Lecture 9

ultraproduct

topics

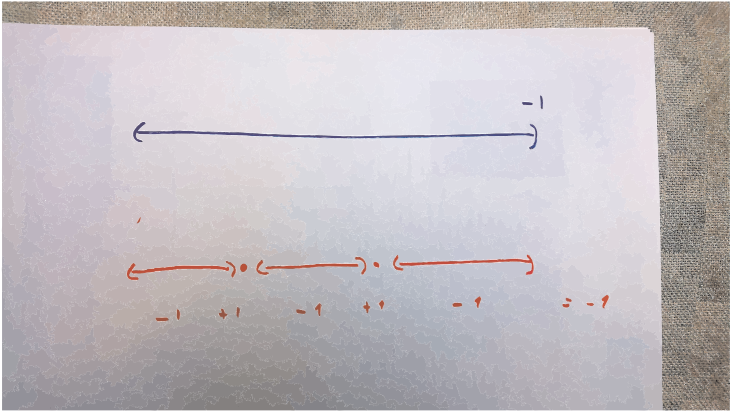
- HW - cell decomposition
- filters and ultrafilters, the Zorn lemma
- ultraproduct
- Łoš's theorem
- ex's: $\mathbf{N}^*$, $\mathbf{R}^*$
- a proof of the compactness thm via ultraproduct

HW-1

task 1: $\chi(U)$ is well-defined on $\mathbf{R}$ and is additive on disjoint unions

$\mathcal{C}$ and $\mathcal{D}$ two cell-decompositions: take their common refinement:

$$A \cap B \;, \quad \text{for} \quad A \cap B \neq \emptyset, A \in \mathcal{C}, B \in \mathcal{D} \;.$$



HW-2

task 2: generalization to $\mathbf{R}^2$

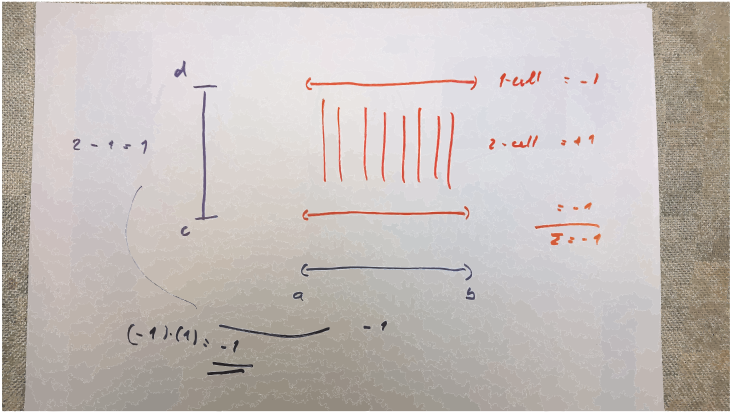
If $W \subseteq \mathbf{R}^2$ decomposes into a 0-cells, b 1-cells and c 2-cells, put:

$$\chi(W) \ := \ a - b + c \ .$$

Ex.:
For $U, V \subseteq \mathbf{R}$ two open intervals and $W := U \times V$:

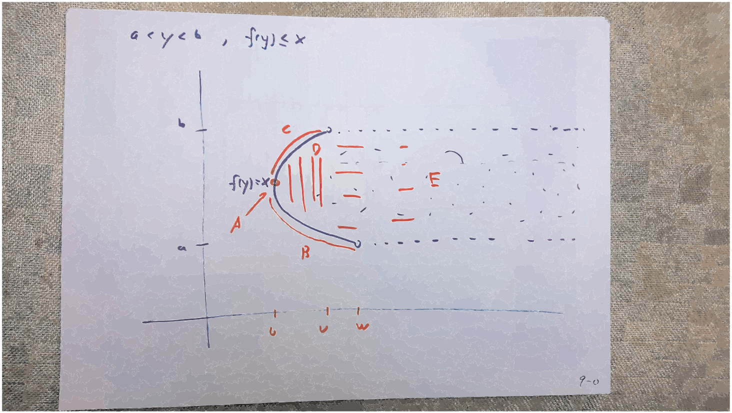
$$\chi(W) = \chi(U) \cdot \chi(V) \ .$$

HW-2: pic



HW-3

task 3: decompose sets in $\mathbf{R}^2$ defined by $a < y < b \wedge f(y) < x < g(y)$ (these are cells rotated by 90 degrees)



motivation

From a given collection of L-structures

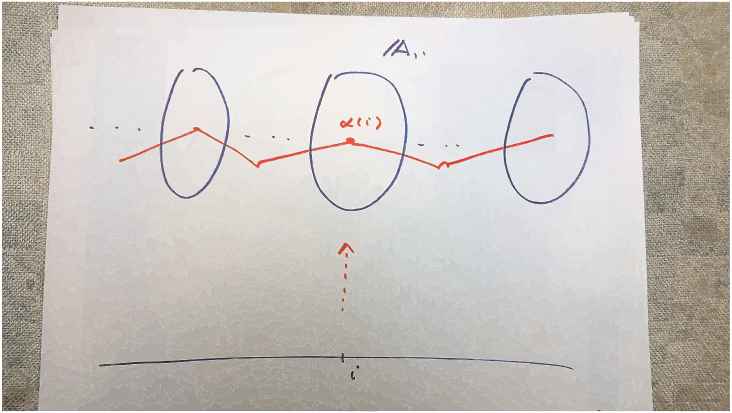
$$\{\mathbf{A}_i\}_{i\in I}$$

construct a new L-structure $\mathbf{A}^*$ that has those FO properties that are

"common to most" $\mathbf{A}_i$.

Generalizes direct product.

idea - pic



filters

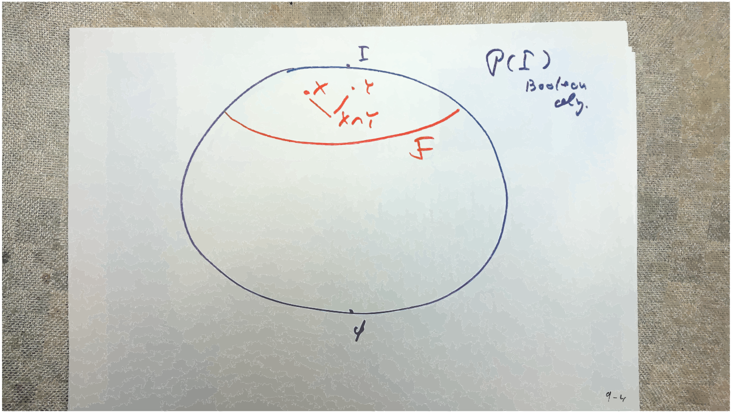
Definition - filter

For $I \neq \emptyset$, a filter on I is $\mathcal{F} \subseteq \mathcal{P}(I)$ s.t.:

- $I \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$ (non-triviality),
- $X \in \mathcal{F}, X \subseteq Y \Rightarrow Y \in \mathcal{F}$ (closed upwards),
- $X, Y \in \mathcal{F} \Rightarrow X \cap Y \in \mathcal{F}$ (closed under intersections).

Ex.:
For I infinite the Fréchet filter consists of all cofinite subsets of I.

filter-pic



more ex's

Ex.:
$I = [0, 1]$
$\mathcal{F}$: all $X \subseteq [0, 1]$ containing a measure 1 set

Ex.:
$I = \mathbf{R}$
$\mathcal{F}$: all $X \subseteq \mathbf{R}$ such that $\mathbf{R} \setminus X$ is countable (or finite)

Ex.:
$I = \mathcal{P}(\mathbf{N})$
$\mathcal{F}$: all $X \subseteq \mathbf{N}$ such that their density

$$\lim_{n\to\infty} \frac{|[0, n] \cap X|}{n+1}$$

exists and goes to 1.

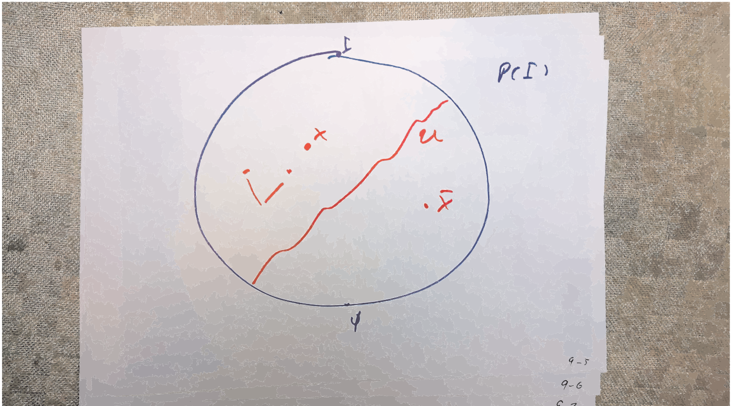
a leap

Definition - ultrafilter

For $I \neq \emptyset$, an ultrafilter on I is a filter $\mathcal{U}$ on I s.t.:

- for all $X \subseteq I$: $X \in \mathcal{U} \vee I \setminus X \in \mathcal{U}$.

When I is clear we shall denote $I \setminus X$ simply $\overline{X}$.



existence

Theorem

For all $I \neq \emptyset$, any filter on I can be extended to an ultrafilter.

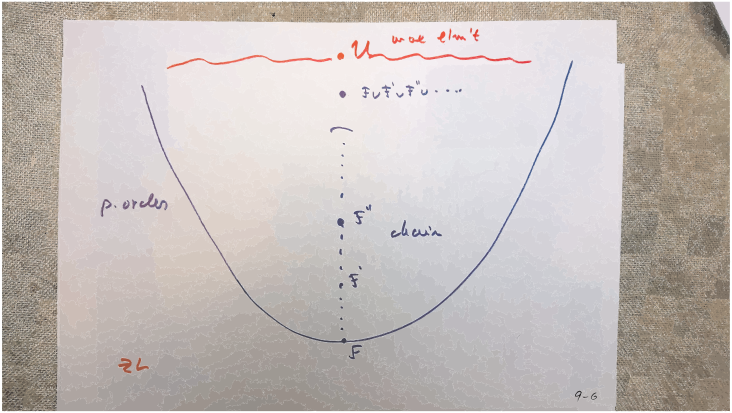
Prf.:
Let $\mathcal{F}$ be a filter. Consider partial ordering $\mathbf{P}$ consisting of all filters extending $\mathcal{F}$ ordered by inclusion.

It satisfies the condition in Zorn's lemma: every chain has an upper bound.

ZL implies that there is a maximal element $\mathcal{U}$ in $\mathbf{P}$: it must be an ultrafilter because if neither X nor $\overline{X}$ were in $\mathcal{U}$ we could extend $\mathcal{U}$.

□

prf-pic



ax's

Ex's of existence statements of set theory:

ZL (Zorn's lemma): Every p.o. in which all chains have an upper bound has a maximal element.

AC (ax. of choice): If all $U_i \neq \emptyset$, $i \in I$, then $\prod_i U_i \neq \emptyset$: there is some function $f : i \in I \to f(i) \in U_i$.

WO (well-ordering principle): Every set can be well-ordered (a strict linear order in which every non-empty set has minimum).

Fact

ZL, AC and WO are equivalent in set theory ZF.

non-principality

Definition

An ultrafilter $\mathcal{U}$ on I is principal iff there is $i_0 \in I$ s.t. for all $X \subseteq I$:

$$X \in \mathcal{U} \quad \text{iff} \quad i_0 \in X \ .$$

Note:

- All ultrafilters on a finite set are principal.
- An ultrafilter is non-principal iff it extends the Frechet filter.

We shall use non-principal ultrafilters in all example constructions.

notation

Given:

- $I \neq \emptyset$,
- L-structures $\mathbf{A}_i$ for $i \in I$,
- an ultrafilter $\mathcal{U}$ on I,

we shall define a new structure denoted

$$\prod_{i \in I} \mathbf{A}_i / \mathcal{U} \ .$$

To ease on notation, when the data above (I, $\mathbf{A}_i$'s and $\mathcal{U}$) are clear from the context, we shall denote the structure just

$$\mathbf{A}^* \ .$$

construction start

We start with the Cartesian product

$$\prod_{i \in I} A_i$$

of the universes. It is non-empty by AC.

Given a formula $\varphi(x_1, \ldots, x_k)$ and $\alpha_1, \ldots, \alpha_k \in \prod_{i \in I} A_i$ define the subset of I:

$$\langle\!\langle \varphi(\alpha_1, \ldots, \alpha_k) \rangle\!\rangle \;:=\; \{ i \in I \mid \mathbf{A}_i \models \varphi(\alpha_1(i), \ldots, \alpha_k(i)) \} \; .$$

equiv.rel.

On $\prod_{i \in I} A_i$ define a relation:

$$\alpha \approx \beta \ \Leftrightarrow_{df} \ \langle\langle \alpha = \beta \rangle\rangle \in \mathcal{U} \ .$$

Lemma

Relation $\approx$ is an equivalence relation.

Prf.:
$\langle\langle \alpha = \alpha \rangle\rangle = I \in \mathcal{U}$ by definition of filters, so $\approx$ is reflexive.

$\langle\langle \alpha = \beta \rangle\rangle = \langle\langle \beta = \alpha \rangle\rangle$, so $\approx$ is symmetric.

$\langle\langle \alpha = \beta \rangle\rangle \cap \langle\langle \beta = \gamma \rangle\rangle \subseteq \langle\langle \alpha = \gamma \rangle\rangle$, so $\approx$ is transitive.

□

universe

Using it define the universe A^* of the future structure by

$$A^* := \prod_{i \in I} A_i / \approx .$$

Notation: $[\alpha]$ is the $\approx$-block of α. I.e.:

$$A^* = \{[\alpha] \mid \alpha \in \prod_{i \in I} A_i\} .$$

interpretation of rel.symbols

Interpret relation symbols of L on A^* as follows:

$$R^{\mathbf{A}^*}([\alpha_1], \ldots, [\alpha_k]) \;\Leftrightarrow_{df}\; \langle\langle R(\alpha_1, \ldots, \alpha_k)\rangle\rangle \in \mathcal{U} \;.$$

Lemma

The interpretation is well-defined: it does not depend on the choice of representants of the $\approx$-blocks:

$$\bigwedge_j [\alpha_j] = [\beta_j] \;\rightarrow R^{\mathbf{A}^*}(\alpha_1, \ldots, \alpha_k) \equiv R^{\mathbf{A}^*}(\beta_1, \ldots, \beta_k) \;.$$

In other words, $\mathbf{A}^*$ satisfies axioms of equality:

$$\bigwedge_j \alpha_j = \beta_j \;\rightarrow R(\alpha_1, \ldots, \alpha_k) \equiv R(\beta_1, \ldots, \beta_k) \;.$$

prf of the lemma

Prf.:

That all $[\alpha_j] = [\beta_j]$ means that all $\langle\langle \alpha_j = \beta_j \rangle\rangle \in \mathcal{U}$ and hence also their intersection is in $\mathcal{U}$.

Then note that

$$\bigcap_j \langle\langle \alpha_j = \beta_j \rangle\rangle \subseteq \langle\langle R(\alpha_1, \ldots, \alpha_k) \equiv R(\beta_1, \ldots, \beta_k) \rangle\rangle \ .$$

Hence $\langle\langle R(\alpha_1, \ldots, \alpha_k) \equiv R(\beta_1, \ldots, \beta_k) \rangle\rangle \in \mathcal{U}$. But this means that

$$\langle\langle R(\alpha_1, \ldots, \alpha_k) \rangle\rangle \in \mathcal{U} \ \text{ iff } \ \langle\langle R(\beta_1, \ldots, \beta_k) \rangle\rangle \in \mathcal{U} \ .$$

□

interpret. cont'd

Now we interpret constants and function symbols of L:

$c^{\mathbf{A}^*} := [\alpha_c]$, where

$$\alpha_c(i) := c^{\mathbf{A}_i^*} .$$

$f^{\mathbf{A}^*}([\alpha_1], \ldots, [\alpha_k]) := \beta$, where

$$\beta(i) := f^{\mathbf{A}_i^*}(\alpha_1(i), \ldots, \alpha_k(i)) .$$

This looks complicated but it simply says that we apply f coordinate wise in each structure $\mathbf{A}_i$ separately.

lemma

Lemma

The interpretation is well-defined: it does not depend on the choice of representants of the $\approx$-blocks and $\mathbf{A}^*$ satisfies axioms of equality:

$$\bigwedge_j \alpha_j = \beta_j \ \rightarrow f(\alpha_1, \ldots, \alpha_k) = f(\beta_1, \ldots, \beta_k) \ .$$

Prf. is analogous to the proof of the previous lemma about the interpretation of relation symbols.

This completes the definition of $\mathbf{A}^*$!

It looks quite complicated and we may worry how shall we ever decide what is true there.

key thm

Łoš's theorem

For any L-formula $\varphi(x_1, \ldots, x_k)$ and any elements $[\alpha_1], \ldots, [\alpha_k] \in A^*$ it holds:

$$\mathbf{A}^* \models \varphi([\alpha_1], \ldots, [\alpha_k]) \text{ iff } \langle\langle \varphi(\alpha_1, \ldots, \alpha_k) \rangle\rangle \in \mathcal{U} .$$

Prf.:
By induction in the complexity of φ.

atomic flas: this is how the structure is defined!

prf cont'd

Boolean connectives:
conjunction:

$$\models \eta \wedge \rho \;\Leftrightarrow\; [\models \eta \;\text{ and }\; \models \rho]$$

We have:

$$\langle\!\langle \eta \wedge \rho \rangle\!\rangle \in \mathcal{U} \;\Leftrightarrow\; \langle\!\langle \eta \rangle\!\rangle \cap \langle\!\langle \rho \rangle\!\rangle \in \mathcal{U} \;\Leftrightarrow_1\; \langle\!\langle \eta \rangle\!\rangle \in \mathcal{U} \wedge \langle\!\langle \rho \rangle\!\rangle \in \mathcal{U} \;.$$

The equivalence $\Leftrightarrow_1$ holds because of the filter definition.

negation:

$$\not\models \eta \;\Leftrightarrow\; \langle\!\langle \eta \rangle\!\rangle \notin \mathcal{U} \;\Leftrightarrow_2\; \langle\!\langle \neg\eta \rangle\!\rangle \in \mathcal{U} \;\Leftrightarrow \models \neg\eta \;.$$

The equivalence $\Leftrightarrow_2$ holds because of the ultrafilter definition.

prf cont'd

disjunction:

$$[\models \eta \ \text{ or } \ \models \rho] \ \Leftrightarrow \models \eta \vee \rho$$

If $\langle\langle\eta\rangle\rangle \in \mathcal{U}$ (or ρ is in) then also $\langle\langle\eta \vee \rho\rangle\rangle \in \mathcal{U}$ because

$$\langle\langle\eta\rangle\rangle \in \mathcal{U} \subseteq \langle\langle\eta \vee \rho\rangle\rangle \ .$$

Opposite direction:

$$[\langle\langle\eta\rangle\rangle \notin \mathcal{U} \ \text{ and } \ \langle\langle\eta\rangle\rangle \notin \mathcal{U}] \ \Leftrightarrow \ [\langle\langle\neg\eta\rangle\rangle \in \mathcal{U} \ \text{ and } \ \langle\langle\neg\eta\rangle\rangle \in \mathcal{U}]$$

hence if also $\langle\langle\eta \vee \rho\rangle\rangle \in \mathcal{U}$ we would have

$$\emptyset \in \mathcal{U}$$

which is a contradiction.

end of prf

∃-quantifier:

$$\models \exists x\eta(x) \;\Leftrightarrow\; \text{for some } \alpha \;\; \models \eta(\alpha)\ .$$

Because for any α

$$\langle\langle \eta(\alpha) \rangle\rangle \subseteq \langle\langle \exists x\eta(x) \rangle\rangle$$

the right-to-left implication follows.

Assume $I_0 = \langle\langle \exists x\eta(x) \rangle\rangle$. Define β by:

- $i \in I_0$: $\alpha(i)$ is some witness for x in $\exists x\eta(x)$,
- $i \notin I_0$: $\alpha(i)$ is arbitrary.

Then

$$\langle\langle \eta(\beta) \rangle\rangle = \langle\langle \exists x\eta(x) \rangle\rangle\ .$$

∀-quantifier: analogous.

□

Corollary

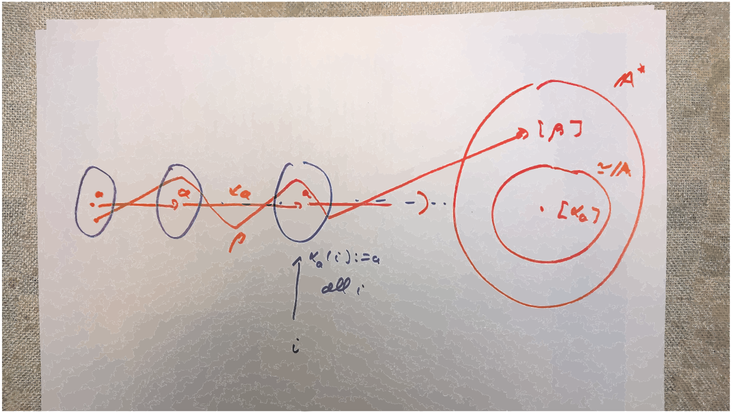
Let $\mathbf{A}$ be an L-structure. Let I be an infinite set, $\mathbf{A}_i = \mathbf{A}$ for all $i \in I$, and assume $\mathcal{U}$ is a non-principal ultrafilter on I.
Then

$$\mathbf{A} \not\preceq \mathbf{A}^* \ .$$

(It is called ultrapower.)

Prf.:
pic next slide.

prf by pic



non-standard $\mathbf{N}$

$I = \omega$
$\mathbf{A}_i := \mathbf{N}$, all $i \in \omega$

$\alpha_k(i) := k$, all $i \in I$ and $k \in \mathbf{N}$ (represents constant k)

$\beta(i) := i$ (represents non-standard element)

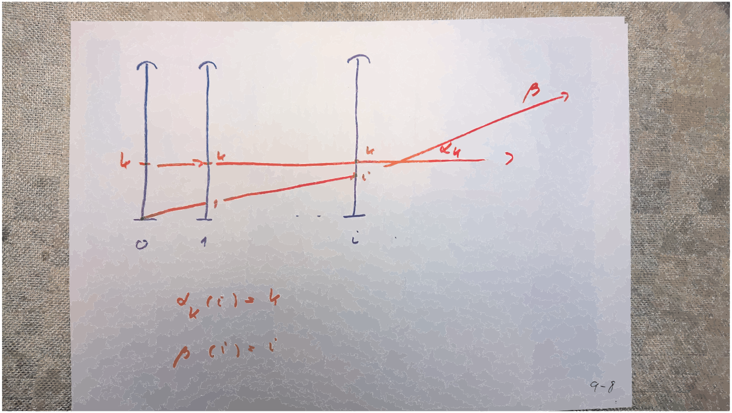
$\mathbf{N}^*$: the ultraproduct

Theorem

Elements $[\alpha_k]$, for $k \in \mathbf{N}$, define a substructure of $\mathbf{N}^*$ isomorphic to $\mathbf{N}$ and in $\mathbf{N}^*$:

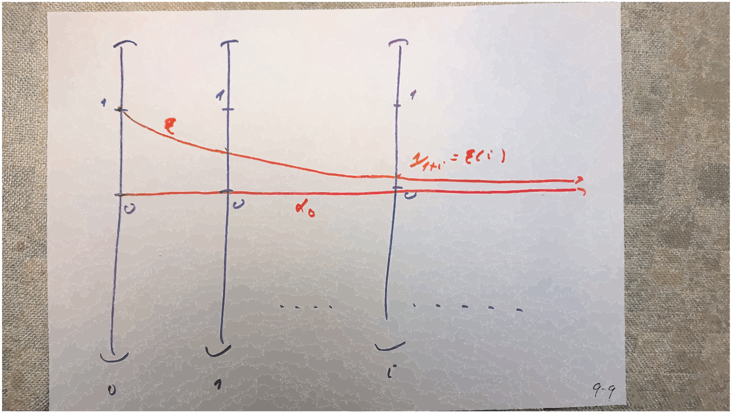
$$\alpha_k < \beta \ , \quad \text{for all } k \in \mathbf{N} \quad .$$

pic



nonstandard reals

$\mathbf{R}^*$: ultrapower of $\mathbf{R}$ as before
$\alpha_r(i) := r$, for all i (represents $r \in R$)
$\epsilon(i) := 1/(1+i)$, for all i (represents an infinitesimal)



compactness via ultraproduct

Given:

- language L,
- an L-theory T,
- for each finite $S \subseteq_{fin} T$ a model $\mathbf{A}_S \models S$.

Want: a model for tho whole of T.

Take:

I: all finite subsets of T, w.l.o.g. we may assume T (and hence I) is infinite

$\mathcal{F}$: a filter generated by all sets for all $S \subseteq_{fin} T$

$$\{Z \in I \mid Z \supseteq S\} .$$

It is non-trivial because the intersection of any finite nb. of them (say determined by $S_1, \ldots, S_\ell$) is non-empty (contains all $Z \supseteq \bigcup_{i \leq \ell} S_i$).

$\mathcal{U}$: a non-principal ultrafilter extending $\mathcal{F}$

Theorem

$\mathbf{A}^* \models T$.

Prf.:

For any $\varphi \in T$ the set of all $Z \in I$ such that $\mathbf{A}_Z \models \varphi$ is in $\mathcal{F} \subseteq \mathcal{U}$: just apply the above definition to $S := \{\varphi\}$. Use Łoš's thm.

□

HW

Two problems to take away:

(1)
Take an ultrapower of $\mathbf{F}_p$ with infinite I and non-principal $\mathcal{U}$.
How does the ultraproduct look like?

(2)
Now take I to be the set of primes and take an ultraproduct of all fields $\mathbf{F}_p$ with a non-principal ultrafilter $\mathcal{U}$.
What can you say about the resulting structure?