Lecture 11

types cont'd

topics

- HW - Prop.4.3.2 (p.138)
- the existence of saturated structures
- $\aleph_1$-saturation via ultraproduct
- isolated types
- the Omitting types thm
- Peano arithmetic PA
- the MacDowell-Specker thm:
  - countable case via omitting types,
  - general case via definable ultrapower.

HW

The task: show that any structure that realizes all 1-types over less than κ parameters is κ-saturated.
Need to show that all n-types over less than κ parameters are realized in $\mathbf{M}$. Prf by induction on n:
Case n=1: this is the hypothesis
Induction step $n \to n = 1$:
Let $p(\overline{x}, y)$ be an $(n+1)$-type over A, $|A| < \kappa$. Define an n-type

$$p'(\overline{x}) \ := \ \{\varphi(\overline{x}) \mid \varphi \in p\} \ .$$

By induction hypothesis p' is realized by some n-tuple $\overline{b} \in M^n$.
Now define a 1-type $q(y)$ over $A' := A \cup \{b_1, \ldots, b_n\}$:

$$\{\psi(\overline{b}, y) \mid \psi \in p\} \ .$$

As still $|A'| < \kappa$, it is realized (by the original hypothesis) by some $c \in M$ and it is easy to check that

$$(\overline{b}, c) \ \text{ realizes type } \ p \ .$$

existence

L: countable

T: complete L-theory with infinite models

Theorem

For all κ, T has an infinite κ^+-saturated model of cardinality at most 2^κ.

Corollaries

- If CH (the continuum hypothesis) holds then there is a saturated model of cardinality $\aleph_1$.
- If GCH (the generalized CH holds, i.e. $\kappa^+ = 2^\kappa$) the then are saturated models of all uncountable successor cardinalities (i.e. of the form κ^+).

We shall prove the thm (and hence the first corollary) for $\kappa = \aleph_0$.

ultraproduct

We shall prove the following statement.

Theorem

Let $\mathbf{M}_i$, $i \in \omega$, be any L-structures and let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then

$$\mathbf{M}^* := \prod_i \mathbf{M}_i/\mathcal{U}$$

is $\aleph_1$-saturated.

To see that this implies the previous thm for $\kappa = \aleph_0$ note:

- $\aleph_0^+ = \aleph_1$,
- $|M^*| \leq \prod_i |M_i|$ which is $\leq \aleph_0^{\aleph_0} = 2^{\aleph_0}$ for countable models $\mathbf{M}_i$,
- and $\mathbf{M}^* \models T$ if all $\mathbf{M}_i \models T$.

prf

Prf.:
Let $A \subseteq M^*$ be a countable set of parameters $[\alpha_j]$, and let

$$p \;:=\; \{\varphi_i(x) \mid i \geq 0\}$$

be any 1-type over A (by the HW it suffices to consider 1-types).

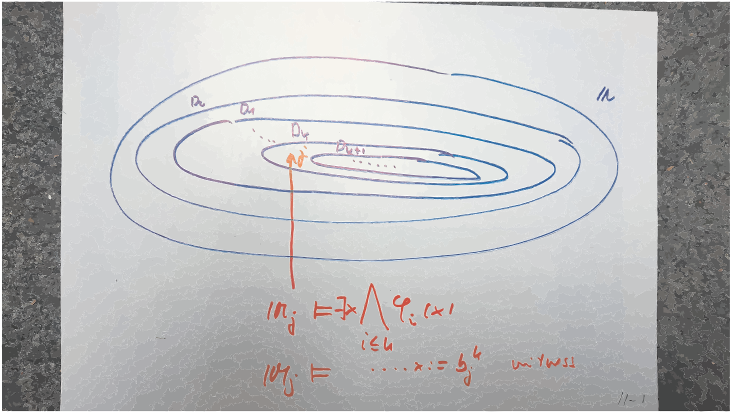
Because p is finitely satisfiable, for all $k \geq 0$ the set

$$D_k \;:=\; \langle\!\langle \exists x \bigwedge_{i \leq k} \varphi_i(x) \rangle\!\rangle$$

is in $\mathcal{U}$. Clearly these set form a descending chain:

$$D_0 \supseteq D_1 \supseteq \dots \; .$$

prf-pic



prf cont'd

Define a function $\gamma \in \prod_i M_i$ by:

$$\gamma(j) \;:=\; \text{any witness to } \exists_{i \leq k} \varphi_i(x) \text{ in } \mathbf{M}_j \text{ if } j \in D_k \setminus D_{k+1} \;.$$

In words: $\gamma(j)$ witnesses as long initial sequence of formulas $\varphi_0(x), \ldots, \varphi_k(s)$ as possible.

For all $i \geq 0$ we have

$$\langle\langle \varphi_i(\gamma) \rangle\rangle \supseteq D_i \in \mathcal{U}$$

and hence

$$\langle\langle \varphi_i(\gamma) \rangle\rangle \in \mathcal{U} \;.$$

By Loš's thm then

$$\mathbf{M}^* \;\models\; \varphi_i([\gamma]) \;, \quad \text{all } i \geq 0 \;.$$

□

countable case

Define the Stone space w.r.t. theory T:

$$S_n(T) \ := \ \text{all complete } n\text{-types consistent with } T \ .$$

It is the same as putting

$$S_n(T) \ := \ S_n^{\mathbf{M}}(\emptyset)$$

for any model **M** of T.

Theorem

T has a countable saturated model iff all $S_n(T)$ are countable, $n \geq 1$.

The only-if direction is immediate, the if-direction is proved by a variant of the Henkin construction used to prove the completeness thm.

isolated types

Definition

A type $p \in S_n(T)$ is isolated ($=$ principal) iff there is a fla $\varphi(\overline{x}) \in p$ such that

$$T \vdash \varphi(\overline{x}) \rightarrow \psi(\overline{x}) \text{ , for all } \psi \in p \text{ .}$$

That is: $\{p\} = [p]$ in the topology of $S_n(T)$.

Lemma

If p is isolated then it is realized in all models of T.

Prf.:

Assume p is isolated by $\varphi(\overline{x}) \in p$. As T is complete $T \vdash \exists \overline{x} \varphi(\overline{x})$ and hence

$$T \vdash \exists \overline{x} \psi(\overline{x}) \text{ , for all } \psi \in p \text{ .}$$

□

Henkin-Orey

The next statement says that being isolated is the only obstruction to omitting a type.

The omitting types theorem (Henkin-Orey)

Let L be countable, T complete and let p_i, $i \geq 0$ be a countable set of non-isolated types.
Then there is a model of T that omits all p_i, $i \geq 0$.

The theorem is proved by a variant of the Henkin construction used usually when proving the Completeness theorem.

PA

Peano arithmetic: an important theory when studying the foundations of mathematics

language L_{PA}: $0, 1, +, \cdot, <$

axioms:

a finite set of axioms called often Robinson's arithmetic Q:

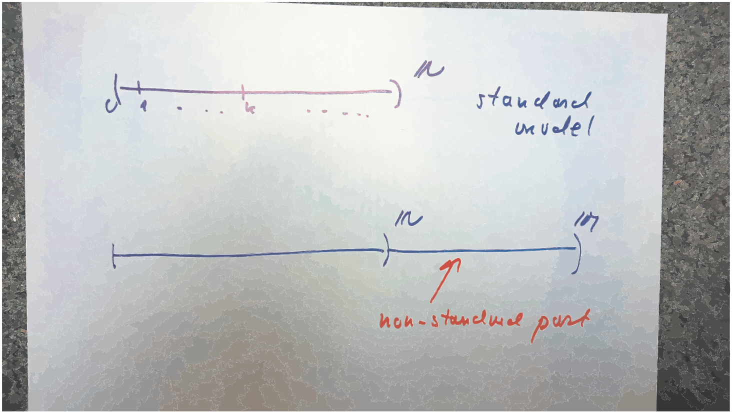
- $x + 0 = x$
- $x + (y + 1) = (x + y) + 1$
- $x \cdot 0 = 0,$
- $x \cdot (y + 1) = (x \cdot y) + x,$
- $x + 1 \neq 0,$
- $x + 1 = y + 1 \rightarrow x = y,$
- the axioms of discrete linear orders for $<$ with $x + 1$ being the successor of x,
- $(x = y \vee x < y) \equiv (\exists z\ x + z = y),$

IND

and by infinitely many instances of the induction scheme IND:

$$[\ \varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x+1))\ ] \ \rightarrow\ \forall x \varphi(x)$$

for all formulas φ that may contain other free variables than x.



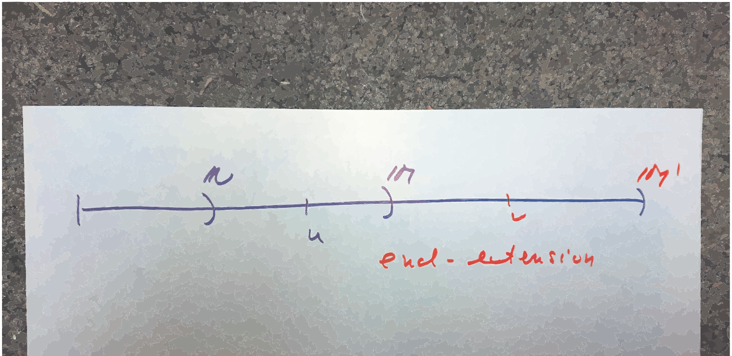
end-extensions

Definition

Let $\mathbf{M} \subseteq \mathbf{M}'$ be two models of PA. Then $\mathbf{M}'$ is a end-extension of $\mathbf{M}$, denoted by $\mathbf{M} \subseteq_e \mathbf{M}'$, iff

$$\forall v \in M' \setminus M \forall u \in M \; \mathbf{M}' \models u < v \; .$$

In words: all elements not in M are at the end.



MacDowell-Specker

Theorem (MacDowell-Specker)

All models of PA have proper end-extensions.

We shall first outline a proof of
- the countable case via the omitting types thm

and then give a proof of
- the general case using definable ultrapowers.

countable case

Prf. outline - countable case:

Let **M** be a countable model of PA. Consider the following theory T:
language: L_{PA} plus names for all elements of M and a new constant c
axioms: axioms of PA with IND in the extended language, and new axioms

$$c > m \ , \quad \text{for all } m \in M \ .$$

Any model of T properly extends M but to arrange that it is an end-extension we need to omit all - countably many - types:

$$p_u \ := \ \{x < u\} \cup \{x \neq m \mid m \in M, \mathbf{M} \models m < u\} \ .$$

The heart of the proof is to show that all these types are non-isolated (this uses some facts about PA).

general case

Let $\mathbf{M} \models PA$. We shall construct its proper end-extension $\mathbf{M}'$ by definable ultrapower, a variant of the earlier ultrapower construction.

index set: $I := M$ (i.e. the model itself)
individual structures: $\mathbf{M}_i := \mathbf{M}$, all $i \in I$
The change in the construction is in how we construct the universe M' of the new structure: we do not start with the set $\prod_i M_i$ of all functions

$$\alpha : I (= M) \to M$$

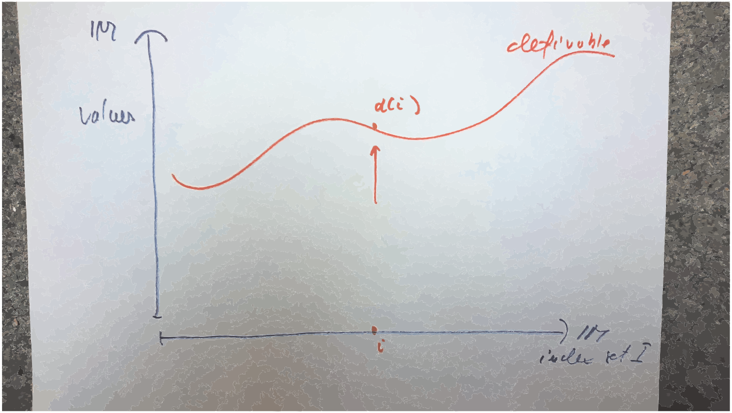
but with the set of definable functions:

$$DefFuc(\mathbf{M}) = \text{ all } \alpha \text{ that are definable in } \mathbf{M}$$

i.e. the graph of α is definable by a fla ψ with parameters from M:

$$\alpha(u) = v \Leftrightarrow_{df} \mathbf{M} \models \psi(u, v, \overline{m}) .$$

universe - pic



universe

Lemma

$\mathit{DefFuc}(\mathbf{M})$ is closed under $+$ and $\cdot$, and for each $m \in M$ it contains function λ_m that is constantly equal to m.

Next we replace the Boolean algebra $\mathcal{P}(M)$ by the algebra of definable subsets:

$$\mathit{Def}(\mathbf{M}) \;:=\; \text{all definable subsets of } M \;.$$

Lemma

$\mathit{Def}(\mathbf{M})$ is a Boolean algebra and it contain all finite and cofinite subsets of M.

ultrafilter

It remains to choose a suitable ultrafilter $\mathcal{U}$ on the Boolean algebra. In earlier constructions it sufficed to take any non-principal $\mathcal{U}$. Here we need a more specific choice.

Definition

$\mathcal{U}$ is **M**-closed iff for all $\alpha \in \mathit{DefFuc}(\mathbf{M})$ and all $m \in M$, if

$$\alpha : M \to [0, m]$$

then for some $u \leq m$, $\alpha^{(-1))}(u) \in \mathcal{U}$.

In words, if M is partitioned definably into m pieces then $\mathcal{U}$ contains at least one: this generalizes the property that $\mathcal{U}$ must contain a set or its complement (that is the case $m = 2$).

ultrafilter

Lemma

A non-principal and **M**-closed $\mathcal{U}$ exists.

This is not proved via Zorn's lemma but by defining $\mathcal{U}$ in **M**. This step uses that we talk about models of PA: PA is strong enough to show that if

$$\alpha : M \rightarrow [0, m]$$

then at least one of the preimages $\alpha^{(-1))}(u)$, $u \leq m$ must be "large". This is a form of pigeon-hole principle.

the structure

Now we are ready to complete the definition of the definable ultrapower $\mathbf{M}'$ (this goes back to Skolem).

universe M':
Take a non-principal $\mathbf{M}$-closed ultrafilter $\mathcal{U}$ and put

$$M' \ := \ \mathit{DefFuc}(\mathbf{M})/\mathcal{U} \ .$$

That is, we identify $\alpha, \beta \in \mathit{DefFuc}(\mathbf{M})$ iff

$$\langle\langle \alpha - \beta \rangle\rangle \in \mathcal{U} \ .$$

Łoš's thm

Łoš's thm goes through in this set-up: the treatment of ax's of equality and of propositional connectives uses just properties of Boolean algebras and ultrafilters. The only non-trivial thing to check are the quantifiers.

Lemma

For any fla $\exists x \psi(x)$ (with parameters from M'):

$$\mathbf{M}' \models \exists x \psi(x) \text{ iff } \langle\langle \exists x \psi(x) \rangle\rangle \in \mathcal{U} \ .$$

Prf.:
The only-if direction is trivi. For the if-direction define $\gamma \in \mathit{DefFuc}(\mathbf{M})$ by:

$$\gamma(i) \ := \ \min\{u \mid \psi(u)\}, \text{ if it exists, and } := 0 \text{ otherwise} \ .$$

This uses IND: it implies the least number principle and hence min u exists and so γ is definable.

□

prf - thm

To conclude the proof of the MacDowell-Specker thm note first that

- **M′** is proper extension:

for $\delta \in \mathit{DefFuc}(\mathbf{M})$ defined by $\delta(u) := u$ we have

$$[\delta] \in M' \setminus M \ .$$

Lemma

$\mathbf{M}'$ is an end-extension of $\mathbf{M}$.

prf - lemma

Prf.:

Let $m \in M$ and $\beta \in DefFuc(\mathbf{M})$, and assume

$$\mathbf{M}' \models [\beta] < m$$

(m is represented by $[\lambda_m]$). Hence

$$D := \langle\langle \beta < m \rangle\rangle \in \mathcal{U} \ .$$

Define

$$\alpha(u) \ := \ \beta(u) \ , \text{ if } u \in D \text{ and } := m, \text{ otherwise.}$$

By the property of $\mathcal{U}$, one of $\alpha^{(-1)}(u)$ for some $u \leq m$ has to be in $\mathcal{U}$. But it cannot be $\alpha^{(-1)}(m)$ because that is $M \setminus D$. So for some $u < m$:

$$\alpha^{(-1)}(u) = \langle\langle \beta = u \rangle\rangle \in \mathcal{U} \ .$$

□