Primitivní funkce I

Nalezněte následující primitivní funkce na maximálních možných intervalech. Určete i tyto intervaly.

1. $\int \left(\frac{1-x}{x}\right)^2 \, \mathrm{d}x$
2. $\int \frac{2^{x+1} - 5^{x-1}}{10^x} \, \mathrm{d}x$
3. $\int \operatorname{tg}^2 x \, \mathrm{d}x$
4. $\int \frac{1}{x^2 - x + 2} \, \mathrm{d}x$
5. $\int \max\{1, x^2\} \, \mathrm{d}x$
6. $\int x\mathrm{e}^{-x^2} \, \mathrm{d}x$
7. $\int \frac{1}{\mathrm{e}^x + \mathrm{e}^{-x}} \, \mathrm{d}x$
8. $\int \mathrm{e}^{3x} \cos 2x \, \mathrm{d}x$
9. $\int \frac{\ln^2 x}{x} \, \mathrm{d}x$
10. $\int \frac{1}{\sqrt{1-x^2}(\arcsin x)^2} \, \mathrm{d}x$
11. $\int \frac{1}{1 + \cos x} \, \mathrm{d}x$
12. $\int \frac{1}{\sin x} \, \mathrm{d}x$
13. $\int \frac{1}{\sin x \cos^3 x} \, \mathrm{d}x$

14. $\int \ln x \, \mathrm{d}x$

15. $\int x^3 a^{-x^2} \, \mathrm{d}x$

16. $\int x \operatorname{arctg}(x+1) \, \mathrm{d}x$

17. $\int x^2 \arccos x \, \mathrm{d}x$

18. $\int \frac{x}{\cos^2 x} \, \mathrm{d}x$

19. $\int \sin(\ln x) \, \mathrm{d}x$

20. $\int \sin^7 x \, \mathrm{d}x$

21. $\int \cos^2 x \, \mathrm{d}x$

22. Nalezněte rekurentní vztah pro $\int \cos^n x \, \mathrm{d}x, \quad n \in \mathbb{N}$

PRIMITIVNÍ FUNKCE

- F je primitivní funkce k f na (a,b), pokud $F' = f$, píšeme $F = \int f\,dx$
- Pro libovolné $c \in \mathbb{R}$ platí $F(x) = \int f(x)\,dx \Rightarrow F(x) + c = \int f(x)\,dx$

  a naopak, je-li $F(x) = \int f(x)\,dx$ a $G(x) = \int f(x)\,dx$, pak $\exists c \in \mathbb{R}: F - G = c$

- Lepení prim. fcí: Nechť f je spojitá na (a,c) a $a < b < c$.

  F_1 je primitivní k f na (a,b)

  F_2 je primitivní k f na (b,c) $\Rightarrow$ Lze "slepit" F_1 a F_2 v bodě b, tj.

  existují limity $\lim_{x \to b-} F_1(x)$ a $\lim_{x \to b+} F_2(x)$ a

$$F(x) = \begin{cases} F_1(x) & x \in (a,b) \\ \lim_{x\to b-} F_1(x) & x = b \\ F_2(x) - \lim_{x \to b+} F_2(x) + \lim_{x \to b-} F_1(x) & x \in (b,c) \end{cases}$$

  je spojitá na (a,c) a $F(x) = \int f(x)\,dx$.

- Součet a násobení konstantou: $F = \int f\,dx$, $G = \int g\,dx$, $\alpha \in \mathbb{R}$

$$\Rightarrow F + G = \int f + g\,dx, \quad \alpha F = \int \alpha f\,dx$$

- Per partes: $\int f'g = fg - \int f g'$ pokud f, g mají vlastní derivace a alespoň jedna z prim. fcí ve vzorečku existuje

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- 1. věta o substituci: Nechť $F = \int f\,dx$ na (a,b) a $\varphi: (\alpha,\beta) \to (a,b)$ má všude v (α,β) vlastní derivaci. Pak $F \circ \varphi = \int (f \circ \varphi)\,\varphi'\,dt$ na (α,β).
- 2. věta o substituci: Nechť $f: (a,b) \to \mathbb{R}$ a nechť $\varphi: (\alpha,\beta) \to (a,b)$ je bijekce, která má všude v (α,β) vlastní nenulovou derivaci. Nechť $\Phi(t) = \int f(\varphi(t))\,\varphi'(t)\,dt$ na (α,β), pak $\Phi \circ \varphi^{-1} = \int f\,dx$, tedy $\Phi(\varphi^{-1}(x)) = \int f(x)\,dx$

Rozdíl v počítání ! $\boxed{\text{1. věta}}$: Poznám v zadání tvar $\int f(\varphi(x))\,\varphi'(x)\,dx$ a píšu

$$I = \int f(\varphi(x))\,\varphi'(x)\,dx = \left| \begin{matrix} t = \varphi(x) \\ dt = \varphi'(x)\,dx \end{matrix} \right| = \int f(t)\,dt$$

a doufám, že $\int f(t)\,dt$ umíme spočítat.

Potom $F(t) = \int f(t)\,dt$ a $I = F(t) = F(\varphi(x))$

$\boxed{\text{2. věta}}$: Mám v zadání $\int f(x)\,dx$. Nevím, co s tím, ale napadne mě zkusit $x = \varphi(t)$.

$$I = \int f(x)\,dx = \left| \begin{matrix} x = \varphi(t) \\ dx = \varphi'(t)\,dt \end{matrix} \right| = \int f(\varphi(t))\,\varphi'(t)\,dt$$

a doufám, že toto umím spočítat.

Pak $G(t) = \int f(\varphi(t))\,\varphi'(t)\,dt$ a $I = G(\varphi^{-1}(x))$

Na rozdíl od derivací neexistuje univerzální návod, něco "nelze zintegrovat" vůbec (typicky $\int e^{-x^2}dx$). Každý příklad vyžaduje invenci a originální přístup, obecné návody jsou jen pro určité typy (viz příští týden). ②

1) $\int\left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2}dx = \int x^{-2}dx - 2\int x^{-1}dx + \int 1\,dx = \frac{x^{-1}}{-1} - 2\ln|x| + x + C$
$= -\frac{1}{x} - 2\ln|x| + x + C$

Intervaly: $(-\infty,0)$ a $(0,\infty)$ [lepit nelze, f není spojitá]

2) $\int \frac{2^{x+1}-5^{x-1}}{10^x}dx = \int \frac{2\cdot 2^x - \frac{1}{5}\cdot 5^x}{10^x}dx = 2\int\left(\frac{1}{5}\right)^x dx - \frac{1}{5}\int\left(\frac{1}{2}\right)^x dx = 2\int e^{x\cdot\ln\frac{1}{5}}dx - \frac{1}{5}\int e^{x\ln\frac{1}{2}}dx =$

$= 2\cdot\frac{e^{x\ln\frac{1}{5}}}{\ln\frac{1}{5}} - \frac{1}{5}\frac{e^{x\ln\frac{1}{2}}}{\ln\frac{1}{2}} + C = -\frac{2\cdot 5^{-x}}{\ln 5} + \frac{2^{-x}}{5\ln 2} + C$ $\quad x\in\mathbb{R}$

3) $\int \mathrm{tg}^2 x\,dx = \int\frac{\sin^2 x}{\cos^2 x}dx = \int\frac{1-\cos^2 x}{\cos^2 x}dx = \int\frac{1}{\cos^2 x}dx - \int 1\,dx = \mathrm{tg}\,x - x + C$

Intervaly: $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k\in\mathbb{Z}$, lepit nelze

4) $\int\frac{1}{x^2-x+2}dx$ 1. krok: Má x^2-x+2 reálné kořeny? $D=(-1)^2-4\cdot1\cdot2 = -7<0$
$\Rightarrow$ žádné reálné kořeny
$\Rightarrow$ povedeme to na arctg z něčeho.

$\|$

$\int\frac{1}{\left(x^2-x+\frac{1}{4}\right)+\frac{7}{4}}dx = \int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}} = \int\frac{dx}{\frac{7}{4}\left(1+\left(\frac{2}{\sqrt7}\cdot\left(x-\frac{1}{2}\right)\right)^2\right)} = \frac{4}{7}\int\frac{dx}{1+\left(\frac{2}{\sqrt7}x-\frac{1}{\sqrt7}\right)^2} = \left| \begin{matrix} t=\frac{2}{\sqrt7}x-\frac{1}{\sqrt7} \\ dt=\frac{2}{\sqrt7}dx \\ \boxed{2.\text{ věta}} \end{matrix}\right| =$

$= \frac{4}{7}\cdot\int\frac{\frac{\sqrt7}{2}dt}{1+t^2} = \frac{2\sqrt7}{7}\int\frac{dt}{1+t^2} = \frac{2\sqrt7}{7}\,\mathrm{arctg}\,t + C = \frac{2\sqrt7}{7}\,\mathrm{arctg}\left(\frac{2x-1}{\sqrt7}\right) + C$ $\quad x\in\mathbb{R}$

Pro srovnání: $\int\frac{1}{x^2-x-2}dx$. 1. krok: Má x^2-x-2 reálné kořeny? $D=9>0$ ANO!
$x_{1,2}=\frac{1\pm3}{2} = \begin{cases}-1\\2\end{cases} \Rightarrow x^2-x-2=(x+1)(x-2)$

$\int\frac{dx}{x^2-x-2} = \int\frac{dx}{(x+1)(x-2)}$. ROZKLAD: $\frac{1}{(x+1)(x-2)} = \frac{A}{x+1}+\frac{B}{x-2}$. Zjistíme A a B

$= \frac{Ax-2A+Bx+B}{(x+1)(x-2)} \Rightarrow (A+B)x-2A+B=1$

$A+B=0$
$-2A+B=1$
$A=-B$
$-2A+B=3B=1$
$B=1/3$
$A=-1/3$

$= -\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{dx}{x-2}$

$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + C$ pro $x\in(-\infty,-1)$
$(-1,2)$
$(2,\infty)$. Lepit nelze

! VÍCE PŘÍŠTĚ !

5) $\int \max\{1, x^2\}\,dx$

$x < -1: \; f_1(x) = x^2 \Rightarrow F_1(x) = \int x^2 dx = \frac{x^3}{3} + C_1$

$x \in (-1,1): \; f_2(x) = 1 \Rightarrow F_2(x) = \int 1\,dx = x + C_2$ f je spojitá => lze lepit

$x \in (1,\infty): \; f_3(x) = x^2 \Rightarrow F_3(x) = \int x^2 dx = \frac{x^3}{3} + C_3$

$\lim_{x \to -1_-} F_1(x) = -\frac{1}{3} + C_1$ $\qquad \lim_{x \to 1_-} F_2(x) = 1 + C_2$

$\lim_{x \to -1_+} F_2(x) = -1 + C_2$ $\qquad \lim_{x \to 1_+} F_3(x) = \frac{1}{3} + C_3$

$\Downarrow \qquad\qquad \Downarrow$

$-\frac{1}{3} + C_1 = -1 + C_2 \qquad 1 + C_2 = \frac{1}{3} + C_3$

$C_2 = \frac{2}{3} + C_1 \qquad C_3 = \frac{2}{3} + C_2 = \frac{4}{3} + C_1$

$$\Rightarrow F(x) = \begin{cases} \frac{x^3}{3} + C_1 & x < -1 \\ x + \frac{2}{3} + C_1 & x \in [-1,1] \\ \frac{x^3}{3} + \frac{4}{3} + C_1 & x \in (1,\infty) \end{cases}$$

je spojitá na $\mathbb{R}$

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6) $\int x e^{-x^2} dx = -\frac{1}{2}\int e^{-x^2}\cdot(-2x)\,dx = \left| \begin{matrix} -x^2 = t \\ -2x\,dx = dt \\ \boxed{1.\text{ věta}} \end{matrix} \right| = -\frac{1}{2}\int e^t dt = -\frac{1}{2}e^t + C = \underline{\underline{-\frac{1}{2}e^{-x^2} + C}}$

$x \in \mathbb{R}$

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7) $\int \frac{dx}{e^x + e^{-x}} = \left| \begin{matrix} e^x = t \\ e^x dx = dt \\ \boxed{2.\text{ věta}} \end{matrix} \right| = \int \frac{\frac{1}{e^x}(e^x dx)}{e^x + e^{-x}} = \int \frac{\frac{1}{t}}{t + \frac{1}{t}} dt = \int \frac{1}{t^2+1} dt = \operatorname{arctg} t + C$

$= \underline{\underline{\operatorname{arctg} e^x + C}}$

$x \in \mathbb{R}$

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8) $I = \int e^{3x} \cos 2x\,dx$. ! Per partes trik !

$\left| \begin{matrix} f' = e^{3x} & g = \cos 2x \\ f = \frac{1}{3}e^{3x} & g' = -\sin 2x \cdot 2 \end{matrix} \right| \quad I = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x}\sin 2x \quad \left| \begin{matrix} f' = e^{3x} & g = \sin 2x \\ f = \frac{1}{3}e^{3x} & g' = 2\cos 2x \end{matrix} \right|$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\cdot\left(\frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\right)$

$\Rightarrow I = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}I$

$\frac{13}{9}I = \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x \Rightarrow \underline{\underline{I = \frac{3}{13}e^{3x}\cos 2x + \frac{2}{13}e^{3x}\sin 2x + C}}$ $\quad x \in \mathbb{R}$

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9) $\int \frac{\ln^2 x}{x} dx = \left| \begin{matrix} \ln x = t \\ \frac{1}{x}dx = dt \\ \boxed{1.\text{ věta}} \end{matrix} \right| = \int t^2 dt = \frac{t^3}{3} + C = \frac{\ln^3 x}{3} + C \quad x \in (0,\infty)$

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10) $\int \frac{dx}{\sqrt{1-x^2}\,\arcsin^2 x} = \left| \begin{matrix} \arcsin x = t \\ \frac{1}{\sqrt{1-x^2}}dx = dt \end{matrix} \right| = \int \frac{dt}{t^2} = \int t^{-2} dt = \frac{t^{-1}}{-1} + C = \underline{\underline{-\frac{1}{\arcsin x} + C}}$

$x \in (-1,0) \cup (0,1)$. Nelze lepit

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11) $\int \frac{1}{1+\cos x} dx = \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx = \int \frac{1\,dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x} dx = -\operatorname{cotg} x - \int \frac{\cos x}{\sin^2 x} dx$

$\boxed{\cos x \neq -1}$ $\qquad \cos x \neq \pm 1 \Leftrightarrow \sin x \neq 0$

$\left| \begin{matrix} t = \sin x \\ dt = \cos x\,dx \\ \boxed{1.\text{ věta}} \end{matrix} \right| = -\operatorname{cotg} x - \int \frac{dt}{t^2} = -\operatorname{cotg} x + t^{-1} + C = -\operatorname{cotg} x + \frac{1}{\sin x} + C = \frac{1-\cos x}{\sin x} + C = \underline{\underline{\frac{\sin x}{1+\cos x} + C}}$

$\sin x \neq 0$ (lze lepit!!) $\qquad \sin x \neq 0 \qquad \cos x \neq -1$

$k \in \mathbb{Z}: \underline{x \in (-\pi + k\pi, \pi + k\pi)}$

12) DÚ

13) $\int \frac{dx}{\sin x \cos^3 x} = \int \frac{1}{\sin x \cos x} \cdot \frac{dx}{\cos^2 x} = \int \frac{1}{\frac{\sin x}{\cos x} \cos^2 x} \cdot \frac{dx}{\cos^2 x} = \int \frac{1}{\operatorname{tg} x \cdot \left(\frac{\cos^2 x}{\cos^2 x + \sin^2 x}\right)} \cdot \frac{dx}{\cos^2 x} =$

$= \int \frac{1 + \operatorname{tg}^2 x}{\operatorname{tg} x} \cdot \frac{dx}{\cos^2 x} = \left| \begin{array}{l} t = \operatorname{tg} x \\ dt = \frac{1}{\cos^2 x} dx \end{array} \right| = \int \frac{1}{t} dt + \int t\, dt = \ln|t| + \frac{t^2}{2} + C = \underline{\underline{\ln|\operatorname{tg} x| + \frac{\operatorname{tg}^2 x}{2} + C}}$

$\boxed{1. \text{ věta}}$

$x \in \left(k\frac{\pi}{2}, \frac{\pi}{2} + k\frac{\pi}{2}\right)$

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14) $\int \ln x\, dx = \left| \begin{array}{ll} f' = 1 & g = \ln x \\ f = x & g' = \frac{1}{x} \end{array} \right| = x \ln x - \int x \cdot \frac{1}{x} dx = \underline{\underline{x \ln x - x + C}} \qquad x > 0$

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15) $\int x^3 a^{-x^2} dx = \left| \begin{array}{l} t = x^2 \\ dt = 2x\,dx \end{array} \right| = \frac{1}{2} \int a^{-x^2} \cdot x^2 \cdot 2x\, dx = \frac{1}{2} \int a^{-t} \cdot t\, dt = \left| \begin{array}{ll} f' = a^{-t} & g = t \\ f = \frac{-a^{-t}}{\ln a} & g' = 1 \end{array} \right| =$

$\boxed{1. \text{ věta}}$

$= \frac{1}{2}\left(-\frac{t a^{-t}}{\ln a} + \int \frac{a^{-t}}{\ln a} dt\right) = -\frac{t a^{-t}}{2 \ln a} - \frac{a^{-t}}{2 \ln^2 a} + C = \underline{\underline{\frac{-x^2 a^{-x^2}}{2 \ln a} - \frac{a^{-x^2}}{2 \ln^2 a} + C}} \qquad x \in \mathbb{R}$

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16) $\int x \operatorname{arctg}(x+1)\, dx = \left| \begin{array}{ll} f' = x & g = \operatorname{arctg}(x+1) \\ f = \frac{x^2}{2} & g' = \frac{1}{1+(x+1)^2} \end{array} \right| = \frac{x^2 \operatorname{arctg}(x+1)}{2} - \frac{1}{2} \int \frac{x^2}{1+(x+1)^2} dx =$

$= \frac{x^2 \operatorname{arctg}(x+1)}{2} - \frac{1}{2}\left(\int \frac{x^2 + 2x + 2}{1+(x+1)^2} - \frac{2x+2}{1+(x+1)^2} dx\right) = \frac{x^2 \operatorname{arctg}(x+1)}{2} - \frac{x}{2} + \int \frac{x+1}{1+(x+1)^2} dx \left| \begin{array}{l} t = (x+1)^2 \\ dt = 2(x+1)dx \end{array} \right|$

$\boxed{1. \text{ věta}}$

$= \frac{x^2 \operatorname{arctg}(x+1)}{2} - \frac{x}{2} + \frac{1}{2} \int \frac{1}{1+t} dt = \underline{\underline{\frac{x^2 \operatorname{arctg}(x+1)}{2} - \frac{x}{2} + \frac{\ln\left((x+1)^2+1\right)}{2} + C}} \qquad x \in \mathbb{R}$

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17) $\int x^2 \arccos x\, dx = \left| \begin{array}{ll} f' = x^2 & g = \arccos x \\ f = \frac{x^3}{3} & g' = -\frac{1}{\sqrt{1-x^2}} \end{array} \right| = \frac{x^3}{3} \arccos x + \frac{1}{3} \int \frac{x^3}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} t = 1 - x^2 \\ dt = -2x\,dx \end{array} \right|$

$\boxed{1. \text{ věta}}$

$= \frac{x^3 \arccos x}{3} + \frac{1}{3}\left(\left(-\frac{1}{2}\right) \int \frac{x^2 \cdot (-2x)\,dx}{\sqrt{1-x^2}}\right) = \frac{x^3 \arccos x}{3} - \frac{1}{6} \int \frac{1-t}{\sqrt{t}} dt = \frac{x^3 \arccos x}{3} - \frac{1}{3} t^{1/2} + \frac{1}{9} t^{3/2} + C$

$= \underline{\underline{\frac{x^3 \arccos x}{3} - \frac{1}{3}\sqrt{1-x^2} + \frac{1}{9}\sqrt{(1-x^2)^3} + C}} \qquad x \in (-1, 1)$

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18) $\int \frac{x}{\cos^2 x} dx = \left| \begin{array}{ll} f' = \frac{1}{\cos^2 x} & g = x \\ f = \operatorname{tg} x & g' = 1 \end{array} \right| = x \operatorname{tg} x - \int \operatorname{tg} x\, dx = x \operatorname{tg} x - \int \frac{\sin x}{\cos x} dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\, dx \end{array} \right| =$

$\boxed{1. \text{ věta}}$

$= x \operatorname{tg} x + \int \frac{1}{t} dt = x \operatorname{tg} x + \ln|t| + C = \underline{\underline{x \operatorname{tg} x + \ln|\cos x| + C}} \qquad x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$

19) DÚ

20) $\int \sin^7 x\,dx = \int \sin^6 x \sin x\,dx = \int (1-\cos^2 x)^3 \sin x\,dx = \left| \begin{array}{l} t=\cos x \\ dt = -\sin x\,dx \end{array} \right| =$ ⑤

$\boxed{1.\text{věta}}$

$= -\int (1-t^2)^3 dt = -\int 1\,dt + 3\int t^2 dt - 3\int t^4 dt + \int t^6 dt = \underline{\underline{\frac{\cos^7 x}{7} - \frac{3\cos^5 x}{5} + \cos^3 x - \cos x + C}}$

$x \in \mathbb{R}$

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21) $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \underline{\underline{\frac{x}{2} + \frac{\sin 2x}{4} + C}} \quad x \in \mathbb{R}$

Jinak: $\int \cos^2 x\,dx = \left| \begin{array}{ll} f' = \cos x & g = \cos x \\ f = \sin x & g' = -\sin x \end{array} \right| = \sin x \cos x + \int \sin^2 x = \frac{\sin 2x}{2} + x - \int \cos^2 x$

$I = \frac{\sin 2x}{2} + x - I \implies \underline{\underline{I = \frac{x}{2} + \frac{\sin 2x}{4} + C}}$

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22) $I_n = \int \cos^n x\,dx$

$I_0 = \int 1\,dx = x + C$

$I_1 = \int \cos x\,dx = \sin x + C$

$n \geq 2: \quad I_n = \int \cos^{n-1} x \cos x\,dx \quad \left| \begin{array}{ll} f' = \cos x & g = \cos^{n-1} x \\ f = \sin x & g' = -(n-1)\cos^{n-2} x \cdot \sin x \end{array} \right|$

$I_n = \cos^{n-1} x \sin x - (n-1)\int \cos^{n-2} x \sin^2 x =$

$= \cos^{n-1} x \sin x + (n-1)\left(\int \cos^{n-2} x - \cos^n x\,dx \right)$

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2} + \cos^{n-1} x \sin x \implies \underline{\underline{I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n}\cos^{n-1} x \sin x}}$

$x \in \mathbb{R}$