Lecture 1

uncountable categoricity, complete theories

Topics

- some remarks on AD
- uncountable categoricity
- ex's of uncountably categorical theories:
  $Th(\mathbf{Z}, suc)$, $Vect_{\mathbf{Q}}$, ACF_p
- complete theories and Vaught's test (we shall prove it in Lect.5)

AD - ax. of determinacy

a type of topological games:

moves: $a_1, b_1, \dots \in N$ (natural numbers)

a play: infinite sequence of natural numbers

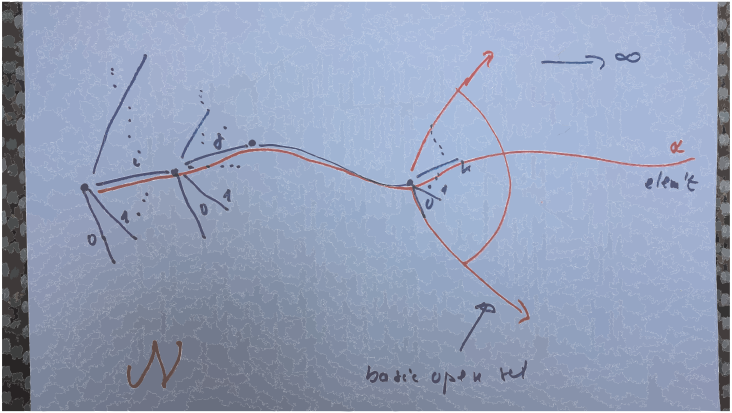
$$(a_1, b_1, \dots, a_i, b_i, \dots) \in \mathcal{N}$$

where $\mathcal{N}$ is the topological Baire space (set th. terminology)
Other notation: $N^N = N^\omega = \omega^\omega$

Fact: $\mathcal{N}$ is homeomorphic to $R \setminus Q$ (the set if irrational numbers)

Warning: $\mathcal{N}$ is the arena for descriptive set theory and they often talk about $\mathcal{N}$ as about "reals".

picture of $\mathcal{N}$



games

a game is defined by any subset $W \subseteq \mathcal{N}$:

$$\text{player I wins play } \alpha = (a_1, b_1, \dots) \text{ iff } \alpha \in W$$

Ehrenfeucht-Fraisse:
for countable universes A, B we can take w.l.o.g. $A = B = N$ and define W to be the set of all plays

$$(a_1, b_1, \dots, a_i, b_i, \dots)$$

such that

$$\{(a_i, b_i) \mid i \geq 1\}$$

is not a partial iso.

AD - formulation

AD (Mycielski-Steinhaus '62

Every game is determined, i.e. one of the players has a winning strategy.

Known facts:

YES for Borel sets W (D.Martin), and some more set theory ...
NO in general: AD contradicts AC

E.x:
AD $\Rightarrow$ all sets of reals are Lebesgue measurable
AC $\Rightarrow$ not all sets ...

many variants in between: take both AD and AC in some restricted forms only

HW problem

a model for RG

Marker (pp.50-51): a generic construction by an infinite process

A specific definition:
universe: N (natural numbers)
edges:

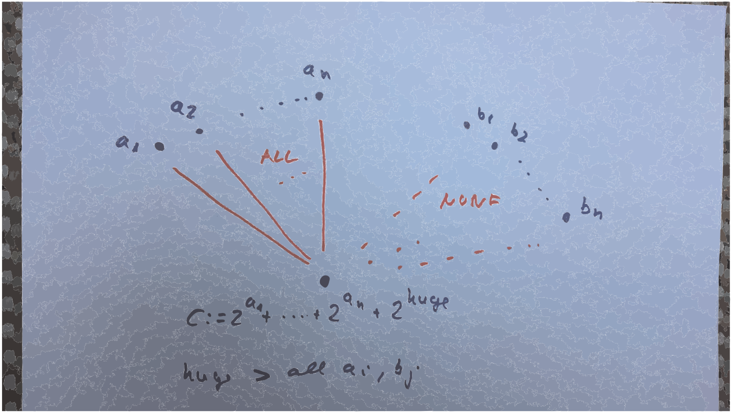
- first define R_0: $R_0(a, b)$ iff

  $$2^a \text{ occurs in the unique expression of } b \text{ as a sum of powers of } 2 \ .$$

- edge relation R: symmetrization of R_0

  $$R(x, y) \ \text{ iff } \ (R_0(x, y) \vee R_0(y, x)) \ .$$

RG model



uncountable categoricity

We consider primarily theories T in a countable language L - this allows for simpler formulations of statements and covers the cases we shall be interested in.

uncountable categoricity: T has unique model (up to iso) in every uncountable power

By Morley's thm we stated earlier this is equal to having a unique model in some uncountable power, so it suffice to think about models having the cardinality of continuum.

successor function

(Z, suc): $suc(x) := x + 1$.

$Th(Z, suc)$

This theory contains as axioms universal closures of the following formulas:

- suc is a bijection:

$$(x \neq y \rightarrow sux(y) \neq suc(y)) \ \wedge \ (\exists z \ suc(z) = x)$$

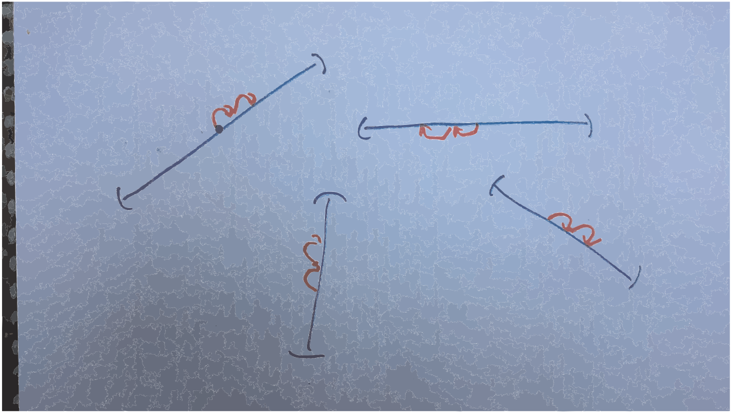
- no finite cycles: for each $k \geq 1$:

$$suc(suc(\ldots(x)\ldots) \neq x$$

  where suc occurs k-times.

Call this theory SUC.

models of SUC



SUC

Theorem

SUC is uncountably categorical.

Prf.: continuum size $\Leftrightarrow$ continuum many blocks

So any two models of this size are isomorphic: put the blocks into a bijection.

$\square$

We will note in Lect.5 - as a corollary to the construction behind Vaught's test - the following statement.

Corollary

SUC axiomatizes $Th(Z, suc)$.

vector spaces

theory $Vect_Q$ of vector spaces of $\mathbf{Q}$

language:

- constant 0 (for the zero vector),
- binary f.symbol $+$ (for the vector addition),
- infinitely many unary f.symbols λ_q, one for each $q \in Q$.

Intended meaning of λ_q: scalar multiplication by q

$$\lambda_q \ : \ x \rightarrow q \cdot x \ .$$

This choice of language is because we do not want to have scalars (i.e. rationals) as elements of our structures and be able to quantify over them - we want to subject to FO logic (and to quantification) only vectors.

$Vect_Q$

axioms:

- axioms forcing that $0, +$ define a commutative group,
- axioms about scalar multiplication, universal closures of formulas:
  - $\lambda_0(x) = 0$ and $\lambda_1(x) = x$,
  - $\lambda_q(x) + \lambda_r(x) = \lambda_{q+r}(x)$,
  - $\lambda_q(x + y) = \lambda_q(x) + \lambda_q(y)$,
  - $\lambda_q(\lambda_r(x)) = \lambda_{q \cdot r}(x)$

Lemma

Models of $Vect_Q$ are exactly vector spaces over $\mathbf{Q}$.

categoricity

Theorem

$Vect_Q$ is uncountably categorical but not countably categorical.

Prf.:
The iso type of a vector space V is determined by its dimension. If the dimension is κ (possibly infinite cardinality), i.e. it has a basis B of size κ, then vectors V are of the form

$$q_1 v_1 + \cdots + q_n v_n$$

with $q_i \in Q$ and $v_i \in B$, and there are

$$\max \aleph_0, \kappa$$

such choices (see next slide). Hence if V us uncountable, it must be that $\kappa = |V|$ and hence all spaces of that cardinality have the same dimension, i.e. are iso.
In the countable case there are more options for dim: $1, 2, \ldots$ or $\aleph_0$.

counting

recall that for infinite cardinalities λ, η it holds:

$$\lambda + \eta = \lambda \cdot \eta = \max \lambda, \eta$$

number of choices qv, with $q \in Q$ and $v \in B$:

$$\aleph_0 \cdot \kappa = \kappa$$

number of choice of n-tuples of such qv:

$$\kappa \cdot \dots \cdot \kappa \ \ (n\text{-times}) \ \ = \kappa$$

sum of these options for all $n \geq 1$:

$$\aleph_o \cdot \kappa = \kappa$$

fields

theory Fields:

language: $0, 1, +, \cdot$ (sometimes also binary $-$ is included)

axioms: universal closures of

- 0 and $=$ form a commutative group:

$$x + 0 = x\ ,\ x + y = y + x\ ,\ x + (y + z) = (x + y) + z\ ,\ \exists y (x + y = 0)$$

- 1 and $\cdot$ form a commutative group on non-zero elements:

$$x \neq 0 \rightarrow x \cdot 1 = x\ ,\ \ldots, x \neq 0 \rightarrow \exists y (x \cdot y = 1)$$

- distributivity:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

algebraic closure

Definition

A field K (i.e $K \models \mathit{Fields}$) is algebraically closed iff all non-constant polynomials $f(x) \in K[x]$ over K have roots in K.

axioms:

$$\forall x_0, \ldots, x_n \exists y \ (x_n \neq 0 \to \sum_{i \leq n} x_i y^i = 0) \ .$$

where y^i abbreviates the term $y \cdot \cdots \cdot y$ (i-times).

theory ACF: Fields + these axioms for all $n \geq 1$.

Key example: the complex field $\mathbf{C}$

alg.closure

Algebraic fact:

For every field K there exists the smallest algebraically closed field containing K: the algebraic closure K^{alg} of K.
It is countable if K is finite and has the cardinality of K if K is infinite.

Ex's:

$\mathbf{R}^{alg} = \mathbf{C}$

$\mathbf{Q}^{alg} \neq \mathbf{C}$

$\mathbf{F}_p^{alg}$, where $\mathbf{F}_p$ is the finite field of counting modulo a prime p, is a countable field

alg. independence

Definition

Elements $a_1, \ldots, a_n \in K$ are algebraically independent iff the only polynomial $f(x_1, \ dots, x_n) \in \mathbf{Z}[\overline{x}]$ for which

$$f(a_1, \ldots, a_n) = 0$$

is the zero polynomial. (A special case of a more general definition.)

Informally: there is no non-trivial algebraic relation among the elements.

This is analogous to the linear independence in vector spaces. And similarly to that situation we have

Definition

$B \subseteq K$ is a transcendence basis iff B is the maximal subset w.r.t $\subseteq$ such that all n-tuples of its elements are algebraically independent.

characteristic

Algebraic fact

The cardinality of all bases of transcendence is the same, the transcendence degree of K.

In vector space the cardinality of a basis determines the space. Here we need additional info:

Definition

The characteristic of K is prime p, *char* $K = p$, iff
$1 + \cdots + 1 \ \ (p\text{-times}) \ = 0$.
It is 0, *char* $K = 0$, iff it is not p for any prime p.

Algebraic fact

The characteristic and the transcendence degree determine K up to iso.

ACF_p

theory ACF_p for p a prime or $p = 0$: ACF plus

- axiom $1 + \cdots + 1 \ \ (p\text{-times}) \ \ = 0$, if p is a prime
- axioms $1 + \cdots + 1 \ \ (q\text{-times}) \ \ \neq 0$ for all primes q, if $p = 0$

Ex's:

$$\mathbf{C} \models ACF_0 \ \text{ and } \ \mathbf{F}_p^{alg} \models ACF_p \ .$$

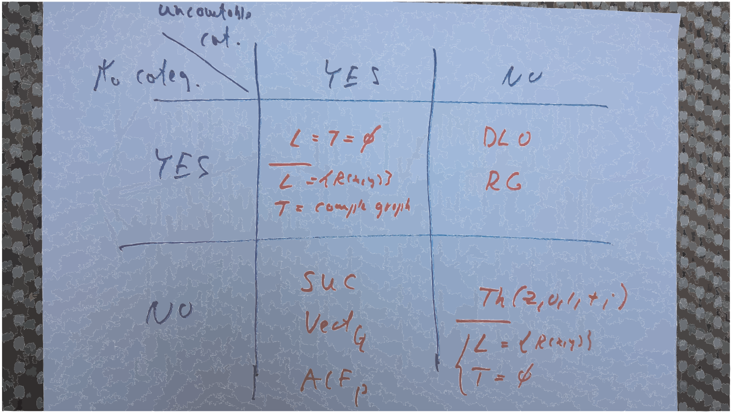
Theorem

Theory ACF_p, p a prime or 0, is uncountably categorical.

Prf.:
Entirely analogous to the case of $Vect_Q$, using the transcendence degree and the characteristic.

□

summary


uncountable cat.
ℵ0 categ.
YES
NO
YES
L = T = ∅
L = {R(x,y)}
T = complete graph
DLO
RG
NO
SUC
Vect_Q
ACF_p
Th(Z,0,1,+,·)
L = {R(x,y)}
T = ∅

So what?

What does follow about T if we know that it is categorical in some power?

A deeper/intrinsic consequence:
it betrays the existence of some invariants that classify structures.
$\Rightarrow$ modern model theory

A more direct consequence: completeness of the theory (under additional cond.'s)

completeness

Definition

An L-theory T is complete iff for all L-sentences φ:

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi\ .$$

Informally: axioms in T already "logically" decide the truth value of all FO statements.

bad news: ZFC is not complete

good news: many theories defining familiar classes of structures are

Vaught's test

Vaught's test

Let T be a satisfiable theory in a countable language that has no finite models.
If T is categorical in some (infinite) power then it is complete.

Proof next time.

Corollary

All theories DLO, RG, SUC, $Vect_Q$ and ACF_p (any p) are complete.

HW problem

A problem to take away:

Take, for example, theory ACF_0 and using the fact that it is complete devise an algorithm that upon receiving φ as input decides if $ACF_0 \models \varphi$ or $ACF_0 \models \neg\varphi$.

Note that because $\mathbf{C} \models ACF_0$, the same algorithm decides what is true or false in the complex field.