(1) With the notation as in the green book (Sec. 5.2 and Chpt. 7, in particular) we want to prove:

<u>Lemma:</u> $K(F_{rud}, G_{rud})$ does not satisfy the LNP for open L^2_n-formulas. //

[We know it does satisfy open IND.]

(2) Pick m s.t. $n = m \cdot \lceil \log m \rceil$ and take m disjoint sets $V_1, \ldots, V_m$ of $\lceil \log m \rceil$ variables each. Identify:

assign's to $\bigcup_i V_i$ $\rightleftarrows$ samples $\omega \in \Omega := P(\{n\})$.

(3) Define $\alpha_i \in F_{rud}$ by low depth trees T_i:

(i) T_i queries exactly all var's in V_i.

(ii) First i leaves (out of m) are labelled by 1, the remaining ones by 0.

(4) Put $\Gamma \in G_{rud}$ to be $\Gamma := (\alpha_1, \ldots, \alpha_m)$.

(5) Simple

<u>Claim:</u> $[\![1 \notin \Gamma]\!] = 1_{\mathbb{B}}$ and $[\![m \in \Gamma]\!] = 1_{\mathbb{B}}$. □

(6) By (5) Lemma (1) will follow from:

$$[\![\exists x \leq m \, (x \in \Gamma \wedge (\forall y (y < x \to y \notin \Gamma)))]\!] < 1_{\mathbb{B}} .$$

(7) Assume for the sake of a contradiction that the value in (6) is $1_{\mathbb{B}}$. By Thm 3.5.2 we get $\alpha \in F_{\mathrm{rud}}$ s.t.:

(*) $[\![\alpha \leq m \wedge \alpha \in \Gamma \wedge \forall y (y < \alpha \to y \notin \Gamma)]\!] = 1_{\mathbb{B}}$.

To ~~prove~~ derive a contradiction (and thus to prove (1)) it suffices to show:

<u>Claim:</u> $\exists \beta \in F_{\mathrm{rud}}$:

$$[\![\beta < \alpha \wedge \beta \in \Gamma]\!] > 0_{\mathbb{B}} .$$

(8) Proof of Claim (7)

(a) Let T be a tree computing α. W.l.o.g. we may assume that whenever T queries some variable $\in V_i$ it queries all of V_i. (This keeps dp(T) low.

(b) Using (T) satisfying (a) we may assume T queries $i \in_? \Gamma_\omega$ rather than var's of ω: This is because $\delta_i, \delta_{i'}$ for $i \neq i'$ are computed on disjoint sets of var's, so values of var's in V_i do not influence value of $\delta_{i'}$.

(c) If P is a path in T from the root to a leaf, let I_P ~~be the set~~ consists of all i s.t. $i \in_? \Gamma_\omega$ is queried on P. For $i \in [m]$ call i:

- certified one, if $i \in I_P$ and $i \in_? \Gamma_\omega$ got answer YES
- certified zero, — " — NO
- undecided, if $i \notin I_P$.

(4.)

(d) Define $\beta \in \mathrm{Fruc}$ by T but with labels changed as follows. If i is the label of the leaf at path P do:

(d1) i is certified ~~zero~~: keep i.

(d2) i is a certified one and $\exists i' < i$ which is also certified one: change i to i' (choose any such i').

(d3) i is undecided: change i to $i-1$.

(d4) i is the min certified one: change i to $i-1$.

(e) Subclaim: $[\![\beta < \alpha \wedge \beta \in \Gamma]\!] > 0_{\mathbb{B}}$.

Proof:

(i) If ω leads to a certified zero then it is not in $\langle\langle \alpha \leq m \wedge \alpha \in \Gamma \rangle\rangle$.

(ii) If ω leads to i as in (d2): i' also satisfies $\omega \in \langle\langle \beta \in \Gamma \wedge \beta < \alpha \rangle\rangle$.

(iii) All ω determining P have the probability $\frac{i}{n}$ to satisfy $\delta_i(\omega) = 1$. For $i-1$ the prob is $\frac{i-1}{n}$, so we loose $\leq 1/n$ - fraction of $\omega \in \Omega$ that lead to a indecided i.a P.

(iv) For (d4) (the key case) note that only

(*) an infinitesimal fraction of $\omega \in \Omega$ determine P labelled by min. certified one i s.t. $i \leq 3/4 \cdot n$.

This is because only a fraction $\frac{i}{n} \leq \frac{3}{4}$ of $\omega \in \Omega$ yield $\delta_i(\omega) = 1$ so if $1/k$-part of Ω ~~leads~~ to (d4) we would loose at least $\frac{1}{4k}$-part of Ω (which is not infinitesimal) and hence $\langle\langle \alpha \in P \rangle\rangle \neq \Omega$ (mod $\mathcal{J}$).

Hence we can change the minimal certified ones $i > 3/4\, n$ to $i-1$. We have:

$$\text{Prob}_{\omega}[\ \delta_{i-1}(\omega)\ ] = \frac{i-1}{n} > 2/3$$

for $i > 3/4 \cdot n$. Hence on these ω's leading to (d4) we can loose $\leq 1/3$-part of Ω.

This shows (*).

(9) Let us summarize the proof of Claim (7):

"cases (d1) and (d4) with $i \le \frac{3}{4}$"

happen with an infinitesimal probability.

In all other cases i is changed to

some $i' < i$. So:

$$\langle\langle \beta < \alpha \rangle\rangle = \Omega \pmod{\mathcal{J}}.$$

Also, in (d2), (d3) and "(d4) with $i > \frac{3}{4}$",

only fractions of 0, 1/4 and 1/3 of Ω

are lost from $\langle\langle \alpha \in \Gamma \rangle\rangle$. So:

$$\mu(\langle\langle \beta \in \Gamma \rangle\rangle) \ge \mu(\langle\langle \alpha \in \Gamma \rangle\rangle) - 1/3 = 2/3 .$$

Thus indeed: $[\![\beta \in \Gamma]\!] > 0_{\mathcal{B}}$.

$\Box$