# Lecture 6

skolemization and full Löwenheim-Skolem theorem

# Topics

- the last HW problem: The reverse of the Ax-Grothendieck thm?
- skolemization of a theory
- a proof of the downwards Löwenheim-Skolem theorem
- the full Löwenheim-Skolem theorem

# last HW problem

Last time we proved the Ax-Grothendieck thm:

$$Inj \quad \Rightarrow \quad Sur$$

for polynomial maps on $C^n$.

The proof goes by showing that

1. if the thm fails then it also fails over $\mathbf{F}_q^{alg}$, for some prime $q$
2. if that happens then the thm actually fails over some finite subfield $K \subseteq \mathbf{F}_q^{alg}$
3. that is impossible as the implication above holds over finite sets

# HW cont'd

The opposite implication

$$Sur \quad \Rightarrow \quad Inj$$

also holds over finite sets, and the item 1 works for any FO sentence. Hence a problem with the argument must occur in item 2.

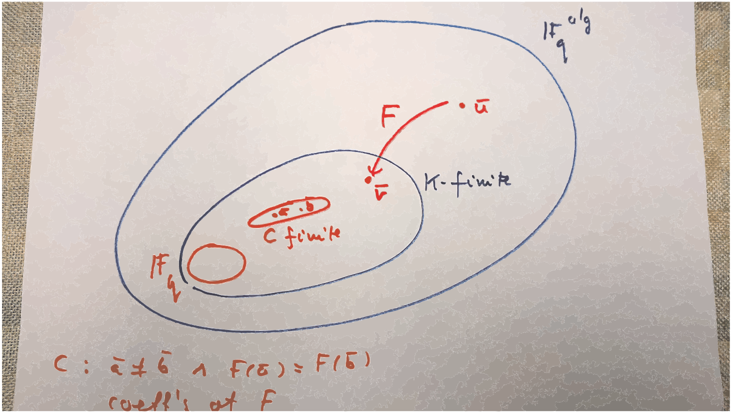
The key step in item 2 is the observations that if a map $F$ is injective on $(\mathbf{F}_q^{alg})^n$ then it is also injective when restricted to $K^n$.

The error in an argument that would attempt to prove the reverse Ax-Grothendieck is that if map $F$ is surjective on $(\mathbf{F}_q^{alg})^n$ then

it does not imply

that it is also surjective when restricted to $K^n$.
(See pic on the next page.)

# HW - pic



$\mathbb{F}_q^{alg}$
F
$\bar{u}$
$\bar{v}$
K-finite
C finite
$\mathbb{F}_q$
C : $\bar{a} \neq \bar{b}$ ∧ F($\bar{a}$) = F($\bar{b}$)
coeff's of F

# L-S thm so far

In Lect.2:

**The Löwenheim-Skolem theorem upwards**

Let $\mathbf{A}$ be an infinite structure in language $L$ and let $\kappa$ be an arbitrary cardinality. Then there is $\mathbf{B}$ such that:

$$\mathbf{A} \preceq \mathbf{B} \quad \text{and} \quad |B| \geq \kappa \ .$$

and in Lect.5:

**The Löwenheim-Skolem thm.**

Let $T$ be a theory in a countable language which has an infinite model. Then $T$ has models of all infinite cardinalities.

# new: L-S down

In this lecture we prove

**The Löwenheim-Skolem theorem downwards**

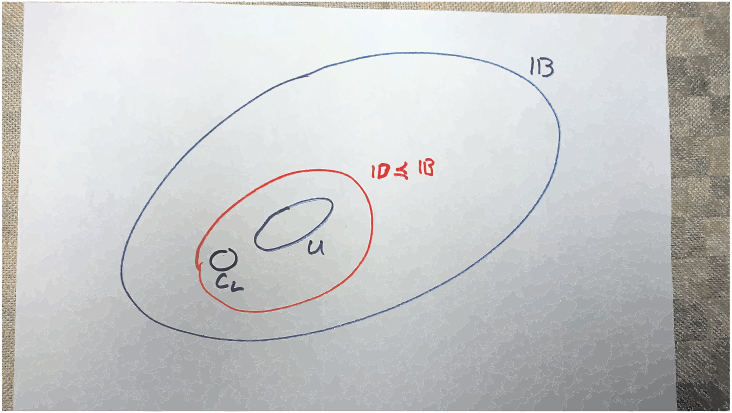
Let $\mathbf{B}$ be an $L$-structure and $U \subseteq B$ be arbitrary. Then there is $\mathbf{D}$ such that:

$$\mathbf{D} \preceq \mathbf{B} \ , \ D \supseteq U \ \text{ and } \ |U| \leq |D| \leq \max(\aleph_0, |L|, |U|).$$

In particular, if $L$ is finite or countable and $U$ is infinite then

$$|D| = |U| \ .$$

# L-S down pic



# prf for Lect.5

Prf. of the L-S thm from Lecture 5:

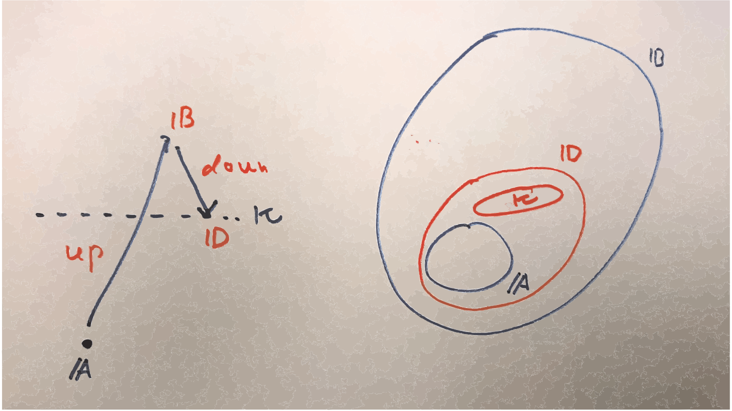
Take $T$ in a countable language that has an infinite model $\mathbf{A}$, and let $\kappa$ be any cardinality. Then do:

- By L-S up get $\mathbf{B}$ s.t. $\mathbf{A} \preceq \mathbf{B}$ and $|B| \geq \kappa$,
- take any $U \subseteq B$ of cardinality precisely $\kappa$,
- by the new L-S down there is $\mathbf{D} \preceq \mathbf{B}$ of cardinality $\kappa$.

We have that all three $\mathbf{A}, \mathbf{B}, \mathbf{D}$ are elementarily equivalent and hence all are models of $T$.

$\square$

# pic



# our task

Given: **B** and $U \subseteq B$

we want: $D$: $U \subseteq D \subseteq B$ such that

$$\mathbf{D} \preceq \mathbf{B} \quad \text{and} \quad |U| \leq |D| \leq \max(\aleph_0, |L|, |U|) \ .$$

In particular,

- $D$ has to be closed under all $L$-functions and contain all $L$-constants, and
- for all $\overline{a} \in D^n$ and any formula $\varphi$:

$$\mathbf{D} \models \varphi(\overline{a}) \quad \text{iff} \quad \mathbf{B} \models \varphi(\overline{a}) \ .$$

# simple case

Assume $\varphi(\overline{x})$ is open and $D$ defines a substructure.

Then for all $\overline{a} \in D^n$:

$$\mathbf{B} \models \varphi(\overline{a})$$
$$\Updownarrow$$
$$\mathbf{D} \models \varphi(\overline{a})$$

because open flas are absolute between structure-substructure (Lect.2).

# idea

An idea how to prove the thm is to reduce to the simple case above; expand $L$ to $L_{Sk} \supseteq L$ such that:

- $\mathbf{B}$ can be expanded to an $L_{Sk}$-structure $\mathbf{B}'$,
- any $L$-fla $\varphi$ is equivalent to an open $L_{Sk}$-fla $\varphi'$,
- $D$ is closed under $L_{Sk}$-functions too.

Then, as before:

$$\mathbf{B} \models \varphi(\bar{a}) \Leftrightarrow \mathbf{B}' \models \varphi(\bar{a}) \Leftrightarrow \mathbf{B}' \models \varphi'(\bar{a}) \Leftrightarrow \mathbf{D} \models \varphi'(\bar{a}) \Leftrightarrow \mathbf{D} \models \varphi(\bar{a}) \ .$$

A subtle point:
Where do the red equivalences $\Leftrightarrow$ of $\varphi$ and $\varphi'$ hold?

# idea technically

We shall implement the informal idea by a construction that will define language $L_{Sk} \supseteq L$ and an $L_{Sk}$-theory $Sk$ such that:

- $|L_{Sk}| \leq \max(\aleph_0, |L|)$,
- $Sk$ is universal,
- $\mathbf{B}$ can be expanded to an $L_{Sk}$-structure $\mathbf{B}' \models \mathrm{Sk}$,
- any $L_{Sk}$-fla $\varphi$ is equivalent to an open $L_{Sk}$-fla $\varphi'$, provably in theory $Sk$,
- $\mathbf{D}$ is an $L_{Sk}$-substructure of $\mathbf{B}'$,
- and finally: $U \subseteq D$ and $|D| \leq \max(\aleph_0, |L|, |U|)$.

# prf of L-S down

Having $L_{Sk}$ and $Sk$ we can prove the L-S thm down as follows:

$$\mathbf{B} \models \varphi(\overline{a}) \Leftrightarrow \mathbf{B}' \models \varphi(\overline{a}) \Leftrightarrow \mathbf{B}' \models \varphi'(\overline{a})$$

because $\mathbf{B}' \models Sk$ the equivalence holds in $Sk$, and then

$$\Leftrightarrow \mathbf{D} \models \varphi'(\overline{a})$$

because $\varphi'$ is open and $\mathbf{D}$ is an $L_{Sk}$-substructure, and

$$\Leftrightarrow \mathbf{D} \models \varphi(\overline{a})$$

because, $Sk$ being universal, holds in $\mathbf{D}$ too.

$\square$

# $\exists$-example

Let $\varphi(\overline{x})$ be an $L$-formula of the form

$$\exists y \; \psi(\overline{x}, y)$$

with $\psi$ open.
Introduce new Skolem function symbol $f_\varphi$ and corresponding Skolem axiom:

$$\psi(\overline{x}, y) \rightarrow \psi(\overline{x}, f_\varphi(\overline{x})) \ .$$

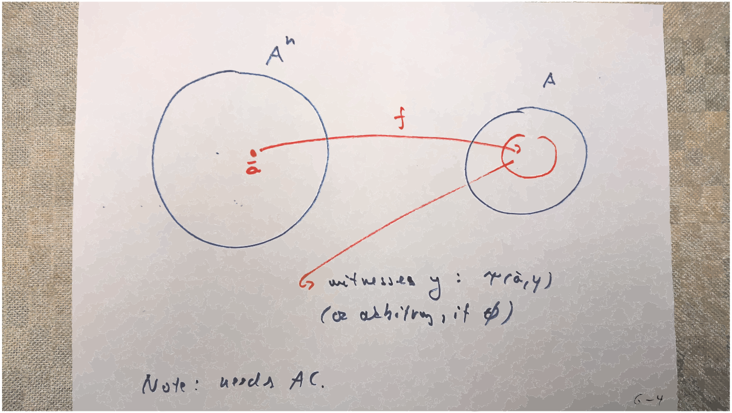
**Lemma**

Formula $\varphi(\overline{x})$ is equivalent to

$$\psi(\overline{x}, f_\varphi(\overline{x}))$$

modulo the Skolem axiom.

# ∃-ex pic



# $\forall$-example

Let $\varphi(\overline{x})$ be now an $L$-formula of the form

$$\forall y \ \psi(\overline{x}, y)$$

with $\psi$ open. Write it as

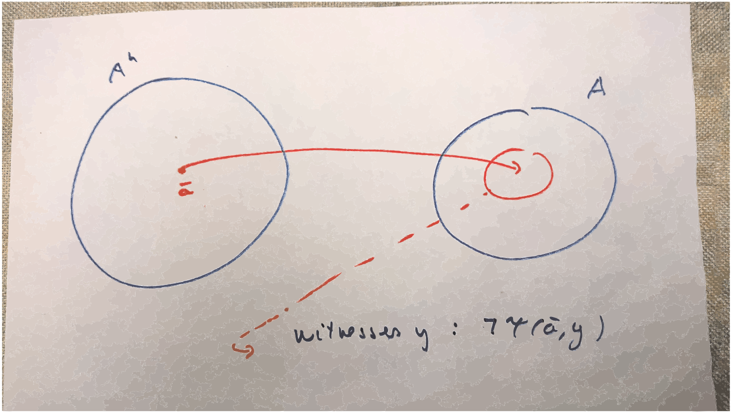
$$\neg\exists y \ \neg\psi(\overline{x}, y) \ .$$

Introduce Skolem function $g$ for $\exists y \ \neg\psi(\overline{x}, y)$ and the corresponding Skolem axiom for $g$:

$$\neg\psi(\overline{x}, y) \rightarrow \neg\psi(\overline{x}, g(\overline{x})) \ .$$

Note: symbol $g$ ought to be $f_{\exists y \neg\psi(\overline{x}, y)}$ but that is typographically cumbersome.

# ∀-ex pic



# ex cont'd

**Lemma**

Formula $\varphi(\overline{x})$ is equivalent to

$$\psi(\overline{x}, g(\overline{x}))$$

modulo the Skolem axiom for $g$.

We have:

$$\exists y \psi(\overline{x}, y) \Leftrightarrow \psi(\overline{x}, f(\overline{x}))$$

and also

$$\forall y \psi(\overline{x}, y) \Leftrightarrow \psi(\overline{x}, g(\overline{x}))$$

which looks identical?!

# $f$ vs. $g$

The point is that $f$ was introduced to find a witness $y$ such that

$$\psi(\overline{x}, y)$$

while $g$ was introduced to find a witness $y$ such that

$$\neg\psi(\overline{x}, y) \ .$$

Informally:
if $\psi(\overline{x}, y)$ fails for some element $y$ then $g$ finds one. Hence if $\psi(\overline{x}, g(\overline{x}))$ holds, there is no such $y$.

# the construction

We shall define the language $L_{Sk}$ and the theory $Sk$ in countably many step, creating chains

$$L_0 \subseteq L_1 \subseteq \ldots \quad \text{and} \quad T_0 \subseteq T_1 \subseteq \ldots$$

and putting

$$L_{Sk} := \bigcup_i L_i \quad \text{and} \quad Sk := \bigcup_i T_i \ .$$

Start:

$$L_0 := L \quad \text{and} \quad T_0 := \emptyset \ .$$

## step $i+1$

language $L_{i+1}$ and theory $T_{i+1}$:

for every $L_i$ formula $\varphi(\overline{x})$ of the form $\exists y \psi(\overline{x}, y)$ with $\psi$ open

- add to $L_i$ new function symbol $f$, and
- add to $T_i$ new Skolem axiom for $f$:

$$\psi(\overline{x}, y) \rightarrow \psi(\overline{x}, f(\overline{x})) \ .$$

Note:

$$|L_{i+1} \setminus L_i| \leq \text{ the nb. of } L_i\text{-flas } \quad \leq \max(\aleph_0, |L_i|) = \max(\aleph_0, |L|)$$

(the last step by induction). So:

$$|L_i| \leq \max(\aleph_0, |L|) \ , \quad \text{for all } i \text{ and hence } \ |L_{Sk}| = \max(\aleph_0, |L|) \ .$$

## the construction cont'd

**Lemma**

1. $|L_{Sk}| \leq \max(\aleph_0, |L|)$,
2. $Sk$ is universal,
3. any $L$-structure $\mathbf{B}$ can be expanded to an $L_{Sk}$-structure $\mathbf{B}' \models$ Sk,
4. any $L_{Sk}$-fla $\varphi$ is equivalent to an open $L_{Sk}$-fla $\varphi'$, provably in theory $Sk$,

Prf.:

Items 1. and 2. are obvious, item 3 is also obvious (but needs AC).

# prf cont'd

Item 4.:

if $\varphi(\overline{x})$ is and $L_i$-formula of the form

$$Q_1 y_1 \dots Q_k y_k \ \psi(\overline{x}, \overline{y})$$

with $Q_i$ either $\exists$ or $\forall$ quantifiers and $\psi$ open:

- use a Skolem function in $L_{i+1}$ and a Skolem axiom in $T_{i+1}$ to write $Q_k y_k \ \psi(\overline{x}, \overline{y})$ as an equivalent open $L_{i+1}$-formula,
- this reduces the nb. of quantifiers in $\varphi$ by 1 at the expenses of rewriting the open kernel as an $L_{i+1}$-fla,
- repeat $k$-times.

# end of the construction

The lemma provides the first four of the six properties of $L_{Sk}$ and $Sk$ we needed:

- $|L_{Sk}| \leq \max(\aleph_0, |L|)$,
- $Sk$ is universal,
- $\mathbf{B}$ can be expanded to an $L_{Sk}$-structure $\mathbf{B}' \models \mathrm{Sk}$,
- any $L_{Sk}$-fla $\varphi$ is equivalent to an open $L_{Sk}$-fla $\varphi'$, provably in theory $Sk$,
- $\mathbf{D}$ is an $L_{Sk}$-substructure of $\mathbf{B}'$,
- $U \subseteq D$ and $|D| \leq \max(\aleph_0, |L|, |U|)$.

To get the last two properties define subseteq $D \subseteq B$ by:

$$D \;:=\; \text{all elements of } B \text{ that are generated from } U \text{ by } L_{Sk}\text{-terms} \quad .$$

$\square_{L-S\ down}$

# HW problem

The following take-away problem is often called the Skolem paradox:

Take set theory ZFC. Assume that it is satisfiable and argue first precisely that it has in infinite model.

Then it follows by the L-S theorem that its has also countable model.

How do you reconcile this with the fact that ZFC proves the existence of an uncountable set?