Lecture 1

a review of first-order logic

prerequisites

If you need to recall basics of first-order logic see the literature recommended for the

Introduction to Mathematical Logic

course at

www.karlin.mff.cuni.cz/˜krajicek/ml.html

I particularly recommend the lecture notes by *Lou van den Dries* available from this page.

topics

FO logic:

- languages (terms, formulas, sentences, ...)
- structures
- satisfiability relation
- theories and their models
- the Completeness and the Compactness theorems
- definable sets and functions

language L

Vocabulary:

- set C_L of constants: $c, d, \ldots$
- set R_L of relation symbols: $R, S, \ldots$, each coming with arity $n_R \geq 1$
- set F_L of function symbols: $f, g, \ldots$, each coming with arity $n_f \geq 1$

Common symbols:

- equality $=$
- logical connectives: $\vee, \wedge, \neg, \rightarrow, \equiv, \ldots$
- variables $x, y, \ldots$
- quantifiers $\exists$ and $\forall$
- brackets of various types: $(,), [,], \ldots$

L-terms

terms:

- variables are terms,
- if $s_1, \ldots, s_k$ are terms and $f \in F_L$ of arity k then

$$f(s_1, \ldots, s_k)$$

  is a term,
- only strings obtained by a finite number of applications of these rules are terms.

Notation:

$$t(x_1, \ldots, x_n)$$

means that all variables occurring in t are among $x_1, \ldots, x_n$

L-formulas

formulas:

- atomic formulas:
  - $t = s$, where t, s are any terms,
  - $R(t_1, \ldots, t_k)$, where $n_R = k$ and t_i are terms.
- formulas are closed under logical connectives; $(\varphi \vee \psi), (\varphi \wedge \psi), \ldots,$
- quantifiers: if φ is a formula, so are $(\exists x \varphi)$ and $(\forall x \varphi)$,
- only strings obtained in a finite nb. of steps via rules above are formulas.

There are always formulas, even if the vocabulary of L is empty.
Ex.:

$$x = y \quad \text{or} \quad (\forall x \; x \neq x)$$

occurrences of variables

A variable x may have free occurrence in a formula, as in

$$x = x \quad \text{or} \quad \exists y \; x \neq y$$

or bounded ($=$ closed), as in

$$\forall x \exists y \; x < y \quad \text{or} \quad \exists x \; x \neq 0$$

Remarks:
- interpret free as meaning "free for substitution"
- x in a quantifier is not assigned either qualification

sentences: formulas without any free occurrence of a variable

Notation: $\varphi(x_1, \ldots, x_n)$ means that all variables with some free occurrence are among $x_1, \ldots, x_n$

theories

L-theory: a set of L-sentences (called *axioms*)

Ex. LO - linear orders

Axioms: the universal closures of formulas

- $\neg x < x$
- $(x < y \wedge y < z) \rightarrow x < z$
- $x < y \vee x = y \vee y < x$

Ex.: DLO - dense linear orders: LO plus

$$x < y \rightarrow \exists z\ (x < z \wedge z < y)$$

L-structures

Ex.: the ordered real closed field:

$$\mathbf{R} = (R, 0, 1, +, \cdot, <)$$

R: the set of reals
$0, 1, +, \cdot, <$: usual meaning

Ex. the countable dense linear order

$$(Q, <)$$

rationals Q with their usual ordering $<$

When we target a particular structure or a class of structures it is customary to use symbols that are established. I.e.:

- we use $+$ for addition and not $x \circ y$ or $f(x, y)$
- we use $<$ for ordering and not just generic $R(x, y)$

L-structures

a general L-structure

$$\mathbf{A} = (A, c^{\mathbf{A}}, \ldots, R^{\mathbf{A}}, \ldots, f^{\mathbf{A}}, \ldots)$$

where

$A \neq \emptyset$ is the universe

and $c^{\mathbf{A}}, \ldots, R^{\mathbf{A}}, \ldots, f^{\mathbf{A}}, \ldots$ interpret L:

- $c^{\mathbf{A}} \in A$
- $R^{\mathbf{A}} \subseteq A^k$, if $n_R = k$
- $f^{\mathbf{A}} : A^k \to A$, if $n_f = k$

Remark: we often skip the superscript $\mathbf{A}$ in $c^{\mathbf{A}}$, etc, when there is no danger of a confusion.

term evaluation

Each term $t(\overline{x})$, where $\overline{x} = (x_1, \ldots, x_n)$, determines

$$t^{\mathbf{A}} \;:\; A^n \to A$$

which is defined by induction on the (syntactic) complexity of t:

- for t a constant this is determined by the interpretation of L
- for $t = f(s_1(\overline{x}), \ldots, s_k(\overline{x}))$ define for $\overline{a} \in A^n$ the value by composition:

$$t^{\mathbf{A}}(\overline{a}) \;:=\; f^{\mathbf{A}}(s_1^{\mathbf{A}}(\overline{a}), \ldots, s_k^{\mathbf{A}}(\overline{a}))$$

satisfiability relation

Definition (Tarski)

For L, $\mathbf{A}$, $\varphi(\overline{x})$ and $\overline{a} \in A^n$ define the satisfiability relation

$$\mathbf{A} \models \varphi(\overline{a})$$

by induction on the complexity of φ:

- $\mathbf{A} \models t(\overline{a}) = s(\overline{a})$ iff $t^{\mathbf{A}}(\overline{a}) = s^{\mathbf{A}}(\overline{a})$
- $\mathbf{A} \models R(\overline{a})$ iff $\overline{a} \in R^{\mathbf{A}}$
- $\models$ commutes with logical connectives:
  $\mathbf{A} \models \varphi(\overline{a}) \wedge \psi(\overline{a})$ iff $\mathbf{A} \models \varphi(\overline{a})$ and $\mathbf{A} \models \psi(\overline{a})$, etc.
- $\mathbf{A} \models \exists y \varphi(\overline{a}, y)$ iff there is $b \in A$ s.t. $\mathbf{A} \models \varphi(\overline{a}, b)$
  and analogously for $\forall$

models of theories

Definition - models

A is a model of theory T iff

$$\mathbf{A} \models \theta$$

for all axioms $\theta \in T$.
T having a model is satisfiable, otherwise it is unsatisfiable.

Ex. $(N, <)$ is a model of LO but not of DLO while $(Q, <)$ is a model of DLO.

Definition - logical consequence

A formula $\varphi(\overline{x})$ is a logical consequence of (or is logically implied by) theory T iff the universal closure $\forall \overline{x} \varphi(\overline{x})$ holds in every model of T.
Notation: $T \models \varphi$.

provability

How else can we establish logical consequences of T? By proofs in predicate calculus:

$$\psi_1, \ldots, \psi_\ell (= \varphi)$$

such that each formula ψ_i is

- an axiom of propositional logic, quantifier ax., ax. of equality or of T,
- or follows from some earlier formulas ψ_j by one of inference rules.

Ex. of axioms: $\alpha \vee \neg\alpha$, $\overline{x} = \overline{y} \to f(\overline{x}) = f(\overline{y})$,
$\varphi(t) \to \exists y \varphi(x)$ (subject to a condition on t), etc.

Ex. of rules:

$$\frac{\alpha \qquad \alpha \to \beta}{\beta} \quad \text{or} \quad \frac{\eta \to \psi(x)}{\eta \to \forall x \psi(x)}$$

the key thms

Notation: $T \vdash \varphi$ iff T proves φ.

Completeness thm - Gödel 1930

$$T \vdash \varphi \ \text{ iff } \ T \models \varphi \ .$$

Alternatively: S is unsatisfiable iff S in inconsistent (proves everything).

A key corollary for logic and for model theory in particular:

Compactness thm. - Gödel, Mal'tsev

$T \models \varphi$ iff there is a finite $T_0 \subseteq T$ such that $T_0 \models \varphi$.
Alternatively: S is unsatisfiable iff there is a finite $S_0 \subseteq S$ that is unsatisfiable.

definable sets

When studying the real closed field in geometry or analysis we often consider more functions and relations than are those in the language: continuous or analytic f's, all open subsets of some R^n, ...
How can this be treated in FO logic? The key notion is:

Definable sets and functions

A subset $U \subseteq A^n$ is definable in $\mathbf{A}$ iff there is a formula

$$\psi(\overline{x}, \overline{z}) \ ,$$

with $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{z} = (z_1, \ldots, z_t)$ and $\overline{b} \in A^t$ (= parameters) s.t. for all $\overline{a} \in A^n$:

$$\overline{a} \in U \quad \text{iff} \quad \mathbf{A} \models \psi(\overline{a}, \overline{b}) \ .$$

A function $h : A^k \to A$ is definable iff its graph is definable.

definable in $\mathbf{R}$

Ex. Sets definable in $\mathbf{R}$ = semialgebraic sets.

There is a trade-off:
- bigger language implies
- more definable sets and functions
- hence more interesting objects are included
- but if the language is too big we cannot obtain a sensible information about the definable sets and functions and may end-up in - essentially - the set theoretic world.

This we do not want: many set-theoretic properties of general sets and functions (even on reals) are not decidable by axioms of contemporary mathematics (= ZFC) and, more importantly, the geometric and algebraic flavor of model theory gets lost.

Ex.: the set-theoretic cardinality of a set versus the topological notion of Euler characteristic