(1) <u>Set-up</u>: same as in my green book in the chapt's about SO structures and the rudimentary model:

$\mathcal{M}$, $\mathcal{M}_n$, $K(F,G)$ extends/expands $\mathcal{M}_n$, etc.

(2) <u>Want</u>: a model $K(F,G)$ s.t.

$$[\![\Sigma_1^b(F,G) - LNP]\!] = 1_{\mathcal{B}}$$

while

$$[\![PHP_n(\Gamma)]\!] < 1_{\mathcal{B}}$$

for some $\Gamma \in G$.

(3) $Maps^{\delta} := \{ \omega : \subseteq [n+1] \xrightarrow[1-to-1]{} [n] \mid |\omega| = \delta \}$

$\Omega := Maps^{n}$

queries: $\bigvee_{i<m} h_i^{\in \mathcal{M}}$, where

$h_i \in Map^{low}$. (low: $n^{\text{infinitesimal}}$), $m \in \mathcal{M}_n$

F: random var's $\alpha : \Omega \to \mathcal{M}_n$ computed by low-depth trees

G: determined by F as in the green book.

(4) <u>Lemma</u>: $[\![\text{Open-IND} \wedge \text{Open-CA}]\!] = 1_{\mathcal{B}}$.

(As in the rudimentary model.)

(5) <u>Lemma</u>: For $\varphi(x)$ an open $L_n(F,G)$-fla and $m \in \mathcal{M}_n$ there is $\Delta \in G$ s.t. for all $\alpha \in F$:

$$[\![\alpha < m \wedge \varphi(\alpha)]\!] = [\![\Delta(\alpha) = 1]\!].$$

<u>Prf</u>: (As in the rud. model.) The truth-value of $\omega \in \langle\langle \dot{i} < m \wedge \varphi(\dot{i}) \rangle\rangle$ is computed by some $\delta_i \in F$, for all $i < m$.

Put: $\Delta = (\delta_0, \ldots, \delta_{m-1})$.

□

(6) Lemma: Same as L5 for $\varphi(x) \in \Sigma_0^b(F, G)$.

Prf: Given $\varphi(x) = (\exists y_1 \le K_1(x)) \cdots (\exists y_k \le K_k(x)) \psi(x, \bar{y})$

we can use L5 to define $\delta_{i, j_1, \dots, j_k}$ for

all $i < n$ and $j_1, \dots, j_k <$ polylog n. Such

$\delta_{i, j_1, \dots, j_k}$ can be combined - for a fixed i -

to evaluate $\varphi(i)$ (the # of $\bar{j}$ is poly $\log n$,

so the combined tree is still low depth). $\square$

(7) Lemma: $[\![\Sigma_1^b(F, G) - \mathrm{LNP}]\!] = 1_{\mathcal{B}}$.

Prf: By L6 we can restrict to $\Sigma_1(F, G)$-fms.

As Σ_1-LNP is equivalent to open-LNP,

L4 concludes the proof. $\square$

(8) Define $\Gamma \in G$ by $\Gamma := (\gamma_1, \dots, \gamma_{n-1})$

where γ_i is defined by a tree T_i

querying first $\bigvee_{j \in [n]} p_{ij}$ and:

- If the answer is YES it uses binary search with queries $\bigvee_{j \in J} p_{ij}$, always halving $J \subseteq [n]$, until j s.t. $p_{ij} = true$ is found. That j labels the corresponding leaf.
- Otherwise label the leaf by 1.

(7) <u>Lemma</u>: $[\![PHP_n(\Gamma)]\!] = 0_{\mathcal{B}}$.

Pf: Analogously to the rudimentary model

$$[\![\forall x \in [n+1], \Gamma(x) \in [n]]\!] = 1_{\mathcal{B}}.$$

We want to establish

$$[\![\forall x, y \in [n+1], x \neq y \to \Gamma_x \neq \Gamma_y]\!] = 1_{\mathcal{B}},$$

i.e. for any two $\alpha, \beta \in F$:

$$[\![\alpha, \beta \in [n+1] \wedge \alpha \neq \beta \to \Gamma_\alpha \neq \Gamma_\beta]\!] = 1_{\mathcal{B}}.$$

We may assume w.l.o.g. that

$$\langle\langle \alpha \neq \beta \wedge \alpha, \beta \in [n+1] \rangle\rangle = \Omega$$

(change some values of α, β if necessary - that moves errors here to errors in $\langle\langle \Gamma(\alpha) \neq \Gamma(\beta) \rangle\rangle$.)

If $\alpha(\omega) = i_1$, $\beta(\omega) = i_2$, with $i_1 \neq i_2 \in [n+1]$, while $\Gamma(\alpha)(\omega) = \Gamma(\beta)(\omega)$, then necessarily $\Gamma(\alpha)(\omega) = \Gamma(\beta)(\omega) = 1$ and we can use queries $(i_1, 1)?$ and $(i_2, 1)?$ to find out which of $\{i_1, i_2\}$ is in $dom(\omega)$ and which is outside. Hence from α, β we construct $\gamma \in F$ s.t.:

- for all ω:

$$\Gamma(\alpha)(\omega) = \Gamma(\beta)(\omega) \longrightarrow \gamma(\omega) \in [n+1] \setminus dom(\omega).$$

To prove the lemma it thus suffices to establish the following key claim.

$\square$

(8) <u>Key claim</u>: For all $g \in F$:

$$\operatorname*{Prob}_{\omega}\left[\, g(\omega) \in [k+1]\text{-dom}(\omega) \,\right]$$

is infinitesimal.

<u>Proof</u>: We shall transform first g into a different computational model where the trees will use list-queries:

$$(h_i)_{i \le m} \in \mathcal{U}$$

Each such $(h_i)_{i \le m}$ is obtained from $(\bigvee_{i<m} h_i)$ by fixing the ordering of terms h_i and appending $h_m := \phi$ at the end.

The answer of ω to $(h_i)_{i \le m}$ is the first $i \le m$ s.t. $h_i \le \omega$.

Let S be the tree computing g (using the original DNF-queries) and let S' be the tree using the list-queries instead, obtained by the transformation above.

Now observe that the levels of list-q's can be collapsed into one list-q. ~~Of~~ Assume.

$$(h_i)_{i \le n}$$

is ~~a~~ the root and $(a_j^i)_{j \le b^i}$ are the list-q's at the respective sons $i \le n$ of the root. Then the answers of ω ~~to~~ at the root and at the resp. son are determined by its answer to one list-q:

$$(h_i a_j^i)_{\substack{i \le n \\ j \le b^i}}$$

ordered lexico-graphically. Repeating this $dp(S)$-many times we get

<u>Claim 1</u>: ~~There~~ There is a list-query $\hat{q} := (u_k)_{k \le s}$ and $(i_k)_{k \le s}$, a list of elements of $[n+2]$, s.t. for all ω:

$$\gamma(\omega) = i_{\hat{q}(\omega)}.$$

To conclude the proof of the key claim (8) (and thus of (7)) it remains to show:

Claim 2: For any $\hat{q}$ and $(b_i)_{i \leq r}$ as in Claim 1:

$$\text{Prob}_{\omega}[\# i_{\hat{q}(\omega)} \in [n+1] \setminus dom(\omega)]$$

is infinitesimal.

Prf-claim 2: We shall use the proof of the PHP-switching lemma as given in my 1995 book. In that proof one takes a partial PHP-restriction $\rho : \subseteq [n+1] \Longleftrightarrow [n]$, $|\rho| = n - n^{\varepsilon}$ (some $\varepsilon > 0$ standard), and plays a game on the sequence

$$u_1{}^{\rho}, u_2{}^{\rho}, \ldots \ldots, u_s{}^{\rho}.$$

The player is faced consequtively with $u_i{}^{\rho}$, $i = 1, \ldots, s$ and builds a map $\tau \supseteq \rho$ that defines values (pre-images) of all items $(u_i \cdot \rho)$ (or vars $(u_i \cdot \rho)$) that either kills $u_i{}^{\rho}$ or contains it. The game stops iff ~~when~~ the latter occurs.

The content of the PHP-switch. l. is that (for a random ρ with probability exp. close to 1) the game stops with $\bar{\tau}$ of six $|\bar{\tau}| \leq n - n^{\delta}$, some $0 < \delta < \varepsilon$ ρ-tractable.

However, for $\omega \supseteq \bar{\tau}$, $\omega \in \Omega$,

$$\operatorname*{Prob}_{\omega \supseteq \bar{\tau}} \left[i_{\hat{q}(\omega)} \notin \operatorname{dom}(\omega) \right] \leq \frac{1}{n^{\delta}}$$

as already $\bar{\tau}$ determines $\hat{q}(\omega)$, (and only for a fraction of $\leq \frac{1}{n^{\delta}}$ $\omega \supseteq \bar{\tau}$ the answer is correct).

The overall probability is thus infinitesimal. □

(9) <u>Remark</u>: It would be interesting (and desirable) to have an elementary proof of the key claim (8) avoiding the PHP-switching lemma.