Lecture 10

types

topics

- HW - ultraproducts of finite fields
- complete and partial types
- ex's over DLO and ACF
- Stone space
- types realized and omitted
- characterization of complete types in terms of elem. extensions
- saturated structures and their properties

HW-1

task 1: $I = \omega$ and $\mathcal{U}$ non-principal

$$\mathbf{F}^* := \prod_{i \in \omega} \mathbf{F}_p$$

Each structure in the product is $\mathbf{F}_p$ and hence satisfies

$$\forall x \, (x = 0 \vee x = 1, \cdots \vee x = p - 1) \text{ and } \mathit{char} = p \, .$$

Hence this is true in all coordinates I and also $I \in \mathcal{U}$.

By Los's thm this holds in $\mathbf{F}^*$ too, i.e. $\mathbf{F}^* \cong \mathbf{F}_p$.

HW-2

task 2: I: the set of primes, $\mathcal{U}$ non-principal

$$\mathbf{F}^* \ := \ \prod_{p \in I} \mathbf{F}_p$$

Each structure in the product is a field (i.e. satisfies FO axioms of fields), so $\mathbf{F}^*$ is a field too.

For any given $p \in I$, in all but one coordinate (namely p itself) the field satisfies

$$\mathbf{F}_q \ \models \ char \neq p \ .$$

All cofinite sets are in $\mathcal{U}$, so by Los's thm this is true in $\mathbf{F}^*$ too: it has characteristic 0.

set-up

What is "the theory" of a single n-tuple from a structure?

set-up:

- L: arbitrary
- $\mathbf{M}$: an L-structure
- $A \subseteq M$: a set of selected parameters
- L_A: L together with names for all elem's of A
  (as in defining diagrams)
- L_A-flas: L-flas using also parameters from A; i.e. such a fla has the form

$$\psi(\overline{x}, \overline{a})$$

  where $\overline{a} \in A^m$ and ψ is an L-fla

types of elements

Definition

The *n-type of $\overline{b} \in M^n$ over A*, denoted

$$\mathrm{tp}^{\mathbf{M}}(\overline{b}/A) \ ,$$

is the set of all L_A-flas $\varphi(\overline{x})$ satisfied in $\mathbf{M}$ by $\overline{b}$.

Remark: earlier we used names like $\mathbf{A}$ and $\mathbf{B}$ for structures, in this lecture (and the next one) I will use $\mathbf{M}$ to be in line with the text in Marker's book: he uses A for the parameter set.

Key property:

$$\mathrm{tp}^{\mathbf{M}}(\overline{b}/A) \ + \ \mathrm{Th}_A(\mathbf{M})$$

is consistent (i.e. satisfiable).

types

Definition

A *n-type over A* (tacitly in $\mathbf{M}$) is any set $p(\overline{x})$ of L_A-formulas with free var's $x_1, \ldots, x_n$ such that

$$p(\overline{x}) \; + \; \mathrm{Th}_A(\mathbf{M})$$

is satisfiable.
The type is complete iff for any L_A-fla $\varphi(\overline{x})$, either $\varphi \in p$ or $\neg\varphi \in p$.

Observations:
(1) Types $\mathrm{tp}^{\mathbf{M}}(\overline{b}/A)$ are complete.
(2) For any finite list $\varphi_1, \ldots, \varphi_k \in p$ the sentence

$$\exists \overline{x} \bigwedge_{i \leq k} \varphi_i(\overline{x})$$

must be true in $\mathbf{M}$. Hence types are sets of L_A-flas that are finitely satisfiable in $\mathbf{M}$.

realized and omitted

Definition

An n-type $p(\overline{x})$ is realized in structure $\mathbf{M}'$, where

$$\mathbf{M} \;\preceq_A\; \mathbf{M}' \,,$$

iff there is $\overline{b} \in (M')^n$ such that

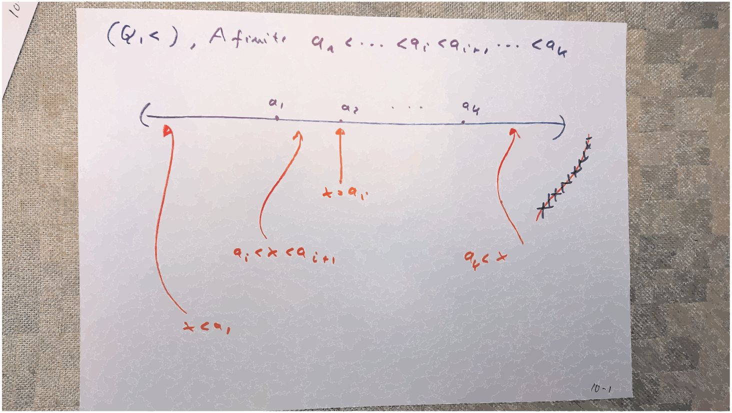
$$p(\overline{x}) \;=\; \mathrm{tp}^{\mathbf{M}'}(\overline{b}/A)$$

and such $\overline{b}$ is said to realize p.

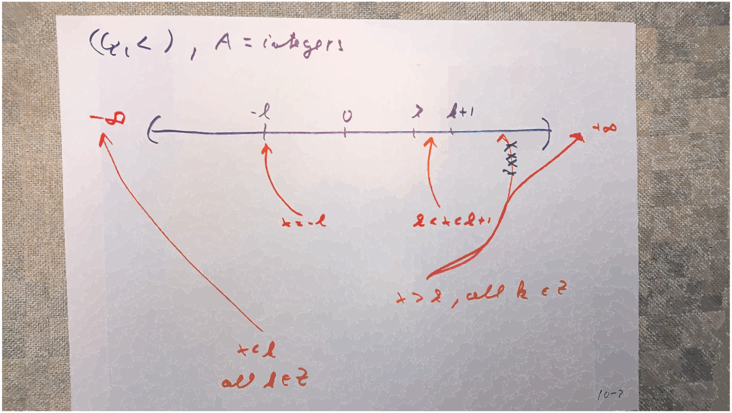
If p is not realized in $\mathbf{M}'$ then it is omitted.

Ex's to follow in four pictures.

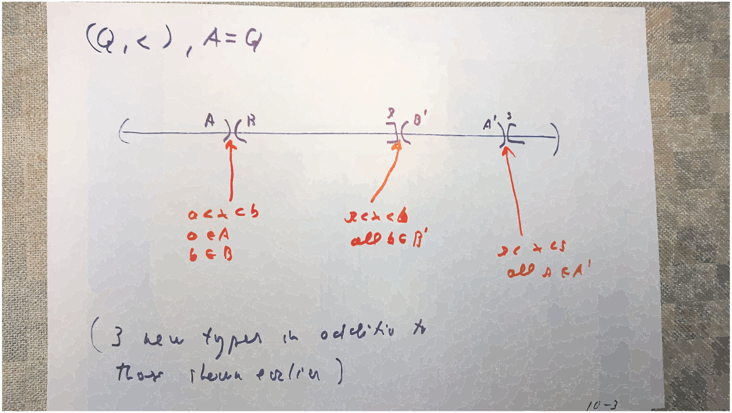
Ex.1



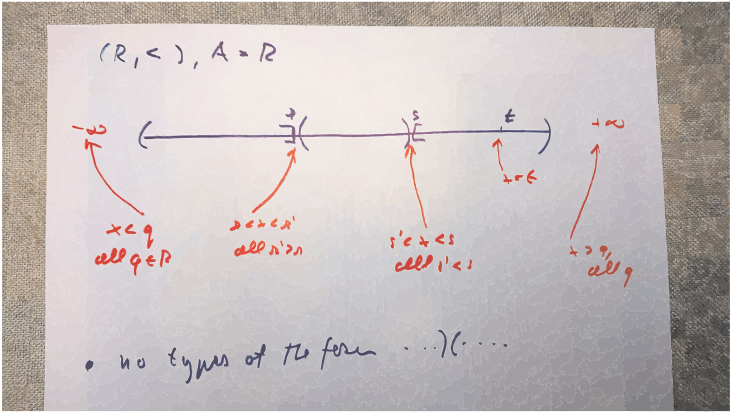
Ex.2



Ex.3


(Q, <), A = Q
A
B
B'
A'
a < x < b
a ∈ A
b ∈ B
all b ∈ B'
all a ∈ A'
(3 new types in addition to those shown earlier)
10-3

Ex.4


(R, <), A = R
−∞
+∞
x < q
all q ∈ R
r < x < r'
all r' > r
s' < x < s
all s' < s
x = t
x > q,
all q
• no types of the form ...)(....

types in ext's

Theorem

Type p is a complete type over A (in $\mathbf{M}$) iff there exists elementary extension

$$\mathbf{M}' \succeq \mathbf{M}$$

that realizes p: there is $\overline{b} \in (M')^n$ s.t.

$$p \;=\; \mathrm{tp}^{\mathbf{M}'}(\overline{b}/A)\;.$$

Prf.:
By the elementarity of the extension we have

$$\mathrm{Th}_A(\mathbf{M}) \;=\; \mathrm{Th}_A(\mathbf{M}')\;.$$

Hence any complete type consistent with $\mathrm{Th}_A(\mathbf{M}')$ is also consistent with $\mathrm{Th}_A(\mathbf{M})$ and thus any complete type over A in $\mathbf{M}'$ is also a type in $\mathbf{M}$.

prf cont'd

For the opposite direction assume p is a complete type over A in $\mathbf{M}$. We shall construct the wanted extension by ultrapower. For simplicity consider that L_A is countable: the argument in the general case is modified analogously to how we proved the compactness thm in Lect.9.

Enumerate p as $\varphi_0, \varphi_1, \ldots$ By the finite satisfiability in $\mathbf{M}$ we have, for each i, some elements $b_1^i, \ldots, b_n^i \in M$ s.t.

$$\mathbf{M} \models \bigwedge_{j \le i} \varphi_j(b_1^i, \ldots, b_n^i) \ .$$

Take the index set to be the set of natural numbers and define an ultrapower $\mathbf{M}^*$ of $\mathbf{M}$ by a non-principal ultrafilter. By a theorem from Lect.9:

$$\mathbf{M}^* \succeq \mathbf{M} \ .$$

prf cont'd

Define elements $[\beta_1], \ldots, [\beta_n]$ of the ultrapower by

$$\beta_\ell(i) \ := \ b_\ell^i \ .$$

By the choice of the elements b_ℓ^i we have that for each j the set

$$\langle\langle \varphi_j(\beta_1, \ldots, \beta_n) \rangle\rangle$$

is cofinite and hence in $\mathcal{U}$. By Los's thm then:

$$\mathbf{M}^* \ \models \ \varphi_j([\beta_1], \ldots, [\beta_n]) \ , \quad \text{for all} \ \ j \geq 0 \ .$$

$\square$

Stone sp.

Definition

The Stone space $S_n^{\mathbf{M}}(A)$ is the set of all complete types over A in $\mathbf{M}$.

We speak of a space because we can endow the set with a topology.

basis open sets:

$$[\varphi] \; := \; \{p \mid \varphi \in p\}.$$

As the complement of $[\varphi]$ is $[\neg\varphi]$, each such set is clopen and the space is totally disconnected:

$$p \neq q \; \Rightarrow \; \exists \varphi \; p \in [\varphi] \wedge q \in [\neg\varphi] \; .$$

Note also:

$$[\varphi] \cup [\psi] \; = \; [\varphi \vee \psi] \;\text{ and }\; [\varphi] \cap [\psi] \; = \; [\varphi \wedge \psi] \; .$$

The space is compact: a consequence of the compactness of FO logic.

ACF ex

Ex.: Let $\mathbf{K}$ be an ACF and $A \subseteq K$ its subfield (w.l.o.g.). We define a map:

$$p \in S_n^{\mathbf{K}}(A) \;\rightarrow\; I_p \;:\; \text{and ideal in } A[\overline{x}]$$

by:

$$I_p := \{f(\overline{x}) \mid f = 0 \in p\} \,.$$

Lemma

The map is a bijection from $S_n^{\mathbf{K}}(A)$ onto the set of prime ideals in $A[\overline{x}]$.

$\Rightarrow$: easy
$\Leftarrow$: needs some algebra
(The map is actually continuous using Zariski topology on the target space.)

realized vs. omitted

If p is realized in $\mathbf{M}$ it will be realized in all elementary extension. But we may try to omit it in some

$$\mathbf{M}' \equiv \mathbf{M} .$$

This will be treated in Lecture 11.

If p is omitted in $\mathbf{M}$ we can realized it in an elem.extension (by an earlier thm).

But can we realize all types over all parameter sets at the same time?

saturation

Definition

Let κ be an infinite cardinality. Structure $\mathbf{M}$ is κ-saturated iff it realizes all types over all parameter sets $A \subseteq M$ of cardinality $|A| < \kappa$.
$\mathbf{M}$ is saturated iff it is $|M|$-saturated.

We cannot allow $|M|$ many parameters as then we could write type:

$$\{x \neq m \mid m \in M\}$$

which can never be realized in $\mathbf{M}$.

Lemma

$\mathbf{M}$ is κ-saturated iff it realizes all 1-types over any parameter set of cardinality less than κ.

Prf.: HW!

uniqueness

We shall study the existence of saturated structures in Lecture 11 but now we shall note three properties such structures would have.

To simplify the cardinality hypotheses of the statements we shall assume that L is countable.

Theorem - uniqueness

Let $\mathbf{M}_1 \equiv \mathbf{M}_2$ and $|M_1| = |M_2|$, and assume both structures are infinite and saturated. Then $\mathbf{M}_1 \cong \mathbf{M}_2$.

Prf.:

This is a variant of the back-and-forth construction as in Cantor's thm. Enumerate the universes of both structures as

$$a_\alpha \text{ and } b_\alpha \text{ , resp. , } \alpha < \lambda$$

where λ is their cardinality.

Then prove - by transfinite induction - that there are partial elementary bijections $h_\beta :\subseteq M_1 \to M_2$ for all $\beta < \lambda$ such that:

$$\{a_\alpha \mid \alpha < \beta\} \subseteq dom(h_\beta) \text{ and } \{b_\alpha \mid \alpha < \beta\} \subseteq rng(h_\beta) \text{ .}$$

prf cont'd

Put:
$h_0 := \emptyset$
$h_\beta := \bigcup_{\gamma<\beta} h_\gamma$, for limit β

key case $h_{\beta+1}$:
Forth-direction: if $a_{\beta+1} \notin dom(h_\beta)$ then consider type

$$\mathrm{tp}^{\mathbf{M}_1}(a_{\beta+1}/dom(h_\beta)) \ .$$

By the saturation of $\mathbf{M}_2$ we can realize this type in $\mathbf{M}_2$ with the parameters $dom(h_\beta)$ replaced by $rng(h_\beta)$: map $a_{\beta+1}$ to any element realizing the type.
Back-direction: analogous.

□

homogeneity

Theorem - homogeneity

Assume $\mathbf{M}$ is saturated, $A \subseteq M$ and $|A| < |M|$. Then any partial elementary map $h : A \to M$ can be extended to an automorphism of $\mathbf{M}$.

In particular, if $\overline{a}, \overline{b} \in M^n$ have the same n-type then there is an automorphism h of $\mathbf{M}$ that maps $\overline{a}$ to $\overline{b}$. (The opposite implication is always true.)

A structure with the property described in the statement is called homogeneous.

Prf.: analogous to the forth-direction in the previous proof.

□

universality

Theorem - universality

Assume **M** is saturated. Then any elementarily equivalent model of cardinality at most $|M|$ can be elementarily embedded into **M**.

A structure with the property described in the statement is called universal.

Monster model $\mathcal{M}$: a saturated model of a "huge cardinality" (e.g. inaccessible cardinal).
It is used in model th. as an ambient universe for all models of a complete theory.