Limity funkcí II

Základní limity

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\mathrm{e}^x-1}{x}=1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

Pro výpočet limit typu “1^∞”:

$$\lim_{x\to a}(f(x))^{g(x)}=\mathrm{e}^{\lim_{x\to a} g(x)\ln(f(x))}.$$

Příklady

1. $\displaystyle\lim_{x\to a}\frac{\operatorname{tg}x-\operatorname{tg}a}{x-a}$, $a\in\mathbb{R}$
2. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x}$
3. $\displaystyle\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{x^3}$
4. $\displaystyle\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$
5. $\displaystyle\lim_{x\to \pi}\frac{\sin nx}{\sin mx}$, $n,\ m\in\mathbb{N}$
6. $\displaystyle\lim_{x\to 1}\frac{\sin \pi x}{1-x}$
7. $\displaystyle\lim_{x\to \frac{\pi}{4}}\operatorname{tg}(2x)\operatorname{tg}\left(\frac{\pi}{4}-x\right)$
8. $\displaystyle\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$, $a\in\mathbb{R}$
9. $\displaystyle\lim_{x\to 0}\frac{\operatorname{cotg}(a+2x)-2\operatorname{cotg}(a+x)+\operatorname{cotg}a}{x^2}$, $\sin a\neq 0$
10. $\displaystyle\lim_{x\to 0^+}\frac{\arccos(1-x)}{\sqrt{x}}$
11. $\displaystyle\lim_{x\to 0^+}\frac{\left(\frac{\pi}{2}-\arcsin\frac{1}{\sqrt{x^2+1}}\right)}{x}$

12. $\lim\limits_{x\to 0} \dfrac{\ln\cos ax}{\ln\cos bx}$, $a,\ b\in\mathbb{R},\ b\neq 0$

13. $\lim\limits_{x\to 0} \dfrac{\ln(a+x)+\ln(a-x)-2\ln a}{x^2}$, $a>0$

14. $\lim\limits_{x\to 0} \dfrac{\ln(\operatorname{tg}(\frac{\pi}{4}+ax))}{\sin bx}$, $a,\ b\in\mathbb{R},\ b\neq 0$

15. $\lim\limits_{x\to 0^+} \ln(x\ln a)\ln\Big(\dfrac{\ln ax}{\ln\frac{x}{a}}\Big)$, $a>0$

16. $\lim\limits_{x\to 0} \dfrac{\ln(1+x\mathrm{e}^x)}{\ln(x+\sqrt{1+x^2})}$

17. $\lim\limits_{x\to 1}(1-x)\log_x 2$

18. $\lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}}$

19. $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{\operatorname{tg} x}$

20. $\lim\limits_{x\to 0}\Big(\dfrac{1+\operatorname{tg} x}{1+\sin x}\Big)^{\frac{1}{\sin^3 x}}$

21. $\lim\limits_{x\to 1}(1+\sin\pi x)^{\operatorname{cotg}\pi x}$

22. $\lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}$

23. $\lim\limits_{x\to 0}(1+x^2)^{\operatorname{cotg}\pi x}$

24. $\lim\limits_{x\to \frac{\pi}{4}}(\operatorname{tg} x)^{\operatorname{tg} 2x}$

25. $\lim\limits_{x\to 1} \dfrac{\sin\pi x^\alpha}{\sin\pi x^\beta}$, $\alpha,\ \beta\in\mathbb{R},\ \beta\neq 0$

26. $\lim\limits_{x\to 0} \dfrac{\mathrm{e}^{\alpha x}-\mathrm{e}^{\beta x}}{\sin\alpha x-\sin\beta x}$, $\alpha,\ \beta\in\mathbb{R},\ \alpha\neq\beta$

27. $\lim\limits_{x\to a} \dfrac{a^x-x^a}{x-a}$, $a\in\mathbb{R}^+$

28. $\lim\limits_{x\to 0}\Big(\dfrac{1+x2^x}{1+x3^x}\Big)^{\frac{1}{x^2}}$

29. $\lim\limits_{x\to 0}\Big(\dfrac{a^{x^2}+b^{x^2}}{a^x+b^x}\Big)^{\frac{1}{x}}$, $a,\ b\in\mathbb{R}^+$

Aritmetika limit: $\lim_{x\to a}\left(f(x) +/-/\cdot/: g(x)\right) = \left(\lim_{x\to a} f(x)\right) +/-/\cdot/: \left(\lim_{x\to a} g(x)\right)$,

pokud má pravá strana smysl a obě limity jsou vlastní

Nejdůležitější věta: Věta o limitě složené funkce.

Nechť: 1) $\lim_{x\to x_0} g(x) = a$

2) $\lim_{y\to a} f(y) = b$

(tj $\forall x\in P$)

3) (V1): Nechť existuje prstencové okolí P bodu x_0 tak, že $g(x)\neq a$ na P

nebo (V2): Nechť f je spojitá funkce v bodě a.

Potom $\lim_{x\to x_0} f(g(x)) = b$.

Použití obou verzí si ukážeme na příkladech.

Nejčastější chyba: Nelze částečně limitit !!! Například:

$\lim_{x\to 0+} (1+x)^{1/x}$ nelze zlimitit nejprve uvnitř na $\lim_{x\to 0+} 1^{1/x} = 1$,

ve skutečnosti je výsledek této limity totiž e.

Základní limity: $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = 1$

$\lim_{x\to 0} \frac{e^x-1}{x} = 1 \qquad \lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$

---

1) $\lim_{x\to a} \frac{\operatorname{tg} x - \operatorname{tg} a}{x-a} = \lim_{y\to 0} \frac{\operatorname{tg}(y+a) - \operatorname{tg} a}{y}$

$y = x-a \mid x = y+a$

$x\to a \Rightarrow y\to 0$

Používáme (V1) takto: vnější fce je $f(y) = \frac{\operatorname{tg}(y+a)-\operatorname{tg} a}{y}$

vnitřní fce je $g(x) = x-a$

$g(x) = x-a$ je lineární $\Rightarrow$ očividně splňuje (V1)

$f(g(x))$ je $\frac{\operatorname{tg} x - \operatorname{tg} a}{x-a}$

$$\lim_{y\to 0} \frac{\operatorname{tg}(y+a)-\operatorname{tg} a}{y} = \lim_{y\to 0} \frac{\frac{\operatorname{tg} y + \operatorname{tg} a}{1-\operatorname{tg} y \operatorname{tg} a} - \operatorname{tg} a}{y} = \lim_{y\to 0} \frac{\operatorname{tg} y + \operatorname{tg} a - \operatorname{tg} a + \operatorname{tg} y \operatorname{tg}^2 a}{y\cdot(1-\operatorname{tg} y \operatorname{tg} a)} =$$

$$= \lim_{y\to 0} \frac{\operatorname{tg} y(1+\operatorname{tg}^2 a)}{y\cdot(1-\operatorname{tg} y\cdot \operatorname{tg} a)} = \lim_{y\to 0} \underbrace{\frac{\sin y}{y}}_{\to 1} \cdot \frac{(1+\operatorname{tg}^2 a)}{\underbrace{\cos y}_{\to 1}\,\underbrace{(1-\operatorname{tg} y\cdot \operatorname{tg} a)}_{\to 1}} = \underline{\underline{1+\operatorname{tg}^2 a}}$$

(2)
$$\lim_{x\to 0}\frac{\sqrt{1-\cos(x^2)}}{1-\cos x}=\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}\cdot\sqrt{1+\cos x^2}}{(1-\cos x)\cdot(1+\cos x)}\cdot\frac{1+\cos x}{\sqrt{1+\cos x^2}}=\lim_{x\to 0}\frac{\sqrt{1-\cos^2 x^2}}{1-\cos^2 x}\cdot\frac{1+\cos x}{\sqrt{1+\cos x^2}}=$$
$$=\lim_{x\to 0}\frac{\sin x^2}{\sin^2 x}\cdot\frac{1+\cos x}{\sqrt{1+\cos x^2}}=\lim_{x\to 0}\underbrace{\frac{\sin x^2}{x^2}}_{\to 1}\cdot\underbrace{\frac{x^2}{\sin^2 x}}_{\to 1}\cdot\frac{1+\cos x}{\sqrt{1+\cos x^2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

ZDE OPĚT (V1) S VNITŘNÍ FCÍ $g(x)=x^2$

---

(3)
$$\lim_{x\to 0}\frac{\mathrm{tg}\,x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{(1-\cos x)(1+\cos x)}{x^2\cos x\,(1+\cos x)}=$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin^2 x}{x^2}\cdot\frac{1}{\cos x(1+\cos x)}=\frac{1}{2}$$

---

(4)
$$\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}=\lim_{x\to 0}\frac{(1-\cos x)\cos 2x\cos 3x+1-\cos 2x\cos 3x}{1-\cos x}=$$
$$=\lim_{x\to 0}\cos 2x\cos 3x+\frac{(1-\cos 2x)\cos 3x+1-\cos 3x}{1-\cos x}=1+\lim_{x\to 0}\cos 3x\,\frac{1-\cos 2x}{1-\cos x}+\lim_{x\to 0}\frac{1-\cos 3x}{1-\cos x}$$
$$=1+\lim_{x\to 0}\underbrace{\cos 3x}_{\to 1}\,\frac{1-\cos^2 2x}{1-\cos^2 x}\cdot\underbrace{\frac{1+\cos x}{1+\cos 2x}}_{\to 1}+\lim_{x\to 0}\frac{1-\cos^2 3x}{1-\cos^2 x}\cdot\underbrace{\frac{1+\cos x}{1+\cos 3x}}_{\to 1}=1+\lim_{x\to 0}\frac{\sin^2 2x}{(2x)^2}\cdot\frac{x^2}{\sin^2 x}\cdot 4+$$
$$+\lim_{x\to 0}\frac{\sin^2 3x}{(3x)^2}\cdot\frac{x^2}{\sin^2 x}\cdot 9=1+4+9=14$$

---

(5)
$$\lim_{x\to\pi}\frac{\sin nx}{\sin mx}=\lim_{y\to 0}\frac{\sin(n\cdot(y+\pi))}{\sin(m\cdot(y+\pi))}=\lim_{y\to 0}\frac{\sin ny\cos n\pi+\cos ny\overbrace{\sin n\pi}^{=0}}{\sin my\cos m\pi+\cos my\underbrace{\sin m\pi}_{=0}}=$$

$y=x-\pi \mid x=y+\pi$

$x\to\pi \Rightarrow y\to 0$

(V1) $g(x)=x-\pi$ lineární, viz 1. příklad

$$=\lim_{y\to 0}\frac{(-1)^n\cdot\sin ny}{(-1)^m\cdot\sin my}=(-1)^{n-m}\lim_{y\to 0}\underbrace{\frac{\sin ny}{ny}}_{\to 1}\cdot\underbrace{\frac{my}{\sin my}}_{\to 1}\cdot\frac{n}{m}=(-1)^{n-m}\frac{n}{m}$$

---

(6)
$$\lim_{x\to 1}\frac{\sin(\pi x)}{1-x}=\lim_{y\to 0}\frac{\sin(\pi y+\pi)}{-y}=\lim_{y\to 0}\frac{\sin\pi y}{y}=\lim_{y\to 0}\frac{\sin\pi y}{\pi y}\cdot\pi=\pi$$

$y=x-1 \mid x=1+y$

$x\to 1 \Rightarrow y\to 0$

(V1) $g(x)=x-1$ lineární, viz 1. příklad

(3)

(7) $\lim_{x\to\pi/4} tg(2x)\,tg(\pi/4 - x) = \lim_{y\to 0} tg(2y+\pi/2)\,tg(-y) = \lim_{y\to 0} -tg(y)\cdot \frac{\sin(2y)\cos\pi/2+\cos(2y)\sin\pi/2}{\cos(2y)\cos\pi/2 - \sin(2y)\sin\pi/2}$

$y = x - \pi/4 \mid x = y + \pi/4$ (V1)

$x\to\pi/4 \Rightarrow y\to 0$

$= \lim_{y\to 0} tg(y)\frac{\cos 2y}{\sin 2y} = \lim_{y\to 0}\frac{\sin y}{y}\cdot\frac{\cos 2y}{\cos y}\cdot\frac{2y}{\sin 2y}\cdot\frac{1}{2} = \underline{\underline{\frac{1}{2}}}$

---

(8) $\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2} = \lim_{x\to 0}\frac{\sin a\cos 2x+\cos a\sin 2x - 2\sin a\cos x - 2\cos a\sin x+\sin a}{x^2} =$

$= \lim_{x\to 0}\frac{\sin a(\cos 2x - 2\cos x+1)+\cos a(\sin 2x - 2\sin x)}{x^2} = \lim_{x\to 0}\frac{\sin a(\cos^2 x - \sin^2 x - 2\cos x+1)+2\cos a\sin x(\cos x - 1)}{x^2}$

$= \lim_{x\to 0}\frac{(2\sin a\cos x+2\cos a\sin x)(\cos x - 1)}{x^2} = \lim_{x\to 0}\frac{2\sin(x+a)\cdot(\cos^2 x-1)}{x^2\cdot(\cos x+1)} = \lim_{x\to 0}\frac{2\sin(x+a)}{\cos x+1}\cdot\left(-\frac{\sin^2 x}{x^2}\right)$

$= \underline{\underline{-\sin a}}$

---

(9) DÚ

---

(10) $\lim_{x\to 0_+}\frac{\arccos(1-x)}{\sqrt{x}} = \lim_{y\to 1_-}\frac{\arccos y}{\sqrt{1-y}} = \lim_{z\to 0_+}\frac{z}{\sqrt{1-\cos z}} = \lim_{z\to 0_+}\frac{z\sqrt{1+\cos z}}{\sqrt{1-\cos^2 z}} = \lim_{z\to 0_+}\frac{z\sqrt{1+\cos z}}{\sin z}$

$y = 1-x \mid x = 1-y$ (V1)

$x\to 0+ \Rightarrow y\to 1_-$

$z = \arccos y \mid y = \cos z$ (V1)

$y\to 1_- \Rightarrow z\to 0_+$

$= \underline{\underline{\sqrt{2}}}$

$\hookrightarrow$ vnější fce $f(z) = \frac{z}{\sqrt{1-\cos z}}$

vnitřní fce $g(y) = \arccos y$

$\Big\}\ f(g(y)) = \frac{\arccos y}{\sqrt{1-y}}$

---

(11) $\lim_{x\to 0}\frac{\pi/2 - \arcsin\left(\frac{1}{\sqrt{x^2+1}}\right)}{x} = \lim_{x\to 0}\frac{\arccos\frac{1}{\sqrt{x^2+1}}}{x} = (*)$

$y = \frac{1}{\sqrt{x^2+1}} \Rightarrow y^2 = \frac{1}{x^2+1} \Rightarrow x^2+1 = \frac{1}{y^2} \Rightarrow x^2 = \frac{1}{y^2}-1$

$\Rightarrow x = \sqrt{\frac{1}{y^2}-1}$ POZOR!! x od začátku může nabývat kladných i záporných hodnot!!

$\Rightarrow$ Pro $x\to 0_+$ máme $x = \sqrt{\frac{1}{y^2}-1}$

Pro $x\to 0_-$ máme $x = -\sqrt{\frac{1}{y^2}-1}$

$g(x) = \frac{1}{\sqrt{x^2+1}}$ splňuje (V1)

$x\to 0 \Rightarrow y\to 1_-$

$(*) = \lim_{y\to 1_-}\frac{\arccos y}{\pm\sqrt{\frac{1}{y^2}-1}} = \lim_{z\to 0_+}\frac{z}{\pm\sqrt{\frac{1-\cos^2 z}{\cos^2 z}}} = \lim_{z\to 0_+}\frac{z\cos z}{\pm\sin z} = \begin{cases} +1 & \text{pro } x\to 0_+ \\ -1 & \text{pro } x\to 0_- \end{cases}$

$z = \arccos y \mid y = \cos z$

$y\to 1_- \Rightarrow z\to 0_+$

Původní limita neexistuje, jednostranné limity jsou různé

(12) $$\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}=\lim_{x\to 0}\frac{\ln\cos ax}{\cos ax-1}\cdot\frac{\cos bx-1}{\ln\cos bx}\cdot\frac{\cos ax-1}{\cos bx-1}=\lim_{x\to 0}\frac{\cos^2 ax-1}{\cos ax+1}\cdot\frac{\cos bx+1}{\cos^2 bx-1}=$$

$$=\lim_{x\to 0}\underbrace{\frac{\sin^2 ax}{(ax)^2}}_{\text{V1}}\cdot\underbrace{\frac{(bx)^2}{\sin^2 bx}}_{\text{V1}}\cdot\frac{a^2}{b^2}\cdot\frac{\cos bx+1}{\cos ax+1}=\underline{\underline{\frac{a^2}{b^2}}}$$

---

(13) $$\lim_{x\to 0}\frac{\ln(a+x)+\ln(a-x)-2\ln a}{x^2}=\lim_{x\to 0}\frac{\ln\frac{a+x}{a}+\ln\frac{a-x}{a}}{x^2}=\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{a}\right)+\ln\left(1-\frac{x}{a}\right)}{x^2}=$$

$$=\lim_{x\to 0}\frac{\ln\left(1-\frac{x^2}{a^2}\right)}{x^2}=\lim\underbrace{\frac{\ln\left(1-\frac{x^2}{a^2}\right)}{-\frac{x^2}{a^2}}}_{\text{V1}}\cdot\frac{-\frac{x^2}{a^2}}{x^2}=\underline{\underline{-\frac{1}{a^2}}}$$

---

(14) $$\lim_{x\to 0}\frac{\ln(\mathrm{tg}(\pi/4+ax))}{\sin bx}=\lim_{x\to 0}\underbrace{\frac{\ln(\mathrm{tg}(\pi/4+ax))}{\mathrm{tg}(\pi/4+ax)-1}}_{\text{V1}}\cdot\frac{\mathrm{tg}(\pi/4+ax)-1}{\sin bx}=\lim_{x\to 0}\frac{\frac{1+\mathrm{tg}\,ax}{1-\mathrm{tg}\,ax}-1}{\sin bx}=$$

$$=\lim_{x\to 0}\frac{2\,\mathrm{tg}\,ax}{\sin bx\,(1-\mathrm{tg}\,ax)}=\lim_{x\to 0}2\cdot\frac{\sin ax}{ax}\cdot\frac{bx}{\sin bx}\cdot\frac{a}{b}\cdot\frac{1}{\cos ax\,(1-\mathrm{tg}\,ax)}=\underline{\underline{\frac{2a}{b}}}$$

---

(15) $$\lim_{x\to 0_+}\ln(x\ln a)\cdot\ln\left(\frac{\ln ax}{\ln x/a}\right)=\lim_{x\to 0_+}\ln(x\ln a)\cdot\ln\left(\frac{\ln x+\ln a}{\ln x-\ln a}\right)=$$

$$=\lim_{x\to 0_+}\ln(x\ln a)\cdot\ln\left(\frac{1+\frac{\ln a}{\ln x}}{1-\frac{\ln a}{\ln x}}\right)=\lim_{x\to 0_+}\ln(x\ln a)\cdot\underbrace{\frac{\ln\left(\frac{1+\frac{\ln a}{\ln x}}{1-\frac{\ln a}{\ln x}}\right)}{\frac{1+\frac{\ln a}{\ln x}}{1-\frac{\ln a}{\ln x}}-1}}_{\text{V1}}\cdot\left(\frac{1+\frac{\ln a}{\ln x}}{1-\frac{\ln a}{\ln x}}-1\right)=$$

$$=\lim_{x\to 0_+}\ln(x\ln a)\cdot\frac{2\ln a}{\ln x-\ln a}=\lim_{x\to 0_+}2\ln a\cdot\frac{\ln x+\ln\ln a}{\ln x-\ln a}=\lim_{x\to 0_+}2\ln a\,\frac{1+\frac{\ln\ln a}{\ln x}}{1-\frac{\ln a}{\ln x}}=\underline{\underline{2\ln a}}$$

---

(16) $$\lim_{x\to 0}\frac{\ln(1+xe^x)}{\ln\left(x+\sqrt{1+x^2}\right)}=\lim_{x\to 0}\underbrace{\frac{\ln(1+xe^x)}{xe^x}}_{\text{V1}}\cdot\underbrace{\frac{x+\sqrt{1+x^2}-1}{\ln\left(x+\sqrt{1+x^2}\right)}}_{\text{V1}}\cdot\frac{xe^x}{\sqrt{1+x^2}-(1-x)}=\lim_{x\to 0}\frac{xe^x\cdot\left(\sqrt{1+x^2}+(1-x)\right)}{1+x^2-(1-x)^2}$$

$$=\lim_{x\to 0}e^x\cdot\frac{\sqrt{1+x^2}+(1-x)}{2}=\underline{\underline{1}}$$

---

(17) $$\lim_{x\to 1}(1-x)\log_x 2=(*)$$

$y=\log_x 2\Rightarrow x^y=2\Rightarrow x=2^{1/y}$

$x\to 1\Rightarrow y\to+\infty$ to nechceme. Jinak: $x^y=2\Rightarrow e^{y\ln x}=e^{\ln 2}\Rightarrow y\ln x=\ln 2\Rightarrow y=\frac{\ln 2}{\ln x}$

$\Rightarrow \log_x 2=\frac{\ln 2}{\ln x}$

$$(*)=\lim_{x\to 1}(1-x)\frac{\ln 2}{\ln x}=\lim_{x\to 1}\frac{x-1}{\ln x}\cdot(-\ln 2)=\underline{\underline{-\ln 2}}$$

(18) $\lim_{x\to 0_+} (1+x)^{1/x} = \lim_{x\to 0_+} \exp\left(\frac{1}{x}\cdot\ln(1+x)\right) \overset{V2}{=} \exp \lim_{x\to 0_+} \frac{\ln(1+x)}{x} = \exp 1 = \underline{\underline{e}}$

(19) $\lim_{x\to \pi/2} (\sin x)^{\operatorname{tg} x} = \lim_{x\to\pi/2} \exp(\operatorname{tg}x\cdot\ln\sin x) \overset{V2}{=} \exp\lim_{x\to\pi/2} \underbrace{\frac{\ln\sin x}{\sin x-1}}_{V1}\cdot(\sin x-1)\operatorname{tg}x =$

$= \exp\lim_{x\to\pi/2} \sin x\cdot\frac{\sin x-1}{\cos x} = \exp\lim_{y\to 0}\frac{\sin(y+\pi/2)-1}{\cos(y+\pi/2)} = \exp\lim_{y\to 0}\frac{\cos y-1}{-\sin y} = \exp\lim_{y\to 0}\frac{1-\cos^2 y}{(\sin y)(1+\cos y)}$

$y = x-\pi/2,\ x\to\pi/2$

$x = y+\pi/2,\ y\to 0$

$= \exp\lim_{y\to 0}\frac{\sin y}{1+\cos y} = \exp 0 = \underline{\underline{1}}$

(20) $\lim_{x\to 0}\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)^{\frac{1}{\sin^3 x}} = \lim_{x\to 0}\exp\left(\frac{\ln\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)}{\sin^3 x}\right) \overset{V2}{=} \exp\lim_{x\to 0}\underbrace{\frac{\ln\left(\frac{1+\operatorname{tg}x}{1+\sin x}\right)}{\frac{1+\operatorname{tg}x}{1+\sin x}-1}}_{V1}\cdot\frac{\frac{1+\operatorname{tg}x}{1+\sin x}-1}{\sin^3 x} =$

$= \exp\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{\sin^3 x\,(1+\sin x)} = \exp\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x\cos x(1+\sin x)} = \exp\lim_{x\to 0}\frac{1-\cos^2 x}{\sin^2 x\cos x(1+\sin x)(1+\cos x)}$

$= \exp\lim_{x\to 0}\frac{1}{\cos x(1+\sin x)(1+\cos x)} = \exp\frac{1}{2} = \underline{\underline{\sqrt{e}}}$

(21) $\lim_{x\to 1}(1+\sin\pi x)^{\operatorname{cotg}\pi x} \overset{V2}{=} \exp\lim_{x\to 1}\operatorname{cotg}\pi x\cdot\ln(1+\sin\pi x) = \exp\lim \underbrace{\frac{\ln(1+\sin\pi x)}{\sin\pi x}}_{V1}\cdot\sin\pi x\cdot\operatorname{cotg}\pi x$

$= \exp\lim_{x\to 1}\cos\pi x = \exp(-1) = \underline{\underline{\frac{1}{e}}}$

(22) $\lim_{x\to 0_+}(\cos\sqrt{x})^{1/x} \overset{V2}{=} \exp\lim_{x\to 0_+}\frac{\ln\cos\sqrt{x}}{x} = \exp\lim_{x\to 0_+}\underbrace{\frac{\ln\cos\sqrt{x}}{\cos\sqrt{x}-1}}_{V1}\cdot\frac{\cos\sqrt{x}-1}{x} = \exp\lim\frac{\cos^2\sqrt{x}-1}{x(\cos\sqrt{x}+1)} =$

$= \exp\lim_{x\to 0_+}\frac{-\sin^2\sqrt{x}}{x}\cdot\frac{1}{(\cos\sqrt{x}+1)} = \exp(-1/2) = \underline{\underline{\frac{1}{\sqrt{e}} = \frac{\sqrt{e}}{e}}}$

(23) $\lim_{x\to 0}(1+x^2)^{\operatorname{cotg}\pi x} \overset{V2}{=} \exp\lim_{x\to 0}\operatorname{cotg}\pi x\cdot\ln(1+x^2) = \exp\lim_{x\to 0}\underbrace{\frac{\ln(1+x^2)}{x^2}}_{V1}\cdot x^2\cdot\operatorname{cotg}\pi x =$

$= \exp\lim_{x\to 0}\frac{\pi x}{\sin\pi x}\cdot\frac{1}{\pi}\cdot x\cdot\cos\pi x = \exp 0 = \underline{\underline{1}}$

(24) DÚ

(6)

(25) $$\lim_{x\to1}\frac{\sin\pi x^\alpha}{\sin\pi x^\beta}=\lim_{x\to1}\underbrace{\frac{\sin\pi x^\alpha}{1-x^\alpha}}_{\to\pi}\cdot\underbrace{\frac{1-x^\beta}{\sin\pi x^\beta}}_{\to\frac{1}{\pi}}\cdot\frac{1-x^\alpha}{1-x^\beta}=\lim_{x\to1}\frac{1-x^\alpha}{1-x^\beta}=\lim_{x\to1}\frac{x^\alpha-1}{x^\beta-1}=$$

(V1), příklad (6)

$$=\lim_{x\to1}\underbrace{\frac{e^{\alpha\ln x}-1}{\alpha\ln x}}_{\text{V1}}\cdot\underbrace{\frac{\beta\ln x}{e^{\beta\ln x}-1}}_{\text{V1}}\cdot\frac{\alpha}{\beta}=\frac{\alpha}{\beta}$$

---

(26) $$\lim_{x\to0}\frac{e^{\alpha x}-e^{\beta x}}{\sin\alpha x-\sin\beta x}=\lim_{x\to0}\frac{e^{\alpha x}-1}{\sin\alpha x-\sin\beta x}-\frac{e^{\beta x}-1}{\sin\alpha x-\sin\beta x}=\lim_{x\to0}\underbrace{\frac{e^{\alpha x}-1}{\alpha x}}_{\text{V1}}\cdot\frac{\alpha x}{\sin\alpha x-\sin\beta x}-$$

$$-\lim_{x\to0}\underbrace{\frac{e^{\beta x}-1}{\beta x}}_{\text{V1}}\cdot\frac{\beta x}{\sin\alpha x-\sin\beta x}=\lim_{x\to0}\frac{1}{\frac{\sin\alpha x}{\alpha x}-\frac{\sin\beta x}{\alpha x}}-\lim_{x\to0}\frac{1}{\frac{\sin\alpha x}{\beta x}-\frac{\sin\beta x}{\beta x}}=$$

$$=\frac{1}{\lim\limits_{x\to0}\left(\frac{\sin\alpha x}{\alpha x}-\frac{\sin\beta x}{\beta x}\cdot\frac{\beta}{\alpha}\right)}-\frac{1}{\lim\limits_{x\to0}\left(\frac{\sin\alpha x}{\alpha x}\cdot\frac{\alpha}{\beta}-\frac{\sin\beta x}{\beta x}\right)}=\frac{1}{1-\frac{\beta}{\alpha}}-\frac{1}{\frac{\alpha}{\beta}-1}=\frac{\alpha}{\alpha-\beta}-\frac{\beta}{\alpha-\beta}=1$$

---

(27) $$\lim_{x\to a}\frac{a^x-x^a}{x-a}=\lim_{x\to a}\frac{a^x-a^a}{x-a}-\lim_{x\to a}\frac{x^a-a^a}{x-a}=\lim_{x\to a}a^a\,\frac{a^{x-a}-1}{x-a}-\lim_{x\to a}a^{a-1}\frac{\left(\frac{x}{a}\right)^a-1}{\frac{x}{a}-1}=$$

$$=\lim_{y\to0}a^a\underbrace{\frac{a^y-1}{y}}_{\text{V1}}-\lim_{z\to1}a^{a-1}\underbrace{\frac{z^a-1}{z-1}}_{\text{V1}}=\lim_{y\to0}a^a\underbrace{\frac{e^{y\ln a}-1}{y\ln a}}_{\text{V1}}\cdot\ln a-\lim_{z\to1}a^{a-1}\underbrace{\frac{e^{a\ln z}-1}{a\ln z}}_{\text{V1}}\cdot\frac{a\ln z}{z-1}$$

$$=a^a\cdot\ln a-a^a=a^a(\ln a-1)$$

---

(28) $$\lim_{x\to0}\left(\frac{1+x2^x}{1+x3^x}\right)^{1/x^2}\underset{\text{V2}}{=}\exp\lim_{x\to0}\ln\left(\frac{1+x2^x}{1+x3^x}\right)\cdot\frac{1}{x^2}=\exp\lim_{x\to0}\underbrace{\frac{\ln\left(\frac{1+x2^x}{1+x3^x}\right)}{\frac{1+x2^x}{1+x3^x}-1}}_{\text{V1}}\cdot\frac{\frac{1+x2^x}{1+x3^x}-1}{x^2}=$$

$$=\exp\lim_{x\to0}\frac{x\cdot(2^x-3^x)}{x^2}=\exp\left(\lim_{x\to0}\frac{2^x-1}{x}-\lim_{x\to0}\frac{3^x-1}{x}\right)=\exp\lim\left(\frac{e^{x\ln2}-1}{x\ln2}\cdot\ln2-\frac{e^{x\ln3}-1}{x\ln3}\cdot\ln3\right)=$$

$$=\exp(\ln2-\ln3)=\exp\ln\frac{2}{3}=\frac{2}{3}$$

---

(29) $$\lim_{x\to0}\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{1/x}\underset{\text{V2}}{=}\exp\lim_{x\to0}\underbrace{\frac{\ln\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)}{\frac{a^{x^2}+b^{x^2}}{a^x+b^x}-1}}_{\text{V1}}\cdot\left(\frac{a^{x^2}+b^{x^2}}{a^x+b^x}-1\right)\cdot\frac{1}{x}=\exp\lim_{x\to0}\frac{a^{x^2}-a^x+b^{x^2}-b^x}{x\cdot(a^x+b^x)}=$$

$$=\exp\lim_{x\to0}\left[\frac{a^{x^2}-1}{x}-\frac{a^x-1}{x}+\frac{b^{x^2}-1}{x}-\frac{b^x-1}{x}\right]\cdot\frac{1}{a^x+b^x}=\exp\lim_{x\to0}\left[\underbrace{\frac{e^{x^2\ln a}-1}{x^2\ln a}}_{\text{V1}}\cdot x\ln a-\underbrace{\frac{e^{x\ln a}-1}{x\ln a}}_{\text{V1}}\cdot\ln a+\right.$$

$$\left.+\underbrace{\frac{e^{x^2\ln b}-1}{x^2\ln b}}_{\text{V1}}\cdot x\ln b-\underbrace{\frac{e^{x\ln b}-1}{x\ln b}}_{\text{V1}}\cdot\ln b\right]\cdot\frac{1}{a^x+b^x}=\exp\left(\frac{-\ln a-\ln b}{2}\right)=\exp\ln(ab)^{-1/2}=(ab)^{-1/2}=\frac{1}{\sqrt{ab}}$$