kořen ($\implies c \le 0$): odmocnina není v $\mathbb{R}$ pro žádné x definována.

$$\boxed{\textbf{(V)} \qquad \int \mathbf{R}(\cos\mathbf{x},\sin\mathbf{x})\mathbf{dx}}$$ `Goniometrické substituce`

Substituce: $y = \operatorname{tg}\frac{x}{2}$ $\qquad x \neq \pi + 2k\pi\,, k\in\mathbb{Z}$

Inverze: $x = 2\operatorname{arctg} y$ $\qquad$ *Tvar derivace:* $dx = \frac{2}{1+y^2}dy$

$$\text{cosinus:} \qquad \cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-y^2}{1+y^2}$$

$$\text{sinus:} \qquad \sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2y}{1+y^2}$$

Zjednodušení:

(1) $R(-\cos x,\sin x) = -R(\cos x,\sin x) \implies$ *Substituce:* $y=\sin x$

(2) $R(\cos x,-\sin x) = -R(\cos x,\sin x) \implies$ *Substituce:* $y=\cos x$

(3) $R(-\cos x,-\sin x) = R(\cos x,\sin x) \implies$ *Substituce:* $y=\operatorname{tg} x,\ x\neq\frac{\pi}{2}+k\pi$

$$\cos^2 x = \frac{\cos^2 x}{\cos^2 x+\sin^2 x} = \frac{1}{1+\operatorname{tg}^2 x} = \frac{1}{1+y^2}$$

$$\sin^2 x = \frac{\sin^2 x}{\cos^2 x+\sin^2 x} = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{y^2}{1+y^2}$$

$$\sin x\cos x = \frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{y}{1+y^2}$$

$$\boxed{\textbf{(VI)} \qquad \int \mathbf{x^m(a+bx^n)^p dx}\,, \qquad \mathbf{m,n,p}\in\mathbb{Q}}$$ `Čebyševovy substituce`

Umíme řešit pomocí elementárních funkcí pouze v následujících třech případech:

(1) $p\in\mathbb{Z}$. Pak položme $m=m'/\ell$, $n=n'/\ell$, kde m', n' a $\ell\in\mathbb{Z}$, $\ell>0$.

Substituce: $t = x^{\frac{1}{\ell}}$

(2) $(m+1)/n\in\mathbb{Z},\ p=k/s,\ k,s\in\mathbb{Z}$

Substituce: $t=(a+bx^n)^{\frac{1}{s}}$

Inverze: $x=\frac{(t^s-a)^{1/n}}{b^{1/n}}$ $\qquad$ *Tvar derivace:* $dx = \frac{1}{nb^{1/n}}(t^s-a)^{\frac{1}{n}-1}st^{s-1}dt.$

Výsledek: $\int x^m(a+bx^n)^p dx = \int \frac{1}{b^{\frac{m}{n}}}(t^s-a)^{\frac{m}{n}}t^k\frac{1}{nb^{\frac{1}{n}}}(t^s-a)^{\frac{1}{n}-1}st^{s-1}dt$

$= \frac{s}{nb^{\frac{m+1}{n}}}\int t^{s+k-1}(t^s-a)^{\frac{m+1}{n}-1}dt$

(3) $\frac{m+1}{n}+p\in\mathbb{Z},\ p=k/s,\ k,s\in\mathbb{Z}$

Substituce: $t=(ax^{-n}+b)^{\frac{1}{s}}$

Inverze: $x=\left(\frac{a}{t^s-b}\right)^{\frac{1}{n}}$ $\qquad$ *Tvar derivace:* $dx = -\frac{a^{1/n}}{n}(t^s-b)^{-\frac{1}{n}-1}st^{s-1}dt$

Výsledek: $\int x^m(a+bx^n)^p dx = \int x^m x^{np}(ax^{-n}+b)^{\frac{k}{s}}dx$

$= \int\left(\frac{a}{t^s-b}\right)^{\frac{m}{n}}t^k\left(\frac{a}{t^s-b}\right)^p\frac{a^{\frac{1}{n}}}{-n}(t^s-b)^{-\frac{1}{n}-1}st^{s-1}dt$

$= -\frac{a^{\frac{m+1}{n}+p}s}{n}\int t^{k+s-1}(t^s-b)^{-(\frac{m+1}{n}+p-1)}dt$

Integrace racionálních funkcí obecně

$R(x) = \frac{P(x)}{Q(x)}$, $P(x), Q(x)$ polynomy. Hledáme $\int R(x)dx$.

1. krok: Pokud je st. $P \geq$ st. Q : částečné dělení, abychom dostali $\frac{P}{Q} = P_1 + \frac{P_2}{Q}$,
kde st. $P_2 <$ st. Q

Dělení polynomu polynomem! Příklad:

$$(x^4 + 2x^2 + 6x + 9) : (2x^2 + 9x + 3) = \frac{1}{2}x^2 - \frac{9}{4}x + \frac{83}{8} - \frac{645x + 177}{8(2x^2+9x+3)}$$

$$\begin{array}{l}
-(x^4 + \frac{9}{2}x^3 + \frac{3}{2}x^2) \\
\hline
-\frac{9}{2}x^3 + \frac{1}{2}x^2 + 6x + 9 \\
-(-\frac{9}{2}x^3 - \frac{81}{4}x^2 - \frac{27}{4}x) \\
\hline
\frac{83}{4}x^2 + \frac{51}{4}x + 9 \\
-(\frac{83}{4}x^2 + \frac{747}{8}x + \frac{249}{8}) \\
\hline
-\frac{645}{8}x - \frac{177}{8}
\end{array}$$

Po 1. kroku umíme najít $\int P_1(x)\,dx$. Zbývá $\int \frac{P_2(x)}{Q(x)}dx$

2. krok: Rozklad na parciální zlomky. $Q(x) = (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k} \cdot (x^2+p_1x+q_1)^{s_1} \cdot \ldots \cdot (x^2+p_\ell x+q_\ell)^{s_\ell}$

Neboli Q má kořeny $\alpha_1, \ldots, \alpha_k$ a každý z nich může mít vyšší násobnost (r_i) a dále nerozložitelné polynomy druhého stupně (se zápornými diskriminanty)

Pak platí, že existují čísla A_i^m, B_j^m, C_j^m tak, že

$$\frac{P(x)}{Q(x)} = \left[\frac{A_1^1}{(x-\alpha_1)} + \frac{A_1^2}{(x-\alpha_1)^2} + \ldots + \frac{A_1^{r_1}}{(x-\alpha_1)^{r_1}}\right] + \ldots + \left[\frac{A_k^1}{(x-\alpha_k)} + \frac{A_k^2}{(x-\alpha_k)^2} + \ldots + \frac{A_k^{r_k}}{(x-\alpha_k)^{r_k}}\right] +$$

$$+ \left[\frac{B_1^1x + C_1^1}{(x^2+p_1x+q_1)} + \ldots + \frac{B_1^{s_1}x + C_1^{s_1}}{(x^2+p_1x+q_1)^{s_1}}\right] + \ldots + \left[\frac{B_\ell^1x + C_\ell^1}{(x^2+p_\ell x+q_\ell)} + \ldots + \frac{B_\ell^{s_\ell}x + C_\ell^{s_\ell}}{(x^2+p_\ell x+q_\ell)^{s_\ell}}\right]$$

Pozor! Každý vícenásobný kořen se rozloží na tolik zlomků, jakou má násobnost!
Totéž pro každý výraz $(x^2+p_ix+q_i)^{s_i}$

Integrace jednotlivých parciálních zlomků:

- $\frac{A}{x-\alpha} \longrightarrow \ln(\ )$
- $\frac{A}{(x-\alpha)^k} \longrightarrow (x-\alpha)^{-k+1}$
- $\frac{Bx+C}{x^2+px+q} \longrightarrow \underbrace{\left(\frac{2x+p}{x^2+px+q}\right)\cdot\frac{B}{2}}_{\longrightarrow \ln(x^2+px+q)} + \underbrace{\frac{\tilde{C}}{x^2+px+q}}_{\longrightarrow \operatorname{arctg}}$

$$(\ldots)^{-k+1}$$
$$\uparrow$$

- $\frac{Bx+C}{(x^2+px+q)^k} \longrightarrow \frac{B}{2}\cdot\frac{(2x+p)}{(x^2+px+q)^k} + \underbrace{\frac{\tilde{C}}{(x^2+px+q)^k}}$ $\longrightarrow$ lin. substituce: $\int\frac{1}{(x^2+1)^k}dx$

Pro integrály $\int \frac{1}{(x^2+1)^k}\,dx$ máme rekurentní formuli

$I_1 = \mathrm{arctg}\, x + C$

$$I_k = \left| \begin{matrix} f'=1 & g=\frac{1}{(1+x^2)^k} \\ f=x & g'=\frac{-2kx}{(1+x^2)^{k+1}} \end{matrix} \right| = \frac{x}{(1+x^2)^k} + 2k\int \frac{x^2+1-1}{(1+x^2)^{k+1}}\,dx = \frac{x}{(1+x^2)^k} + 2kI_k - 2kI_{k+1}$$

$$\Longrightarrow \underbrace{I_{k+1} = \frac{2k-1}{2k} I_k + \frac{1}{2k}\frac{x}{(1+x^2)^k}}$$

1) $$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = \int \frac{x^3-5x^2+6x}{x^3-5x^2+6x} + \frac{5x^2-6x+1}{x^3-5x^2+6x}\,dx = x + \underbrace{\int \frac{5x^2-6x+1}{x^3-5x^2+6x}\,dx}_{I}$$

$Q(x) = x^3-5x^2+6x = x(x-2)(x-3)$

(*) $$\frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

2 možnosti: 1) dát PS na společný jmenovatel a porovnat čitatele.
2) tzv. "zakrývací metoda"

Princip "zakrývací metody": vynásobit (*) $(x-\alpha_i)$ a dosadit $x=\alpha_i$, kde α_i jsou kořeny

$\longrightarrow$ $$\frac{5x^2-6x+1}{(x-2)(x-3)} = A + x\cdot\frac{B}{x-2} + x\cdot\frac{C}{x-3}$$ a dosaď $x=0$

$$\underline{\frac{1}{6} = A}$$

$\longrightarrow$ $$\frac{5x^2-6x+1}{x\cdot(x-3)} = (x-2)\frac{A}{x} + B + (x-2)\frac{C}{x-3}$$ a dosaď $x=2$: $\underline{\frac{-9}{2} = B}$

$\longrightarrow$ $$\frac{5x^2-6x+1}{x(x-2)} = (x-3)\frac{A}{x} + (x-3)\frac{B}{x-2} + C$$ a dosaď $x=3$: $\underline{\frac{28}{3} = C}$

Pozor: Tato metoda dá koeficienty jen u nejvyšších mocnin parc. zlomků a nefunguje pro kvadratické členy ve jmenovateli

$$I = \frac{1}{6}\int\frac{dx}{x} - \frac{9}{2}\int\frac{dx}{x-2} + \frac{28}{3}\int\frac{dx}{x-3}$$ a proto $$\int\frac{x^3+1}{x^3-5x^2+6x}\,dx = \underline{x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C}$$

pro $x \in (-\infty,0) \cup (0,2) \cup (2,3) \cup (3,\infty)$

2) $\int \frac{1}{(x^3+1)^2}\,dx = \int \frac{dx}{(x+1)^2(x^2-x+1)^2} = \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} + \frac{Ex+F}{(x^2-x+1)^2}\,dx = (*)$ ③

Zakrývací metoda: $B = \frac{1}{9}$. Nic dalšího z ní nedostaneme

$\Rightarrow$ Roznásobení: $1 = A\cdot(x+1)(x^2-x+1)^2 + \frac{1}{9}\cdot(x^2-x+1)^2 + (Cx+D)(x^2-x+1)\underbrace{(x+1)^2}_{(x^2+2x+1)} + (Ex+F)\underbrace{(x+1)^2}_{(x^2+2x+1)}$

$$1 = A\cdot(x+1)(x^4-2x^3+3x^2-2x+1) + \frac{1}{9}(x^4-2x^3+3x^2-2x+1) + (Cx+D)(x^4+x^3+x+1) + Ex^3 + (2E+F)x^2 + (E+2F)x + F$$

$$1 = Ax^5 - Ax^4 + Ax^3 + Ax^2 - Ax + A + \frac{1}{9}(x^4-2x^3+3x^2-2x+1) + Cx^5 + (C+D)x^4 + Dx^3 + Cx^2 + (C+D)x + D + Ex^3 + (2E+F)x^2 + (E+2F)x + F$$

x^5: $A + C = 0$	x^2: $A + \frac{1}{3} + C + 2E + F = 0$
x^4: $-A + \frac{1}{9} + C + D = 0$	x: $-A - \frac{2}{9} + C + D + E + 2F = 0$
x^3: $A - \frac{2}{9} + D + E = 0$	1: $A + \frac{1}{9} + D + F = 1$

$A = -C$:
$$2C + D = -1/9$$
$$-C + D + E = 2/9$$
$$2E + F = -1/3$$
$$2C + D + E + 2F = 2/9$$
$$-C + D + F = 8/9$$

$\Rightarrow -1/9 + E + 2F = 2/9 \Rightarrow E + 2F = 1/3$
$$2E + F = -1/3$$
$$\Sigma:\ 3(E+F) = 0$$
$$E = -F$$

$\boxed{F = 1/3 \quad E = -1/3}$

Dosaď E, F: $2C + D = -1/9$, $-C + D - 1/3 = 2/9$
$$2C + D = -1/9$$
$$-C + D = 5/9$$
$$3C = -2/3$$
$$C = -2/9 \Rightarrow \underline{D = 1/3}$$
$$\Downarrow$$
$$\underline{A = 2/9}$$

$$(*) = \frac{2}{9}\int \frac{dx}{x+1} + \frac{1}{9}\int \frac{dx}{(x+1)^2} + \int \frac{-\frac{2}{9}x + 1/3}{x^2-x+1}\,dx + \int \frac{-1/3 x + 1/3}{(x^2-x+1)^2}\,dx = I_1 + I_2 + I_3 + I_4$$

$I_1 = \frac{2}{9}\ln|x+1|$ $\qquad I_2 = -\frac{1}{9}\cdot\frac{1}{(x+1)}$

$$I_3 = -\frac{1}{9}\int \frac{(2x-3)\,dx}{x^2-x+1} = -\frac{1}{9}\int \frac{(2x-1)\,dx}{x^2-x+1} - \frac{1}{9}\int \frac{-2}{x^2-x+1}\,dx = -\frac{1}{9}\ln(x^2-x+1) + \overbrace{\frac{2}{9}\int \frac{dx}{(x^2-x+\frac{1}{4})+\frac{3}{4}}}^{I_3'}$$

$$I_3' = \frac{2}{9}\cdot\frac{4}{3}\cdot\int \frac{dx}{\left(\frac{2}{\sqrt{3}}(x-\frac{1}{2})\right)^2+1} = \left| \begin{array}{l} t = \frac{2x-1}{\sqrt{3}} \\ dt = \frac{2}{\sqrt{3}}dx \end{array} \right| = \frac{8}{27}\cdot\frac{\sqrt{3}}{2}\cdot\int \frac{dt}{t^2+1} = \frac{4\sqrt{3}}{27}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$I_4 = -\frac{1}{3}\int \frac{x-1}{(x^2-x+1)^2}\,dx = -\frac{1}{6}\int \frac{2x-2}{(x^2-x+1)^2}\,dx = -\frac{1}{6}\int \frac{2x-1}{(x^2-x+1)^2}\,dx + \frac{1}{6}\int \frac{1}{(x^2-x+1)^2} = +\frac{1}{6}\cdot\frac{1}{x^2-x+1} + \overbrace{\frac{1}{6}\int \frac{1}{\left((x-\frac{1}{2})^2+\frac{3}{4}\right)^2}\,dx}^{I_4'}$$

$$I_4' = \frac{1}{6}\cdot\left(\frac{4}{3}\right)^2\cdot\int \frac{1}{\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)^2}\,dx = \frac{16}{54}\cdot\frac{\sqrt{3}}{2}\int \frac{dt}{(t^2+1)^2} = \frac{4\sqrt{3}}{27}\cdot\frac{1}{2}\operatorname{arctg} t + \frac{4\sqrt{3}}{27}\cdot\frac{1}{2}\cdot\frac{t}{1+t^2}$$

Vše dohromady:

$$\int \frac{1}{(x^3+1)^2}\,dx = \frac{2}{9}\ln|x+1| - \frac{1}{9}\cdot\frac{1}{x+1} - \frac{1}{9}\ln(x^2-x+1) + \frac{4\sqrt3}{27}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right) + \frac{1}{6}\,\frac{1}{x^2-x+1} + \frac{2\sqrt3}{27}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right)$$
$$+\frac{2\sqrt3}{27}\cdot\frac{\frac{2x-1}{\sqrt3}}{1+\left(\frac{2x-1}{\sqrt3}\right)^2} + C$$

$$= \frac{2}{9}\ln|x+1| - \frac{1}{9}\ln(x^2-x+1) + \frac{2\sqrt3}{9}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right) - \frac{1}{9}\cdot\frac{1}{x+1} + \frac{1}{6}\cdot\frac{1}{x^2-x+1} + \frac{2}{27}\cdot\frac{2x-1}{\frac{4}{3}x^2-\frac{4}{3}x+\frac{4}{3}} + C$$

$$= \frac{2}{9}\ln|x+1| - \frac{1}{9}\ln(x^2-x+1) + \frac{2\sqrt3}{9}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right) - \frac{1}{9}\cdot\frac{1}{x+1} + \frac{1}{6}\cdot\frac{1}{x^2-x+1} + \frac{1}{18}\cdot\frac{2x-1}{x^2-x+1} + C$$

$$= \frac{2}{9}\ln|x+1| - \frac{1}{9}\ln(x^2-x+1) + \frac{2\sqrt3}{9}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right) - \frac{1}{9}\cdot\frac{1}{x+1} + \frac{1}{9}\cdot\frac{x+1}{x^2-x+1} + C$$

$$= \frac{2}{9}\ln|x+1| - \frac{1}{9}\ln(x^2-x+1) + \frac{2\sqrt3}{9}\operatorname{arctg}\left(\frac{2x-1}{\sqrt3}\right) + \frac{1}{9}\cdot\left(\frac{3x}{x^3+1}\right) + C \qquad x\in(-\infty,-1)\cup(1,\infty)$$

3) Substituce vedoucí na racionální funkce : viz přiložená tabulka

$$I = \int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} \quad \dots \text{ typ } R(x, x^{1/6}) \text{ tj. (III). Substituce } t = x^{1/6}$$

$$dt = \frac{1}{6}x^{-5/6}\,dx$$
$$dx = 6t^5\,dt$$

$$I = \int \frac{6t^5\,dt}{t^6\cdot(1+2t^3+t^2)} = 6\int\frac{dt}{t(2t^3+t^2+1)} = 6\int\frac{dt}{t(t+1)(2t^2-t+1)} = 6\cdot\int \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{2t^2-t+1}$$

Zakrývací metoda: $A=1$, $B=-1/4$. Roznásobení:

$$1 = 1\cdot(t+1)(2t^2-t+1) - \frac{1}{4}t(2t^2-t+1) + (Ct+D)(t+1)t$$

$$t^3:\quad 0 = 2-\frac{1}{2}+C \Rightarrow C = -3/2$$

$$t^2:\quad 0 = 1+1/4+C+D \Rightarrow D = -5/4+3/2 = 1/4$$

$$\Rightarrow I = 6\ln|x^{1/6}| - \frac{3}{2}\ln|x^{1/6}+1| + \underbrace{6\cdot\int\frac{-3/2\,t+1/4}{2t^2-t+1}\,dt}_{I'}$$

$$I' = \frac{6}{8}\cdot\int\frac{-6t+1}{t^2-t/2+1/2}\,dt = -\frac{6}{8}\cdot 3\cdot\int\frac{2t-1/3}{t^2-t/2+1/2} = -\frac{9}{4}\cdot\int\frac{2t-1/2\,dt}{t^2-t/2+1/2} - \frac{9}{4}\int\frac{1/6}{t^2-t/2+1/2}\,dt$$

$$= -9/4\ln(t^2-t/2+1/2) - \frac{3}{8}\underbrace{\int\frac{dt}{(t^2-t/2+1/16)+7/16}}_{I''}$$

$$I'' = -\frac{3}{8}\cdot\frac{16}{7}\cdot\int\frac{dt}{\left(\frac{4t-1}{\sqrt7}\right)^2+1} = \left|\begin{array}{l} s = \frac{4t-1}{\sqrt7} \\ ds = \frac{4}{\sqrt7}dt\end{array}\right| = -\frac{6}{7}\cdot\frac{\sqrt7}{4}\cdot\operatorname{arctg}\left(\frac{4t-1}{\sqrt7}\right)$$

Dohromady: $I = \ln x - \frac{3}{2}\ln|x^{1/6}+1| - \frac{9}{4}\ln\left(x^{1/3}-\frac{1}{2}x^{1/6}+1/2\right) - \frac{3\sqrt7}{14}\operatorname{arctg}\left(\frac{4x^{1/6}-1}{\sqrt7}\right) + C$, $x\in(0,\infty)$

4) $\int x\sqrt{x^2-2x+2}\,dx$ Eulerova substituce, $a>0$ (5)

$I=$

$\Rightarrow \sqrt{x^2-2x+2} = x+t$

$x^2-2x+2 = x^2+2xt+t^2$

$2-t^2 = 2x(1+t) \Rightarrow x = \dfrac{2-t^2}{2\cdot(1+t)}$

$x+t = \dfrac{2-t^2}{2\cdot(1+t)}+t = \dfrac{t^2+2t+2}{2\cdot(1+t)}$

$dx = \frac{1}{2}\cdot\dfrac{-2t(1+t)-(2-t^2)}{(1+t)^2}dt$

$= \frac{1}{2}\cdot\dfrac{-t^2-2t-2}{(1+t)^2}dt$

$I=\int \dfrac{2-t^2}{2\cdot(1+t)}\cdot\dfrac{t^2+2t+2}{2\cdot(1+t)}\cdot\frac{1}{2}\cdot(-1)\cdot\dfrac{t^2+2t+2}{(1+t)^2}\,dt$

$= -\frac{1}{8}\cdot\int\dfrac{(2-t^2)(t^2+2t+2)^2}{(1+t)^4}\,dt$ .... To by šlo dopočítat, ale vypadá to hrozně. Nešlo by to trošku lehčeji?

$I=\frac{1}{2}\int(2x-2)\sqrt{x^2-2x+2}\,dx + \int\sqrt{x^2-2x+2}\,dx = \frac{1}{2}\int\sqrt{y}\,dy + \int\sqrt{x^2-2x+2}\,dx$ $\quad y=x^2-2x+2$

$= \frac{1}{3}(x^2-2x+2)^{3/2} + \overbrace{\int\sqrt{x^2-2x+2}\,dx}^{I'}$

Vsuvka: speciální odmocniny lze substituovat i jinak než Eulerem:

- $\sqrt{1-x^2}$: $x=\sin t$ nebo $x=\cos t$
- $\sqrt{1+x^2}$: $x=\sinh t$
- $\sqrt{x^2-1}$: $x=\cosh t$

$I'=\int\sqrt{x^2-2x+2}\,dx = \left| \begin{matrix} f'=1 & g=\sqrt{x^2-2x+2} \\ f=x & g'=\frac{1}{2}\cdot\frac{1}{\sqrt{x^2-2x+2}}\cdot(2x-2)\end{matrix}\right| = x\cdot\sqrt{x^2-2x+2} - \underbrace{\int\dfrac{x(x-1)}{\sqrt{x^2-2x+2}}\,dx}_{I''}$

$I''= -\int\dfrac{x^2-2x+2\,dx}{\sqrt{x^2-2x+2}} - \frac{1}{2}\int\dfrac{2x-2\,dx}{\sqrt{x^2-2x+2}} + \int\dfrac{1}{\sqrt{x^2-2x+2}}\,dx = -I' - \frac{1}{2}\cdot 2\cdot\sqrt{x^2-2x+2} + \int\dfrac{1}{\sqrt{(x-1)^2+1}}\,dx$

$= -I' - \sqrt{x^2-2x+2} + \operatorname{argsinh}(x-1)$

$I' = (x-1)\sqrt{x^2-2x+2} + \operatorname{argsinh}(x-1) - I' \Rightarrow I' = \frac{1}{2}(x-1)\sqrt{x^2-2x+2} + \operatorname{argsinh}(x-1) + C$

$\Rightarrow I = \left(\frac{1}{3}(x^2-2x+2)+\frac{1}{2}(x-1)\right)\sqrt{x^2-2x+2} + \operatorname{argsinh}(x-1) + C \qquad x\in\mathbb{R}$

---

5) $I=\int\dfrac{x+\sqrt{x^2+x+1}}{1+x+\sqrt{x^2+x+1}}\,dx$ Euler, $a>0$: $\sqrt{x^2+x+1} = t+x$

$x^2+x+1 = x^2+2tx+t^2$

$x(1-2t) = t^2-1$

$x = \dfrac{t^2-1}{1-2t}$ $\qquad t+x = t+\dfrac{t^2-1}{1-2t} = \dfrac{-t^2+t-1}{1-2t}$

$dx = \dfrac{2t(1-2t)-(t^2-1)\cdot(-2)}{(1-2t)^2}dt = \dfrac{-2t^2+2t-2}{(1-2t)^2}dt$

6

$$I=\int \frac{\frac{t^2-1}{1-2t}+\frac{-t^2+t-1}{1-2t}}{1+\frac{t^2-1}{1-2t}+\frac{-t^2+t-1}{1-2t}}\cdot\frac{(-2t^2+2t-2)}{(1-2t)^2}\,dt=\int\frac{-2t\cdot(t^2-t+1)}{(1-2t)^2\cdot(-t-1)}\,dt=2\int\frac{t^3-t^2+t}{(t+1)(1-2t)^2}\,dt$$

$$=\int\frac{2t^3-2t^2+2t}{4t^3-3t+1}\,dt=\frac{1}{2}\int\frac{4t^3-4t^2+4t}{4t^3-3t+1}\,dt=\frac{1}{2}\left(\int\frac{4t^3-3t+1}{4t^3-3t+1}+\frac{-4t^2+7t-1}{4t^3-3t+1}\,dt\right)$$

$$=\frac{1}{2}t+\frac{1}{2}\int\frac{-4t^2+7t-1}{4t^3-3t+1}=\frac{1}{2}t-\frac{1}{2}\underbrace{\int\frac{4t^2-7t+1}{(t+1)(1-2t)^2}\,dt}_{I'}$$

$$I'=-\frac{1}{2}\cdot\int\frac{A}{t+1}+\frac{B}{1-2t}+\frac{C}{(1-2t)^2}\,dt$$

Zakrývací metoda: $A=\frac{12}{9}=\frac{4}{3}$

$C=\frac{-3/2}{3/2}=-1$

$$4t^2-7t+1=\frac{4}{3}(1-2t)^2+B(t+1)(1-2t)-1(t+1)$$

$$t^2:\ 4=\frac{16}{3}-2B\quad\Rightarrow B=\frac{2}{3}$$

$$I'=-\frac{1}{2}\cdot\frac{4}{3}\cdot\ln|t+1|-\frac{1}{2}\cdot\frac{2}{3}\cdot\left(-\frac{1}{2}\right)\cdot\ln|1-2t|-\frac{1}{2}\cdot(-1)\cdot\left(-\frac{1}{2}\right)\cdot(-1)\cdot\frac{1}{1-2t}$$

$$I=\frac{1}{2}\left(\sqrt{x^2+x+1}-x\right)-\frac{2}{3}\ln\left|\sqrt{x^2+x+1}-x+1\right|+\frac{1}{6}\ln\left|1+2x-2\sqrt{x^2+x+1}\right|+\frac{1}{4}\cdot\frac{1}{1+2x-2\sqrt{x^2+x+1}}+C$$

$x\in\mathbb{R}$

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6) $I=\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$

Euler: $\sqrt{x^2+3x+2}=x+t$

$x^2+3x+2=x^2+2xt+t^2$

$x(3-2t)=t^2-2$

$x=\frac{t^2-2}{3-2t}$

$x+t=\frac{-t^2+3t-2}{3-2t}$

$dx=\frac{2t(3-2t)-(t^2-2)\cdot(-2)}{(3-2t)^2}\,dt=\frac{-2t^2+6t-4}{(3-2t)^2}\,dt$

$x^2+3x+2=(x+1)(x+2)\Rightarrow x\in(-\infty,-2)\cup(-1,\infty)$

$x=-\sqrt{x^2+3x+2}:\ 3x+2=0\Rightarrow x\neq-2/3$

$\Rightarrow \underline{x\in(-\infty,-2)\cup(-1,-2/3)\cup(-2/3,+\infty)}$

$$I=\int\frac{-t}{\frac{t^2-2}{3-2t}+\frac{-t^2+3t-2}{3-2t}}\cdot\frac{(-2t^2+6t-4)}{(3-2t)^2}\,dt$$

$$=2\int\frac{t^3-3t^2+2t}{(3t-4)(3-2t)}\,dt=2\int\frac{t^3-3t^2+2t}{-6t^2+17t-12}\,dt=-\frac{1}{3}\int\frac{6t^3-18t^2+12t}{6t^2-17t+12}\,dt=-\frac{1}{3}\int\frac{t\cdot(6t^2-17t+12)-t^2}{6t^2-17t+12}$$

$$=-\frac{1}{3}\frac{t^2}{2}+\frac{1}{3}\int\frac{t^2}{6t^2-17t+12}\,dt=-\frac{1}{6}t^2+\frac{1}{18}\int\frac{6t^2-17t+12}{6t^2-17t+12}+\frac{17t-12}{6t^2-17t+12}\,dt=-\frac{t^2}{6}+\frac{t}{18}+$$

$$+\frac{1}{18}\underbrace{\int\frac{17t-12}{(3t-4)(2t-3)}\,dt}_{I'}$$

$$I'=\frac{1}{18}\int\frac{A}{3t-4}+\frac{B}{2t-3}\,dt$$

Zakrývací metoda: $A=\frac{\frac{17\cdot4}{3}-12}{\frac{2\cdot4}{3}-3}=\frac{32}{-1}=-32$

$B=\frac{17\cdot3/2-12}{3\cdot3/2-4}=\frac{27}{1}=27$

$$=1/18\cdot(-32)\cdot\frac{1}{3}\ln|3t-4|+\frac{1}{18}\cdot27\cdot\frac{1}{2}\cdot\ln|2t-3|$$

$$I=-\frac{1}{6}\cdot\left(\sqrt{x^2+3x+2}-x\right)^2+\frac{1}{18}\left(\sqrt{x^2+3x+2}-x\right)-\frac{16}{27}\ln\left|3\sqrt{x^2+3x+2}-3x-4\right|+\frac{3}{4}\ln\left|2\sqrt{x^2+3x+2}-2x-3\right|+C$$

7) $\int \frac{\sin^2 x}{1+\sin^2 x}\,dx$ $\quad\Bigg|\quad$ $y=\operatorname{tg} x$, $\quad x=\operatorname{arctg} y$, $\quad dx=\frac{1}{1+y^2}\,dy$

I''

$$y^2=\frac{\sin^2 x}{\cos^2 x}=\frac{\sin^2 x}{1-\sin^2 x}$$

$$y^2(1-\sin^2 x)=\sin^2 x$$

$$y^2=\sin^2 x\cdot(1+y^2)$$

$$\sin^2 x=\frac{y^2}{1+y^2}$$

$$I=\int \frac{\frac{y^2}{1+y^2}}{1+\frac{y^2}{1+y^2}}\cdot\frac{1}{1+y^2}\,dy=\int\frac{y^2}{(1+2y^2)(1+y^2)}\,dy=\int \frac{Ay+B}{1+2y^2}+\frac{Cy+D}{1+y^2}\,dy$$

$$y^2=Ay^3+By^2+Ay+B+2Cy^3+2Dy^2+Cy+D \Rightarrow \begin{array}{l} A+2C=0\\ B+2D=1\\ A+C=0\\ B+D=0\end{array} \quad A=C=0,\quad D=1,\ B=-1$$

$$I=-\int\frac{dy}{1+2y^2}+\int\frac{dy}{1+y^2}=\operatorname{arctg} y-\frac{1}{\sqrt2}\operatorname{arctg}(\sqrt2 y)+C$$

$$=\underline{x-\frac{1}{\sqrt2}\operatorname{arctg}(\sqrt2\operatorname{tg}x)+C_k\qquad x\in(-\tfrac{\pi}{2}+k\pi,\ \tfrac{\pi}{2}+k\pi)}$$

Lze lepit!!

$$\lim_{x\to(\frac{\pi}{2}+k\pi)_-}\dots=\frac{\pi}{2}+k\pi-\frac{1}{\sqrt2}\operatorname{arctg}(+\infty)+C_k=\frac{\pi}{2}+k\pi-\frac{\pi}{2\sqrt2}+C_k$$

$$\lim_{x\to(\frac{\pi}{2}+k\pi)_+}=\frac{\pi}{2}+k\pi-\frac{1}{\sqrt2}\operatorname{arctg}(-\infty)+C_{k+1}=\frac{\pi}{2}+k\pi+\frac{\pi}{2\sqrt2}+C_{k+1}$$

$$\Rightarrow \underline{C_{k+1}=C_k-\frac{\pi}{\sqrt2}}$$

C_0 lze zvolit libovolně, další C_k nutno volit podle

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8) $I=\int\frac{1}{2\sin x-\cos x+5}\,dx$ $\quad\Bigg|\quad$ $y=\operatorname{tg}\frac{x}{2}$, $\quad x=2\operatorname{arctg} y$, $\quad dx=\frac{2}{1+y^2}\,dy$, $\quad \sin x=\frac{2y}{1+y^2}$, $\quad \cos x=\frac{1-y^2}{1+y^2}$

$$I=\int\frac{1}{\frac{4y}{1+y^2}-\frac{1-y^2}{1+y^2}+5}\cdot\frac{2}{1+y^2}\,dy=2\int\frac{dy}{4y-1+y^2+5+5y^2}=2\int\frac{dy}{6y^2+4y+4}=\int\frac{dy}{3y^2+2y+2}$$

$$=\frac13\int\frac{dy}{y^2+\frac23 y+\frac23}=\frac13\int\frac{dy}{y^2+\frac23y+\frac19+\frac59}=\frac35\int\frac{dy}{1+\left(\frac{3y+1}{\sqrt5}\right)^2}=\underline{\frac{\sqrt5}{5}\operatorname{arctg}\left(\frac{3\operatorname{tg}\frac{x}{2}+1}{\sqrt5}\right)+C_k}$$

$$x\in(-\pi+2k\pi,\ \pi+2k\pi)$$

Lze lepit:

$$\left.\begin{array}{l}\lim\limits_{x\to(\pi+2k\pi)_-}\dots=\frac{\sqrt5}{5}\cdot\frac{\pi}{2}+C_k\\ \lim\limits_{x\to(\pi+2k\pi)_+}\dots=\frac{\sqrt5}{5}\cdot\left(-\frac{\pi}{2}\right)+C_{k+1}\end{array}\right\}\quad \underline{C_{k+1}=C_k+\frac{\sqrt5}{5}\pi}$$

9, DÚ

10, $\int \frac{\sin^3 x}{\cos^4 x}\,dx = I \quad \left| \begin{array}{l} y = \cos x \\ dy = -\sin x\,dx \end{array} \right.$

$$I = \int \frac{(1-\cos^2 x)\sin x\,dx}{\cos^4 x} = \int \frac{y^2-1}{y^4}\,dy = \int y^{-2}\,dy - \int y^{-4}\,dy = \underline{\underline{-\frac{1}{\cos x} + \frac{1}{3\cos^3 x} + C}}$$

$x \in (-\pi/2 + k\pi, \pi/2 + k\pi)$

11, DÚ

12, $\int \sqrt{a^2+x^2}\,dx = I \quad a=0: \; I = \int |x|\,dx = \begin{cases} \frac{x^2}{2} + C_1 & \text{pro } x>0 \\ -\frac{x^2}{2} + C_2 & \text{pro } x<0 \end{cases}$ $\quad C_1 = C_2$ lepení

$$a \neq 0: \; I = |a| \int \sqrt{1+\frac{x^2}{a^2}}\,dx \left| \begin{array}{l} y = \frac{x}{a} \\ dy = \frac{1}{a}\,dx \end{array} \right| = a|a| \int \sqrt{1+y^2}\,dy \left| \begin{array}{l} y = \sinh t \\ dy = \cosh t\,dt \end{array} \right.$$

$$I = a|a| \int \sqrt{1+\sinh^2 t}\,\cosh t\,dt = a|a| \int \cosh^2 t\,dt \quad \left| \begin{array}{ll} f' = \cosh t & g = \cosh t \\ f = \sinh t & g' = \sinh t \end{array} \right|$$

$$= a|a| \left(\sinh t \cosh t - \int \sinh^2 t\,dt \right)$$

$$= a|a| \left(\sinh t \cosh t + t - \int \cosh^2 t\,dt \right) \Rightarrow I = \frac{a|a|}{2} \operatorname{argsinh} \frac{x}{a} + \frac{a|a|}{2} \cdot \frac{x}{a} \cdot \cosh\left(\operatorname{argsinh} \frac{x}{a}\right) + C$$

$$I = \frac{a|a|}{2} \operatorname{argsinh} \frac{x}{a} + \frac{x|a|}{2} \cdot \sqrt{1+\frac{x^2}{a^2}} + C = \underline{\underline{\frac{a|a|}{2} \operatorname{argsinh} \frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + C}}$$

$x \in \mathbb{R}$