Lecture 8

QE continued

topics

- HW: QE for $Vect_Q$
- completeness of ACF_p from the QE for ACF
- definable sets in ACF, strong minimality
- algebraic geometry, Chevalley's theorem
- combinatorial pregeometries
- theory RCF and QE
- the Tarski-Seidenberg thm
- o-minimality
- definable sets and cell decomposition

HW

The task: establish the QE for $Vect_Q$

Atomic flas in $\overline{x}, y$ can be put into the form

$$\sum_i q_i x_i = 0 \ \text{ or } \ \sum_i q_i x_i = y \ .$$

Hence a primitive fla $\psi(\overline{x}, y)$ with one $\exists y$ quantifier asserts that a system of equations as above, and of inequations of the form

$$\sum_i q_i x_i \neq 0 \ \text{ or } \ \sum_i q_i x_i \neq y$$

has a solution for y.

direct elimination

Case 1: the system $\psi(\overline{x}, y)$ contains some equation

$$\sum_i q_i x_i = y \ .$$

Substitute everywhere $\sum_i q_i x_i$ for y: the resulting system $\psi'(\overline{x})$ is equivalent to $\exists y \psi(\overline{x}, y)$.

Case 2: otherwise let $\psi'(\overline{x})$ be the system of (in)equations in ψ in $\overline{x}$ only:

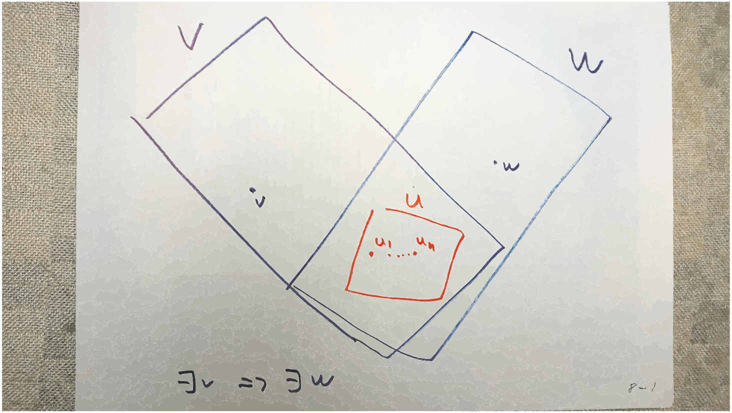
$$\sum_i q_i x_i = 0 \quad \text{and} \quad \sum_i q_i x_i \neq 0 \ .$$

If it is solvable we may take for y any vector except those finitely many ruled out by the inequations

$$\sum_i q_i x_i \neq y$$

in the system. Hence again to $\exists y \psi(\overline{x}, y)$ is equivalent to $\psi'(\overline{x})$.

model-th. criterion



QE $\Rightarrow$ completeness

Let φ be a sentence.

By QE, $ACF \vdash \varphi \equiv \alpha$, where α is a q-free sentence: involves only closed terms built from 0 and 1, i.e. their evaluation is done inside the prime field $\mathbf{F}_p$ or $\mathbf{Q}$, resp.
Assume $char = p$ (the case $char = 0$ is analogous) and let $\mathbf{A}$ and $\mathbf{B}$ be two ACF_p fields. Then we have:

$$\mathbf{A} \models \varphi \Leftrightarrow \mathbf{A} \models \alpha \Leftrightarrow$$

$$\mathbf{F}_p \models \alpha \Leftrightarrow$$

$$\mathbf{B} \models \alpha \Leftrightarrow \mathbf{B} \models \varphi$$

So:

$$\mathbf{A} \equiv \mathbf{B} \; .$$

definable sets in ACF

Each fla $\varphi(\overline{x}, \overline{y})$ is equivalent to a Boolean combination of atomic flas

$$t(\overline{x}, \overline{y}) = s(\overline{x}, \overline{y})$$

and that is equivalent in fields to

$$p(\overline{x}, \overline{y}) = 0$$

where $p \in \mathbf{Z}[\overline{x}, \overline{y}]$.
Hence for any ACF $\mathbf{A}$ and any parameters $\overline{a} \in A^m$, $\varphi(\overline{x}, \overline{a})$ defines a subset of A^n ($n =$ the nb. of x-vars) that is a Boolean combination of sets defined by polynomial equations over A:

$$p(\overline{x}, \overline{a}) = 0 \ .$$

alg.geometry terminology

Subsets of **A** defined by polynomial equations over A:

$$p(\overline{x}, \overline{a}) = 0$$

are in algebraic geometry called Zariski closed.

Their Boolean combinations are called constructible sets.

Corollary of QE for ACF

Definable sets in any ACF are exactly constructible sets.

Chevalley

Theorem (Chevalley)

The image of a constructible set in a polynomial map is constructible.

Prf.:

$$F \ : \ \overline{x} \in A^n \to \overline{y} \in A^\ell$$

If $U \subseteq A^n$ is constructible then it is definable by some fla $\varphi(\overline{x})$ (with parameters from A).
Using this definition we can define $F(U)$ by

$$\exists \overline{x} \ (\varphi(\overline{x}) \wedge F(\overline{x}) = \overline{y}) \ .$$

Hence $F(U)$ is definable and thus also constructible.

$\square$

Note that this thm is equivalent to QE: projections are special poly maps.

minimality

Definition: minimality

A structure **D** is minimal iff all definable subsets of D are finite or cofinite.
A theory T is strongly minimal iff all its models are minimal.

Theorem

ACF is strongly minimal.

Prf.:
A polynomial $p(x)$ in 1 variable has no roots (if p is a non-zero constant), a finite nb. of roots, or all elements are roots (if p is the zero poly). Hence atomic flas define finite or cofinite sets.
The class of finite or cofinite sets is closed under Boolean operations. By QE then all definable subsets of the field are finite or cofinite (we claim nothing about definable subsets of Cartesian powers).

□

a closure operation

Definition: acl

Let $\mathbf{A}$ be a structure and $U \subseteq A$. The model-theoretic algebraic closure of U, denoted by $acl(U)$, is the set of all $b \in A$ such that there is a formula $\psi(x, \overline{y})$ and parameters $\overline{a} \in A^n$ satisfying:

- $\mathbf{A} \models \psi(b, \overline{a})$,
- there are finitely many $c \in A$ s.t. $\mathbf{A} \models \psi(c, \overline{a})$.

Fact

In ACF this notion coincides with the algebraic notion:
$b \in acl(U)$ iff b is in the alg.closure of the subfield of $\mathbf{A}$ generated by U.

properties

Fact

In any structure **A** the closure operation *acl* satisfies the following properties:

- $U \subseteq acl(U) = acl(acl(U))$,
- $U \subseteq V \Rightarrow acl(U) \subseteq acl(V)$,
- (finite character) $b \in acl(U)$ iff $\exists U_0 \subseteq_{fin} U \; b \in acl(U_0)$.

If **A** is strongly minimal (i.e. its elementary diagram is) then also:

- (exchange property) If $a \in U$ and $b \in acl(U) \setminus acl(U \setminus \{a\})$ then $a \in acl((U \setminus \{a\}) \cup \{b\})$.

pregeometries

Any set $X \neq \emptyset$ together with an operation

$$cl \ : \ \mathcal{P}(X) \to \mathcal{P}(X)$$

satisfying the properties from the previous slide is called a pregeometry (or a matroid). Using cl we can define a number of geometric notions:

- $U \subseteq X$ is independent iff for all $u \in U$, $u \notin acl(U - u)$,
- $B \subseteq X$ is a basis iff B is maximal independent,
- dimension $dim(X)$ is the cardinality of a basis,
- $U \subseteq X$ is closed iff $U = cl(U)$,
- ...

It is a deep model-th fact that the iso-type of a strongly minimal structure is determined by its theory plus the dimension of the pregeometry it defines.

geometric model th: much more

reals

In the ordered real closed filed **R** consider a fla stating that a quadratic poly has a root:

$$\exists y \; ay^2 + by + c = 0$$

(I use a, b, c instead of x_2, x_1, x_0 as it is a custom).

Namely, this formulas with free var's a, b, c is equivalent to the following q-free formula:

$$[a = 0 \wedge (b \neq 0 \vee c = 0)] \vee [a \neq 0 \wedge b^2 \geq 4ac] \; .$$

This is not an accident.

axiomatization

theory RCF (real closed fields):

language: $0, 1, +, \cdot, <$

axioms:

- ax's of commutative fields,
- ax's stating that $<$ is a strict linear ordering,
- ax's of ordered fields:

$$x < y \rightarrow x + z < y + z \quad \text{and} \quad (x < y \wedge 0 < z) \rightarrow x \cdot z < y \cdot z$$

- ax of squares: $x > 0 \rightarrow \exists y \; y \cdot y = x$,
- odd degree polys have roots; for all $n = 1, 3, 5, \ldots$:

$$x_n \neq 0 \rightarrow \exists y \sum_{i \leq n} x_i y^i = 0 \ .$$

Tarski's thm

Theorem (Tarski)

RCF has QE and is complete and decidable.

semi-algebraic geometry: a subset of R^n is semialgebraic iff it is defined by a Boolean combination of strict inequalities

$$p(\overline{x}) > 0 \ .$$

Because $t = s$ is equivalent to $t - s = 0$ which is equivalent to

$$\neg(t - s > 0 \vee s - t > 0)$$

the QE implies that all definable sets are semialgebraic.

Corollary (Tarski-Seidenberg)

A projection of a semialgebraic set is also semialgebraic.

A proof is analogous to the one for Chevalley's thm (slide 9).

Ex's

Ex.: the topological closure of a definable set is definable.

Assume $U \subseteq R^n$ is definable by $\varphi(\overline{x})$ with parameters from R. Then its closure is defined by $\eta(\overline{y})$:

$$\forall \epsilon > 0 \exists \overline{x} \; (\bigwedge_i (x_i - y_i)^2 < \epsilon) \wedge \varphi(\overline{x})$$

and where

$$(x_i - y_i)^2 < \epsilon)$$

is an abbreviation for

$$x_i \cdot x_i + y_i \cdot y_i < \epsilon + (1 + 1) \cdot x_i \cdot y_i \; .$$

More ex's in Marker's book.

o-minimality

Corollary

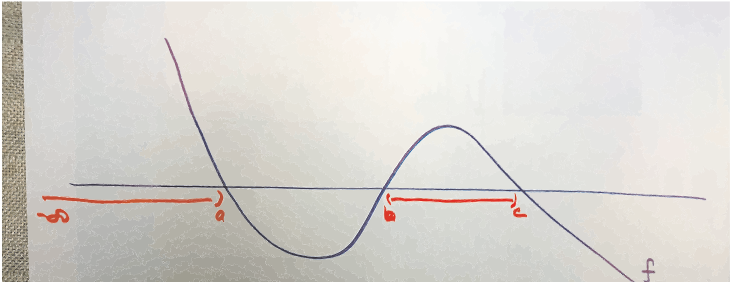
Every definable subset $U \subseteq R$ is a finite union of open intervals and points.

This property is called o-minimality (order-min).

Prf.:

$p(x) = 0$ may define an empty set or a finite set of whole R, and $p(x) > 0$ may define an empty set, the whole R or a proper open interval.

$\square$



definable sets

The o-minimality alone implies over the theory of ordered fields (i.e. no specific RCF ax's are needed) a structure theory for definable subsets of all R^n.

Key notion: cell decomposition.

Case $n = 1$:

0-cells: single points
1-cells: open intervals (a, b), where a and b can be also $+/-\infty$, resp.

Corollary of QE

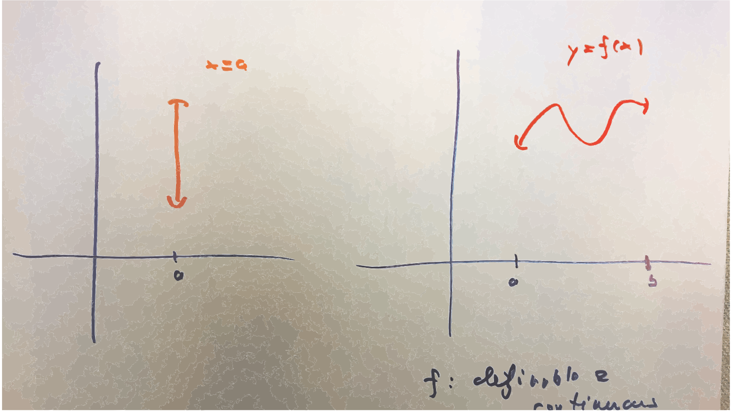
Every definable subset of R is a finite disjoint union on 0- and 1- cells.

$n = 2$

Case $n = 2$:

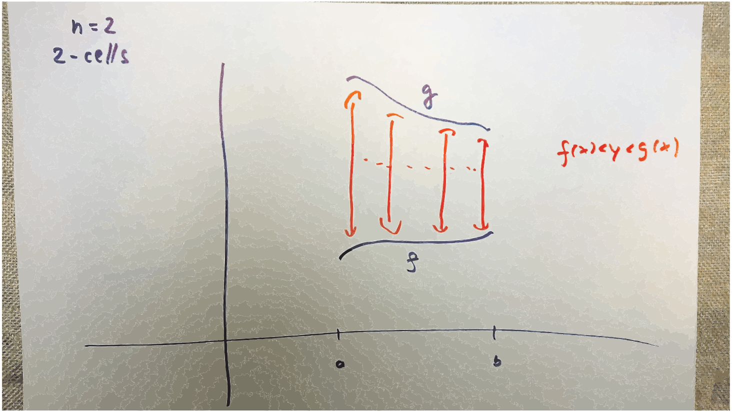
0-cells: single points

1-cells:



2-cells

2-cells:



Theorem - cell decomposition for $n = 2$

Every definable subset of R^2 is a finite disjoint union on 0-, 1- and 2- cells.

tame topology

Definable sets in $\mathbf{R}$ are rich enough to include a lot of mathematics (and semialgebraic geometry in particular) but they avoid set-th pathologies. Terminology: tame topology. See van den Dries's book with the same title.

Ex.: a use of cell-decomp. in defining Euler characteristic.
Case $n = 1$: for a definable set $U \subseteq R$ define $\chi(U) \in \mathbf{Z}$ as follows:

- express U as a disjoint union of some finite family $\mathcal{C}$ of 0- and 1-cells,
- put $\chi(U) := k - \ell$, where k (resp. ℓ) is the nb. of 0-cells (resp. 1-cells) in $\mathcal{C}$.

Fact

The value of $\chi(U)$ does not depend on the choice of $\mathcal{C}$ and

$$\chi(U \cup V) = \chi(U) + \chi(V)$$

for disjoint U, V.

HW

Problems to take away:

(1) Prove the fact from the previous slide.

(2) How would you generalize it to $n = 2$?

(1) Show how the set of $(x, y) \in R^2$ satisfying

$$a < y < b \wedge f(y) < x < g(y)$$

where f and g are definable can be decomposed into cells.