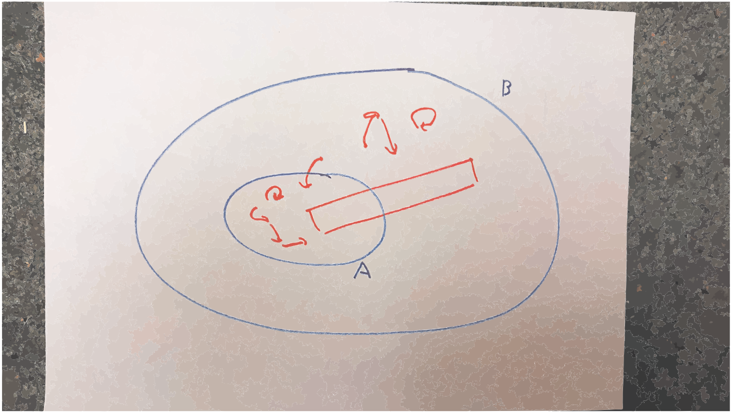
Lecture 1

relations between structures, applications of compactness

topics

- substructures, preservation thms
- elementary substructure
- embedding and isomorphism
- elementary equivalence
- non-standard models of theories of $\mathbf{N}$ and $\mathbf{R}$
- The Löwenheim-Skolem theorem up
- categoricity

substructures



substructures

Definition: substructures

$\mathbf{A} \subseteq \mathbf{B}$ ($\mathbf{A}$ is a substructure of $\mathbf{B}$) iff

- $A \subseteq B$,
- $R^{\mathbf{A}}$ is $R^{\mathbf{B}}$ restricted to A,
- $f^{\mathbf{A}}$ is $f^{\mathbf{B}}$ restricted to A and A is closed under $f^{\mathbf{B}}$.

Ex. $\mathbf{Q} = (Q, 0, 1, +, \cdot, <) \subseteq \mathbf{R} = (R, 0, 1, +, \cdot, <)$.

Ex. $([0,1], 0, 1, +, \cdot, <) \not\subseteq \mathbf{R} = (R, 0, 1, +, \cdot, <)$.

absoluteness of open flas

Lemma

Assume $\mathbf{A} \subseteq \mathbf{B}$. Let $\psi(\overline{z})$ be an open ($=$ quantifier-free) formula, $\overline{a} \in A^n$. Then:

$$\mathbf{A} \models \psi(\overline{a}) \quad \text{iff} \quad \mathbf{B} \models \psi(\overline{a}) \; .$$

Prf.:
For atomic flas this is from the definition and for their propositional combinations it follows from Tarski's definition of $\models$.

$\square$

existential preservation up

Lemma

Assume $\mathbf{A} \subseteq \mathbf{B}$. Let $\psi(\overline{x}, y)$ be an open formula and $\overline{a} \in A^n$. Then:

$$\mathbf{A} \models \exists y \psi(\overline{a}, y) \implies \mathbf{B} \models \exists y \psi(\overline{a}, y) \ .$$

Prf.:
$\mathbf{A} \models \exists y \psi(\overline{a}, y)$
implies
$\mathbf{A} \models \psi(\overline{a}, a')$ for some $a' \in A$
implies by the previous lemma
$\mathbf{B} \models \psi(\overline{a}, a')$ for the same $a' \in A \subseteq B$
implies
$\mathbf{B} \models \exists y \psi(\overline{a}, y)$.

□

universal preservation down

The lemma cannot be literally reversed:

$\mathbf{R} \models \exists y(y \cdot y = 1 + 1)$ but $\sqrt{2}$ does not exist in $\mathbf{Q}$.

But it can be reversed if $\exists$ is changed into $\forall$:

Lemma

Assume $\mathbf{A} \subseteq \mathbf{B}$. Let $\psi(\overline{x}, y)$ be an open formula and $\overline{a} \in A^n$. Then:

$$\mathbf{B} \models \forall y \psi(\overline{a}, y) \implies \mathbf{A} \models \forall y \psi(\overline{a}, y) \ .$$

Ex.
$\mathbf{R} \models \forall y(y \cdot y + 1 \neq 0)$ and indeed $\sqrt{-1}$ does not exist in $\mathbf{Q}$ either.

elementary substructures

When all flas are preserved we have a stronger notion:

Definition - elem.substructure

$\mathbf{A} \preceq \mathbf{B}$ ($\mathbf{A}$ is elementary substructure of $\mathbf{B}$) iff
for all formulas $\varphi(\overline{x})$ and all $\overline{a} \in A^n$:

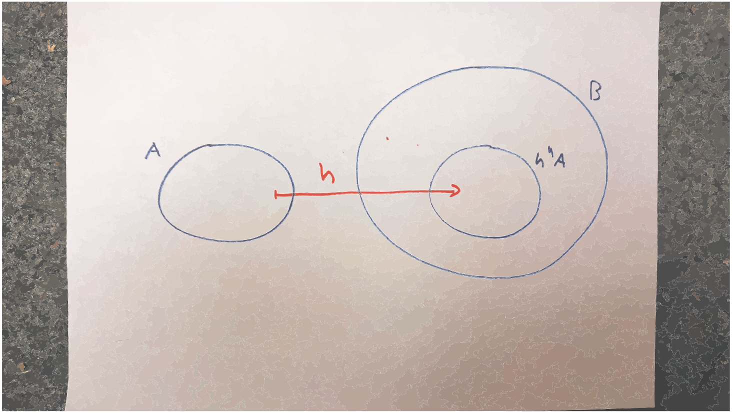
$$\mathbf{A} \models \varphi(\overline{a}) \ \text{ iff } \ \mathbf{B} \models \varphi(\overline{a}) \ .$$

Ex. $\mathbf{Q}$ is not elem.substructure of $\mathbf{R}$ but

$$(Q, <) \preceq (R, <) \ .$$

This needs a proof and we shall prove this later.

embedding



embedding and isomorphism

The following notion generalizes the notion of a substructure to the case when A is not literally a subset of B.

Definition - embedding

Map $h : A \to B$ is embedding of $\mathbf{A}$ into $\mathbf{B}$ iff

- h is 1-to-1,
- $\overline{a} \in R^{\mathbf{A}} \Leftrightarrow h(\overline{a}) \in R^{\mathbf{B}}$,
- $h(f^{\mathbf{A}}(\overline{a})) = f^{\mathbf{B}}(h(\overline{a}))$.

That is, for all open flas $\psi(\overline{x})$:

$$\mathbf{A} \models \psi(\overline{a}) \text{ iff } \mathbf{B} \models \psi(h(\overline{a})) .$$

$h(\overline{a}) := (h(a_1, \dots, h(a_n))$.

Definition - isomorphism

Isomorphism $=$ embedding $+$ onto. Notation: $\mathbf{A} \cong \mathbf{B}$.

isomorphism and elem.equivalence

Isomorphic structures are often just identified. In fact:

Lemma

Assume $\mathbf{A} \cong \mathbf{B}$ via map h. Let $\varphi(\overline{x})$ be any fla and $\overline{a} \in A^n$. Then:

$$\mathbf{A} \models \varphi(\overline{a}) \text{ iff } \mathbf{B} \models \varphi(h(\overline{a})) \ .$$

Prf.:

By ind. on the complexity of φ. The key step is: $\mathbf{B} \models \exists y \psi(h(\overline{a}), y)$ implies

$$\mathbf{B} \models \psi(h(\overline{a}), b) \ , \quad \text{for some } b \in B \quad .$$

But any b is in the range of h, so $b = h(a')$ and we have:

$$\mathbf{B} \models \psi(h(\overline{a}), h(a')) \ .$$

By ind. hypothesis $\mathbf{A} \models \psi(\overline{a}, a')$ and $\mathbf{A} \models \exists y \psi(\overline{a}, y)$ follows.

□

theory of a structure

Corollary

Assume $\mathbf{A} \cong \mathbf{B}$ via map h. Then for all sentences θ:

$$\mathbf{A} \models \theta \quad \text{iff} \quad \mathbf{B} \models \theta \ .$$

This statement can be elegantly phrased using the following notions

Definition: elem. equivalence and theory of a structure

Theory of $\mathbf{A}$: $Th(\mathbf{A}) \ := \ $ the set of all sentences true in $\mathbf{A}$.
Two structures $\mathbf{A}, \mathbf{b}$ (in a common lang.) are elementarily equivalent, $\mathbf{A} \equiv \mathbf{B}$, iff

$$Th(\mathbf{A}) = Th(\mathbf{B}) \ .$$

$$\mathbf{A} \cong \mathbf{B} \ \Rightarrow \ \mathbf{A} \equiv \mathbf{B} \ .$$

a question

What about $\mathbf{A} \equiv \mathbf{B} \;\Rightarrow\; \mathbf{A} \cong \mathbf{B}$?

Our first applications of the compactness will be several counter-examples to this implication.

A problem to take away: Show that this is true whenever $\mathbf{A}$ is a finite structure in a finite language.

Set up:

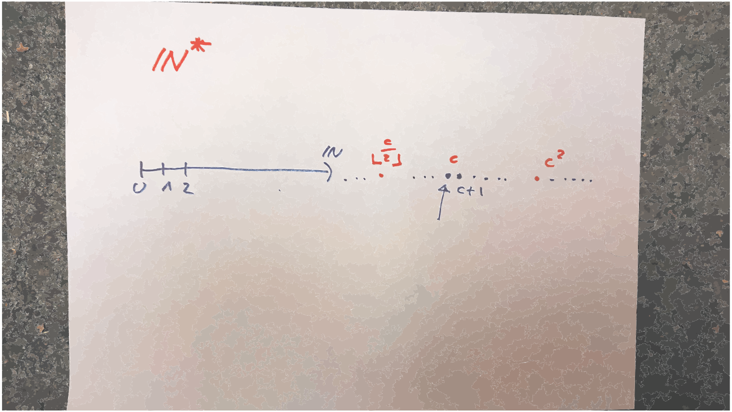
- L: $0, 1, +, \cdot, <$
- $\mathbf{N} := (N, 0, 1, +, \cdot, <)$
- c: a new constant
- theory $T \;:=\; Th(\mathbf{N}) \;\cup\; \{c > 1 + \cdots + 1 \;\; (n \text{ times}) \;\mid n \geq 1\}$.

non-standard integers

The compactness implies:

Lemma

T is satisfiable.



infinitesimal reals

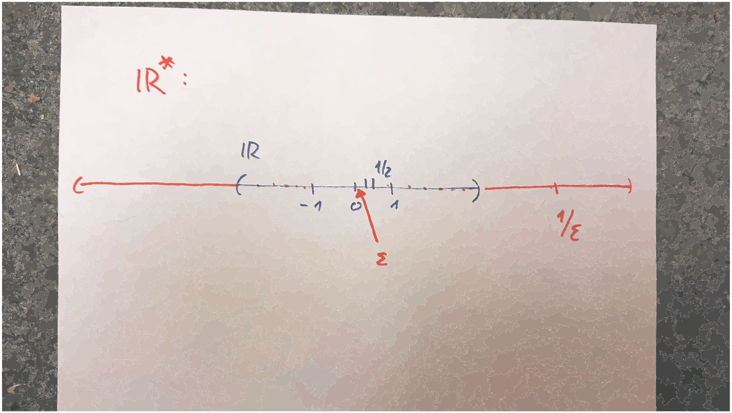
A bit harder example. Take the same L and $\mathbf{R}$ and define:

- ϵ: a new constant,
- new constants c_r, one for each real $r \in R$,
  L_R is L plus all these constants c_r,
- $\mathbf{R}'$: an expansion of $\mathbf{R}$ by interpreting each constant c_r by r,
- $Th_R(\mathbf{R})$: L_R sentences true in $\mathbf{R}'$,
- $T := Th_R(\mathbf{R}) \cup \{0 < \epsilon\} \cup \{1 > epsilon + \ldots \epsilon \text{ (n times)} \mid n \geq 1\}$.

In $\mathbf{N}$ we could use numerals $1 + \cdots + 1$ to name each element of the universe. In $\mathbf{R}$ this is impossible and the role of the new constants c_r is to name all reals. E.g. statement $\pi^2 < 20$ is represented by $c_\pi \cdot c_\pi < c_{20}$.

Lemma

T is satisfiable.


IR*:
IR
1/2
-1
0
1
1/ε
ε

going up

L: any
A: any infinite
L_A: L with names c_u for all $u \in A$ (as before)
$Th_A(\mathbf{A})$: as before
D: an arbitrary set of new constants
$T \ := \ Th_A(\mathbf{A}) \cup \{d \neq d' \mid \text{ and two different } \ d, d' \in D\}$

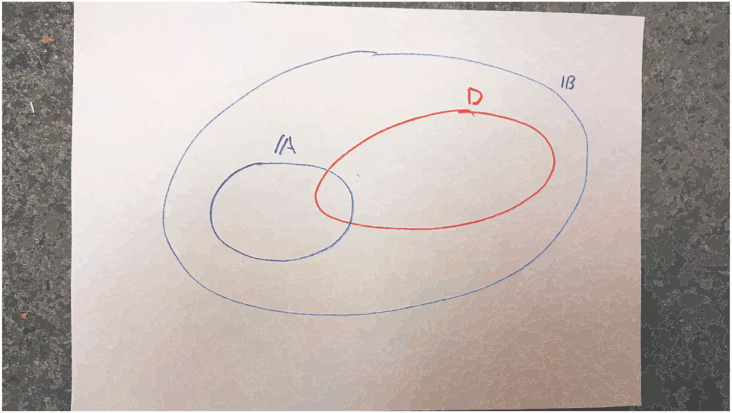
Lemma

T is satisfiable.

Prf.:
Any finite number of constants from D can be interpreted in **A** by different elements because it is infinite.

□

huge model



Löwenheim-Skolem up

The Löwenheim-Skolem theorem upwards

Let $\mathbf{A}$ be an infinite structure in language L and let κ be an arbitrary cardinality. Then there is $\mathbf{B}$ such that:

$$\mathbf{A} \preceq \mathbf{B} \quad \text{and} \quad |B| \geq \max \kappa \ .$$

Informally: cardinalities of elem. extensions of an infinite structure are unbounded.

Prf.:
Take D of cardinality κ and any model $\mathbf{B}$ of T from previous slide.

□

Note that we do not know that the model has cardinality exactly κ.

categoricity

It follows that the theory of no infinite **A** can determine **A** up to isomorphism. The next best thing we can hope for is that

- *Th*(**A**) determines all its models in some particular cardinality (i.e. the theory plus the cardinality determines the structure up to iso).

Definition - categoricity

Let κ be any infinite cardinality and let T be a theory with a model of cardinality κ.
Then T is κ-categorical iff T has a unique model in cardinality κ up to isomorphism.

Morley's thm

This looks like a chaotic situation where many combinations can occur. But fortunately the picture is much simpler for countable T.

Morley's theorem

Let L and T be countable. If T is κ-categorical for some uncountable κ then it is categorical all uncountable cardinalities.

Hence for countable L, T there are only four options, all combinations of:

- T is/is not countably categorical,
- T is/is not uncountably categorical.

We shall not prove Morley's thm but we shall see examples of theories of all four categories.

Vaught's conjecture

Assume L, T are countable, T complete with infinite models. Define:

$$I(T,\kappa) \;:=\; \text{the number of cardinality } \kappa \text{ models of } T \text{ up to iso} \quad .$$

What are possible values of $I(T,\aleph_0)$?

Finite case: any $n \geq 1$ can appear except 2!
Infinite case: easy examples with $I(T,\aleph_0) = \aleph_0$ and $I(T,\aleph_0) = 2^{\aleph_0}$.

Vaught's conjecture

No other infinite cardinality is possible.

Informally: Continuum Hypothesis holds as long as you look at structures rather than sets.
The only known general result is:

$$I(T,\aleph_0) > \aleph_1 \;\rightarrow I(T,\aleph_0) = 2^{\aleph_0} \; .$$